Tannakian Categories

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Contents

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Introduction

In the first section, it is shown how to introduce on an abstract category operations of tensor products and duals having properties similar to the familiar operations on the category Vec_k of finite-dimensional vector spaces over a field k. What complicates this is the necessity of including enough constraints so that, whenever an obvious isomorphism, for example,

$$U \otimes (V \otimes W) \xrightarrow{\simeq} (V \otimes U) \otimes W,$$

exists in Vec_k, a unique isomorphism is constrained to exist also in the abstract setting.

The next section studies the category $\mathsf{Rep}_k(G)$ of finite-dimensional representations of an affine group scheme G over k and demonstrates necessary and sufficient conditions for a category C with a tensor product to be equivalent to $\mathsf{Rep}_k(G)$ for some G; such a category C is then called a neutral Tannakian category.

A fibre functor on a Tannakian category C with values in a field $k' \supset k$ is an exact k-linear functor $\mathsf{C} \to \mathsf{Vec}_{k'}$ that commutes with tensor products. For example, the forgetful functor is a fibre functor on $\mathsf{Rep}_k(G)$. In the third section it is shown that the fibre functors on $\mathsf{Rep}_k(G)$ are classified by the torsors of G. Also, the general notion of a (nonneutral) Tannakian category is introduced and discussed.

The fourth section studies the notion of a polarization (compatible families of sesquilinear forms having certain positivity properties) on a Tannakian category, and the fifth studies the notion of graded Tannakian category.

In the sixth section, motives are defined using absolute Hodge cycles, and the related motivic Galois groups discussed. In an appendix, some terminology from non-abelian cohomology is reviewed.

We note that the introduction to Saavedra Rivano 1972 is an excellent summary of the theory of Tannakian categories except that two changes are necessary: Théorème 3 only becomes correct when the condition "End($\mathbb{1}$) $= k$" is added to the definition of a Tannakian category over k; in the statement of Théorème 4 the condition that G be abelian or connected can be dropped (see §4 below).

Notations and Terminology

Functors between additive categories are assumed to be additive. All rings have a 1, and in general they are commutative except in §2. A morphism of functors is also called a functorial or natural morphism. A strictly full subcategory is a full subcategory containing with any X, all objects isomorphic to X. Isomorphisms are denoted $\approx$ and canonical (or given) isomorphisms $\simeq$. The empty set is denoted by $\emptyset$.

Our notations agree with those of Saavedra Rivano 1972 except for some simplifications: what would be called a $\otimes$-widget AC unifère by Saavedra here becomes a tensor widget, and $\underline{\mathrm{Hom}}^{\otimes,1}$ becomes $\underline{\mathrm{Hom}}^{\otimes}$.

Some categories:

Mod_R	Finitely generated R-modules.
Proj_R	Finitely generated projective R-modules.
$\mathsf{Rep}_k(G)$	Linear representations of G on finite-dimensional k-vector spaces.
Set	Category of sets.
Vec_k	Finite-dimensional k-vector spaces.

1. Tensor Categories

Let C be a category and let

$$\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C},\quad (X,Y)\rightsquigarrow X\otimes Y$$

be a functor. An ***associativity constraint*** for $(\mathsf{C},\otimes)$ is a functorial isomorphism

$$\phi_{X,Y,Z}\colon X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$$

such that, for all objects X,Y,Z,T, the diagram

$$\begin{tikzcd}
 & X\otimes(Y\otimes(Z\otimes T)) \arrow[dl, "1\otimes\phi"'] \arrow[dr, "\phi"] & \\
X\otimes((Y\otimes Z)\otimes T) \arrow[d, "\phi"'] & & (X\otimes Y)\otimes(Z\otimes T) \arrow[d, "\phi"] \\
(X\otimes(Y\otimes Z))\otimes T \arrow[rr, "\phi\otimes 1"] & & ((X\otimes Y)\otimes Z)\otimes T
\end{tikzcd} \tag{1.0.1}$$

is commutative (this is the ***pentagon axiom***, Saavedra Rivano 1972, I, 1.1.1.1; Mac Lane 1998, p. 162). Here, as in subsequent diagrams, we have omitted the obvious subscripts on the maps; for example, the ϕ at top-right is $\phi_{X,Y,Z\otimes T}$. A ***commutativity constraint*** for $(\mathsf{C},\otimes)$ is a functorial isomorphism

$$\psi_{X,Y}\colon X\otimes Y\to Y\otimes X$$

such that, for all objects X,Y,

$$\psi_{Y,X}\circ\psi_{X,Y}\colon X\otimes Y\to X\otimes Y$$

is the identity morphism on $X\otimes Y$ (Saavedra Rivano 1972, I, 1.2.1). An associativity constraint ϕ and a commutativity constraint ψ are ***compatible*** if, for all objects X,Y,Z, the diagram

$$\begin{tikzcd}
 & X\otimes(Y\otimes Z) \arrow[r, "\phi"] \arrow[dl, "1\otimes\psi"'] & (X\otimes Y)\otimes Z \arrow[dr, "\psi"] & \\
X\otimes(Z\otimes Y) \arrow[dr, "\phi"'] & & & Z\otimes(X\otimes Y) \arrow[dl, "\phi"] \\
 & (X\otimes Z)\otimes Y \arrow[r, "\psi\otimes 1"] & (Z\otimes X)\otimes Y &
\end{tikzcd} \tag{1.0.2}$$

is commutative (this is the ***hexagon axiom***, Saavedra Rivano 1972, I, 2.1.1.1; Mac Lane 1998, p. 184). A pair (U,u) comprising an object U of C and an isomorphism $u\colon U \to U\otimes U$ is an ***identity object*** of $(\mathsf{C},\otimes)$ if $X \rightsquigarrow U\otimes X\colon \mathsf{C}\to\mathsf{C}$ is an equivalence of categories.

DEFINITION 1.1. A system $(\mathsf{C},\otimes,\phi,\psi)$, in which ϕ and ψ are compatible associativity and commutativity constraints, is a ***tensor category*** if there exists an identity object.

EXAMPLE 1.2. The category Mod_R of finitely generated modules over a commutative ring R becomes a tensor category with the usual tensor product and the obvious constraints. (If one perversely takes ϕ to the negative of the obvious isomorphism, then the pentagon (1.0.1) fails to commute by a sign.) A pair (U,u_0) comprising a free R-module of rank 1 and a basis element u_0 determines an identity object (U,u) of Mod_R — take u to be the unique isomorphism $U\to U\otimes U$ mapping u_0 to $u_0\otimes u_0$. Every identity object is of this form.

For other examples, see the end of this section.

PROPOSITION 1.3. *Let (U,u) be an identity object of the tensor category $(\mathsf{C},\otimes)$.*

(a) *There exists a unique functorial isomorphism*

$$l_X\colon X\to U\otimes X$$

such that l_U is u and the diagrams

$$\begin{array}{ccc} X\otimes Y & \xrightarrow{l} & U\otimes(X\otimes Y) \\ \| & & \downarrow{\scriptstyle\phi} \\ X\otimes Y & \xrightarrow{l\otimes 1} & (U\otimes X)\otimes Y \end{array} \qquad \begin{array}{ccc} X\otimes Y & \xrightarrow{l\otimes 1} & (U\otimes X)\otimes Y \\ \downarrow{\scriptstyle 1\otimes l} & & \downarrow{\scriptstyle \psi\otimes 1} \\ X\otimes(U\otimes Y) & \xrightarrow{\phi} & (X\otimes U)\otimes Y \end{array}$$

commute.

(b) *If (U',u') is a second identity object of $(\mathsf{C},\otimes)$, then there is a unique isomorphism $a\colon U\to U'$ making*

$$\begin{array}{ccc} U & \xrightarrow{u} & U\otimes U \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a\otimes a} \\ U' & \xrightarrow{u'} & U'\otimes U' \end{array}$$

commute.

PROOF. (a) We confine ourselves to defining l_X — see Saavedra Rivano 1972, I, 2.5.1, 2.4.1, for more details. As $X\rightsquigarrow U\otimes X$ is an equivalence of categories, it suffices to define $1\otimes l_X\colon U\otimes X\to U\otimes(U\otimes X)$; this we take to be

$$U\otimes X \xrightarrow{u\otimes 1} (U\otimes U)\otimes X \xrightarrow{\phi^{-1}} U\otimes(U\otimes X).$$

(b) The map

$$U \xrightarrow{l_U} U'\otimes U \xrightarrow{\psi_{U',U}} U\otimes U' \xrightarrow{(l_{U'})^{-1}} U'$$

has the required properties. □

The functorial isomorphism

$$r_X \stackrel{\text{def}}{=} \psi_{U,X} \circ l_X : X \to X \otimes U$$

has analogous properties to l_X. We shall often use $(\mathbb{1}, e)$ to denote a (the) identity object of $(\mathsf{C}, \otimes)$.

REMARK 1.4. (a) There is no standard definition of "tensor category" in the literature. Rather, authors adopt the definition most convenient for their purposes.

(b) In the language of the category theorists, our tensor categories are symmetric monoidal categories (i.e., a monoidal categories equipped with a symmetric braiding).

(c) Our notion of a tensor category is the same as that of a "$\otimes$-catégorie AC unifère" in Saavedra Rivano 1972 and, because of 1.3(b), is essentially the same as the notion of "$\otimes$-catégorie ACU" defined ibid. I, 2.4.1 (cf. ibid. I, 2.4.3).

Extending $\otimes$

Let ϕ be an associativity constraint for $(\mathsf{C}, \otimes)$. Any functor $\mathsf{C}^n \to \mathsf{C}$ defined by repeated application of $\otimes$ is called an ***iterate*** of $\otimes$. If $F, F' : \mathsf{C}^n \to \mathsf{C}$ are iterates of $\otimes$, then it is possible to construct an isomorphism of functors $\tau : F \to F'$ out of ϕ and ϕ^{-1}. The significance of the pentagon axiom is that it implies that τ is unique: any two iterates of $\otimes$ to C^n are isomorphic by a unique isomorphism constructed out of ϕ and ϕ^{-1} (Mac Lane 1963; Mac Lane 1998, VII, 2). In other words, there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes_{i=1}^n : \mathsf{C}^n \to \mathsf{C}$ when $n \geq 1$. Similarly, when $(\mathsf{C}, \otimes)$ is a tensor category, there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes_{i \in I} : \mathsf{C}^I \to \mathsf{C}$ where I is any unordered finite set: the tensor product of any finite family of objects of C is well-defined up to a unique isomorphism (Mac Lane 1963). We can make this statement more precise.

PROPOSITION 1.5. *The tensor structure on a tensor category* $(\mathsf{C}, \otimes)$ *admits an extension as follows: for each finite set* I *there is a functor*

$$\bigotimes\nolimits_{i \in I} : \mathsf{C}^I \to \mathsf{C},$$

and for each map $\alpha : I \to J$ *of finite sets, there is a functorial isomorphism*

$$\chi(\alpha) : \bigotimes\nolimits_{i \in I} X_i \to \bigotimes\nolimits_{j \in J} \left(\bigotimes\nolimits_{i \mapsto j} X_i \right)$$

satisfying the following conditions:

(a) *if* I *consists of a single element, then* $\bigotimes_{i \in I}$ *is the identity functor* $X \rightsquigarrow X$*; if* α *is a map between single-element sets, then* $\chi(\alpha)$ *is the identity automorphism of the identity functor;*

(b) *the isomorphisms defined by maps* $I \xrightarrow{\alpha} J \xrightarrow{\beta} K$ *give rise to a commutative diagram*

$$\begin{tikzcd}
\bigotimes_{i \in I} X_i \arrow[rr, "\chi(\alpha)"] \arrow[d, "\chi(\beta\alpha)"] & & \bigotimes_{j \in J} \left(\bigotimes_{i \mapsto j} X_i \right) \arrow[d, "\chi(\beta)"] \\
\bigotimes_{k \in K} \left(\bigotimes_{i \mapsto k} X_i \right) \arrow[r, "\otimes(\chi(\alpha | I_k))"] & \bigotimes_{k \in K} \left(\bigotimes_{j \mapsto k} \left(\bigotimes_{i \mapsto j} X_i \right) \right). &
\end{tikzcd}$$

where $I_k = (\beta\alpha)^{-1}(k)$.

PROOF. Apply Mac Lane 1963; 1998 VII 2. □

By $(\bigotimes_{i\in I}, \chi)$ being an extension of the tensor structure on C, we mean that $\bigotimes_{i\in I} X_i = X_1 \otimes X_2$ when $I = \{1,2\}$ and that the isomorphisms

$$X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$$
$$X\otimes Y\to Y\otimes X$$

induced by χ are equal to ϕ and ψ respectively. It is automatic that $(\bigotimes_{\emptyset} X_i, \chi(\emptyset\to\{1,2\})$ is an identity object and that $\chi(\{2\}\hookrightarrow\{1,2\})$ is $l_X\colon X\to \mathbb{1}\otimes X$. If $(\bigotimes'_{i\in I}, \chi')$ is a second such extension, then there is a unique system of functorial isomorphisms $\bigotimes_{i\in I} X_i \to \bigotimes'_{i\in I} X_i$ compatible with χ and χ' and such that, when $I=\{i\}$, the isomorphism is id_{X_i}.

Whenever a tensor category $(\mathsf{C},\otimes)$ is given, we shall always assume that an extension as in (1.5) has been made. (We could, in fact, have defined a tensor category to be a system as in (1.5).)

Invertible objects

Let $(\mathsf{C},\otimes)$ be a tensor category. An object L of C is ***invertible*** if

$$X\rightsquigarrow L\otimes X\colon \mathsf{C}\to\mathsf{C}$$

is an equivalence of categories. Thus, if L is invertible, there exists an L' such that $L\otimes L' = \mathbb{1}$; the converse assertion is also true. An ***inverse*** of L is any pair (L^{-1},δ) where

$$\delta\colon \bigotimes_{i\in\{\pm\}} X_i \xrightarrow{\approx} \mathbb{1},\quad X_+ = L,\quad X_- = L^{-1}.$$

Note that this definition is symmetric: (L,δ) is an inverse of L^{-1}. If (L_1,δ_1) and (L_2,δ_2) are both inverses of L, then there is a unique isomorphism $\alpha\colon L_1\to L_2$ such that the composite

$$\delta_2\circ(1\otimes\alpha)\colon L\otimes L_1\to L\otimes L_2\to\mathbb{1}$$

is δ_1. For example, an object L of Mod_R is invertible if and only if it is projective of rank 1 (Saavedra Rivano 1972, I, 0.2.2.2).

Internal Hom

Let $(\mathsf{C},\otimes)$ be a tensor category.

DEFINITION 1.6. If the functor

$$T\rightsquigarrow \mathrm{Hom}(T\otimes X, Y)\colon \mathsf{C}^{\mathrm{opp}}\to\mathsf{Set}$$

is representable, then we denote by $\underline{\mathrm{Hom}}(X,Y)$ the representing object and by

$$\mathrm{ev}_{X,Y}\colon \underline{\mathrm{Hom}}(X,Y)\otimes X\to Y$$

the morphism corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$.

Thus, to a morphism $g\colon T \otimes X \to Y$ there corresponds a unique morphism $f\colon T \to \underline{\mathrm{Hom}}(X,Y)$ such that $\mathrm{ev}_{X,Y} \circ (f \otimes \mathrm{id}_X) = g$,

$$\begin{tikzcd} T \arrow[d, dashed, "f"'] \\ \underline{\mathrm{Hom}}(X,Y) \end{tikzcd} \qquad \begin{tikzcd} T \otimes X \arrow[d, dashed, "f\otimes \mathrm{id}"'] \arrow[dr, "g"] & \\ \underline{\mathrm{Hom}}(X,Y) \otimes X \arrow[r, "\mathrm{ev}"] & Y. \end{tikzcd} \tag{1.6.1}$$

In other words,

$$\mathrm{Hom}(T, \underline{\mathrm{Hom}}(X,Y)) \simeq \mathrm{Hom}(T \otimes X, Y).$$

For example, in Mod_R, $\underline{\mathrm{Hom}}(X,Y)$ exists and equals $\mathrm{Hom}_R(X,Y)$ regarded as an R-module, because for any R-modules X, Y, T,

$$\mathrm{Hom}_R(T, \mathrm{Hom}_R(X,Y)) \simeq \mathrm{Hom}_R(T \otimes_R X, Y)$$

(Bourbaki Algèbre, II 4.1). In this case, $\mathrm{ev}_{X,Y}$ is

$$f \otimes x \mapsto f(x)\colon \mathrm{Hom}_R(X,Y) \otimes X \to Y,$$

whence its name.

We now assume that $\underline{\mathrm{Hom}}(X,Y)$ exists for every pair (X,Y) of objects in C. Then there is a composition map

$$\underline{\mathrm{Hom}}(X,Y) \otimes \underline{\mathrm{Hom}}(Y,Z) \to \underline{\mathrm{Hom}}(X,Z), \tag{1.6.2}$$

corresponding to

$$\underline{\mathrm{Hom}}(X,Y) \otimes \underline{\mathrm{Hom}}(Y,Z) \otimes X \xrightarrow{\mathrm{ev}} \underline{\mathrm{Hom}}(Y,Z) \otimes Y \xrightarrow{\mathrm{ev}} Z,$$

and an isomorphism

$$\underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X,Y)) \to \underline{\mathrm{Hom}}(Z \otimes X, Y) \tag{1.6.3}$$

inducing, for every object T,

$$\begin{aligned} \mathrm{Hom}(T, \underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X,Y))) &\xrightarrow{\simeq} \mathrm{Hom}(T \otimes Z, \underline{\mathrm{Hom}}(X,Y)) \\ &\xrightarrow{\simeq} \mathrm{Hom}(T \otimes Z \otimes X, Y) \\ &\xrightarrow{\simeq} \mathrm{Hom}(T, \underline{\mathrm{Hom}}(Z \otimes X, Y)). \end{aligned}$$

Note that

$$\mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}(X,Y)) \simeq \mathrm{Hom}(\mathbb{1} \otimes X, Y) = \mathrm{Hom}(X,Y). \tag{1.6.4}$$

The ***dual*** $X^\vee$ of an object X is defined to be $\underline{\mathrm{Hom}}(X, \mathbb{1})$. There is therefore a map $\mathrm{ev}_X\colon X^\vee \otimes X \to \mathbb{1}$ inducing a functorial isomorphism

$$\mathrm{Hom}(T, X^\vee) \to \mathrm{Hom}(T \otimes X, \mathbb{1}). \tag{1.6.5}$$

The morphism $X \mapsto X^\vee$ can be made into a contravariant functor: to $f\colon X \to Y$ we attach the unique morphism ${}^t f\colon Y^\vee \to X^\vee$ rendering commutative

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{{}^t f \otimes \mathrm{id}} & X^\vee \otimes X \\ \downarrow{\scriptstyle \mathrm{id}\otimes f} & & \downarrow{\scriptstyle \mathrm{ev}_X} \\ Y^\vee \otimes Y & \xrightarrow{\mathrm{ev}_Y} & \mathbb{1}. \end{array} \tag{1.6.6}$$

For example, in Mod_R, $X^\vee = \mathrm{Hom}_R(X, R)$ and ${}^t f$ is determined by the equation

$$\langle {}^t f(y), x\rangle_X = \langle y, f(x)\rangle_Y, \quad y \in Y^\vee \quad x \in X,$$

where we have written $\langle\ ,\ \rangle_X$ and $\langle\ ,\ \rangle_Y$ for ev_X and ev_Y.

When f is an isomorphism, we let $f^\vee = ({}^t f)^{-1}\colon X^\vee \to Y^\vee$, so that

$$\mathrm{ev}_Y \circ (f^\vee \otimes f) = \mathrm{ev}_X\colon X^\vee \otimes X \to \mathbb{1}. \tag{1.6.7}$$

For example, in Mod_R,

$$\langle f^\vee(x'), f(x)\rangle_Y = \langle x', x\rangle_X, \quad x \in X^\vee, x \in X.$$

Let $i_X\colon X \to X^{\vee\vee}$ be the morphism corresponding in (1.6.5) to $\mathrm{ev}_X \circ \psi\colon X \otimes X^\vee \to \mathbb{1}$. If i_X is an isomorphism, then X is said to be ***reflexive***. If X has an inverse $(X^{-1}, \delta\colon X^{-1} \otimes X \xrightarrow{\approx} \mathbb{1})$, then X is reflexive and the map $X^{-1} \to X^\vee$ determined by δ (see 1.6.1) is an isomorphism.

For any finite families of objects $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$, there is a morphism

$$\bigotimes_{i\in I} \underline{\mathrm{Hom}}(X_i, Y_i) \to \underline{\mathrm{Hom}}(\bigotimes_{i\in I} X_i, \bigotimes_{i\in I} Y_i) \tag{1.6.8}$$

corresponding in (1.6.1) to

$$\Big(\bigotimes_{i\in I} \underline{\mathrm{Hom}}(X_i, Y_i)\Big) \otimes \Big(\bigotimes_{i\in I} X_i\Big) \xrightarrow{\simeq} \bigotimes_{i\in I} (\underline{\mathrm{Hom}}(X_i, Y_i) \otimes X_i) \xrightarrow{\otimes \mathrm{ev}} \bigotimes_{i\in I} Y_i.$$

In particular, there are morphisms

$$\bigotimes_{i\in I} X_i^\vee \to \Big(\bigotimes_{i\in I} X_i\Big)^\vee \tag{1.6.9}$$

and

$$X^\vee \otimes Y \to \underline{\mathrm{Hom}}(X, Y) \tag{1.6.10}$$

obtained respectively by taking $Y_i = \mathbb{1}$ all i, and $X_1 = X$, $X_2 = \mathbb{1} = Y_1$, $Y_2 = Y$.

Rigid tensor categories

DEFINITION 1.7. A tensor category $(\mathsf{C}, \otimes)$ is said to be ***rigid***[1] if

(a) $\underline{\mathrm{Hom}}(X, Y)$ exists for all objects X and Y,

[1]There is an alternative definition of rigidity (Deligne 1990, § 2). Let $(\mathsf{C}, \otimes)$ be a tensor category, and let $(\mathbb{1}, e)$ be an identity object for $(\mathsf{C}, \otimes)$. If $\underline{\mathrm{Hom}}(X, \mathbb{1})$ exists, then $(\underline{\mathrm{Hom}}(X, \mathbb{1}), \mathrm{ev}_{X,\mathbb{1}})$ is a dual for X (in the sense of 1.6.5). Thus, in a rigid tensor category, all objects admit duals. Conversely, assume that all objects in

(b) the morphisms (1.6.8)

$$\underline{\mathrm{Hom}}(X_1,Y_1)\otimes\underline{\mathrm{Hom}}(X_2,Y_2)\to\underline{\mathrm{Hom}}(X_1\otimes X_2,Y_1\otimes Y_2)$$

are isomorphisms for all X_1,X_2,Y_1,Y_2, and

(c) all objects of C are reflexive.

In fact, these conditions imply that the morphisms (1.6.8) are isomorphisms for all finite families.

Let $(\mathsf{C},\otimes)$ be a rigid tensor category. The functor

$$\{X,f\}\rightsquigarrow\{X^\vee,{}^tf\}:\mathsf{C}^{\mathrm{opp}}\to\mathsf{C}$$

is an equivalence of categories because its composite with itself is isomorphic to the identity functor. It is even an equivalence of tensor categories in the sense defined below — note that $\mathsf{C}^{\mathrm{opp}}$ has an obvious tensor structure for which $\otimes X_i^{\mathrm{opp}}=(\otimes X_i)^{\mathrm{opp}}$. In particular,

$$f\mapsto{}^tf:\mathrm{Hom}(X,Y)\to\mathrm{Hom}(Y^\vee,X^\vee)\tag{1.7.1}$$

is an isomorphism. There is also a canonical isomorphism

$$\underline{\mathrm{Hom}}(X,Y)\to\underline{\mathrm{Hom}}(Y^\vee,X^\vee),\tag{1.7.2}$$

namely, the composite of the isomorphisms

$$\underline{\mathrm{Hom}}(X,Y)\xleftarrow{1.6.10}X^\vee\otimes Y\xrightarrow{\simeq}X^\vee\otimes Y^{\vee\vee}\xrightarrow{\simeq}Y^{\vee\vee}\otimes X^\vee\xrightarrow{1.6.10}\underline{\mathrm{Hom}}(Y^\vee,X^\vee).$$

For any object X of C, there is an isomorphism

$$\underline{\mathrm{Hom}}(X,X)\xrightarrow{1.6.10}X^\vee\otimes X\xrightarrow{\mathrm{ev}}\mathbb{1}.$$

On applying the functor $\mathrm{Hom}(\mathbb{1},-)$ to this, we obtain (see 1.6.4) a morphism

$$\mathrm{Tr}_X:\mathrm{End}(X)\to\mathrm{End}(\mathbb{1})\tag{1.7.3}$$

called the ***trace morphism***. The ***rank***, $\mathrm{rank}(X)$, of X is defined to be $\mathrm{Tr}_X(\mathrm{id}_X)$. There are the following formulas (Saavedra Rivano 1972, I, 5.1.4):

$$\begin{aligned}\mathrm{Tr}_{X\otimes X'}(f\otimes f') &= \mathrm{Tr}_X(f)\cdot\mathrm{Tr}_{X'}(f')\\ \mathrm{Tr}_{\mathbb{1}}(f) &= f.\end{aligned}\tag{1.7.4}$$

In particular,

$$\begin{aligned}\mathrm{rank}(X\otimes X') &= \mathrm{rank}(X)\cdot\mathrm{rank}(X')\\ \mathrm{rank}(\mathbb{1}) &= \mathrm{id}_{\mathbb{1}}.\end{aligned}\tag{1.7.5}$$

C admit a dual. Then the pair $(X^\vee\otimes Y,\mathrm{ev}_{X,Y})$ with $\mathrm{ev}_{X,Y}$ the composite

$$X^\vee\otimes Y\otimes X\simeq X^\vee\otimes X\otimes Y\xrightarrow{\mathrm{ev}_X\otimes\mathrm{id}_Y}\mathbb{1}\otimes Y\simeq Y$$

is an internal Hom, $\underline{\mathrm{Hom}}(X,Y)$, for X and Y. The map (1.6.8) is

$$X_1^\vee\otimes Y_1\otimes X_2^\vee\otimes Y_2\xrightarrow{\simeq}(X_1\otimes X_2)^\vee\otimes Y_1\otimes Y_2.$$

Finally, in a *symmetric* monoidal category, the definition of a dual is symmetric between X and $X^\vee$: X is the dual of $X^\vee$, and so is reflexive.

Tensor functors

Let $(\mathsf{C}, \otimes)$ and $(\mathsf{C}', \otimes')$ be tensor categories.

DEFINITION 1.8. A ***tensor functor*** $(\mathsf{C}, \otimes) \to (\mathsf{C}', \otimes')$ is a pair (F, c) comprising a functor $F : \mathsf{C} \to \mathsf{C}'$ and a functorial isomorphism $c_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ with the following properties:

(a) for all $X, Y, Z \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{tikzcd} FX \otimes (FY \otimes FZ) \arrow[r, "\mathrm{id}\otimes c"] \arrow[d, "\phi'"] & FX \otimes F(Y \otimes Z)c \arrow[r] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\phi)"] \\ (FX \otimes FY) \otimes FZ \arrow[r, "c\otimes\mathrm{id}"] & F(X \otimes Y) \otimes FZc \arrow[r] & F((X \otimes Y) \otimes Z) \end{tikzcd}$$

commutes;

(b) for all $X, Y \in \mathrm{ob}(\mathsf{C})$, the diagram

$$\begin{tikzcd} FX \otimes FY \arrow[r, "c"] \arrow[d, "\psi'"] & F(X \otimes Y) \arrow[d, "F(\psi)"] \\ FY \otimes FX \arrow[r, "c"] & F(Y \otimes X) \end{tikzcd}$$

commutes.

(c) if (U, u) is an identity object of C, then $(F(U), F(u))$ is an identity object of C'.

In Saavedra Rivano 1972, I, 4.2.3, a tensor functor is called a "$\otimes$-foncteur AC unifère".

Let (F, c) be a tensor functor $(\mathsf{C}, \otimes) \to (\mathsf{C}', \otimes')$. The conditions (a), (b), (c) imply that, for every finite family $(X_i)_{i \in I}$ of objects in C, c gives rise to a well-defined isomorphism

$$c : \bigotimes_{i \in I} F(X_i) \to F(\bigotimes_{i \in I} X_i).$$

Moreover, for every map $\alpha : I \to J$, the diagram

$$\begin{tikzcd} \bigotimes_{i\in I} F(X_i) \arrow[rr, "c"] \arrow[d, "\chi'(\alpha)"] & & F\left(\bigotimes_{i\in I} X_i\right) \arrow[d, "F(\chi(\alpha))"] \\ \bigotimes_{j\in J}(\bigotimes_{i\mapsto j} F(X_i)) \arrow[r, "c"] & \bigotimes_{j\in J}(F(\bigotimes_{i\mapsto j} X_i)) \arrow[r, "c"] & F(\bigotimes_{j\in J}(\bigotimes_{i\mapsto j} X_i)) \end{tikzcd}$$

is commutative. In particular, (F, c) maps inverse objects to inverse objects. If $\underline{\mathrm{Hom}}(X, Y)$ exists, then the morphism

$$F(\mathrm{ev}_{X,Y}) : F(\underline{\mathrm{Hom}}(X, Y)) \otimes F(X) \to F(Y)$$

gives rise to morphisms $F_{X,Y} : F(\underline{\mathrm{Hom}}(X, Y)) \to \underline{\mathrm{Hom}}(FX, FY)$; in particular, if $X^\vee \overset{\text{def}}{=} \underline{\mathrm{Hom}}(X, \mathbb{1})$ exists, then $F(\mathrm{ev}_X)$ defines a morphism $F_X : F(X^\vee) \to F(X)^\vee$.

PROPOSITION 1.9. *Let $(F, c) : (\mathsf{C}, \otimes) \to (\mathsf{C}', \otimes')$ be a tensor functor of rigid tensor categories. Then $F_{X,Y} : F(\underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(FX, FY))$ is an isomorphism for all $X, Y \in \mathrm{ob}(\mathsf{C})$.*

PROOF. It suffices to show that F preserves duality, but this is obvious from the following characterization of the dual of X: it is a pair $(Y, Y \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1})$ for which there exists $\epsilon\colon \mathbb{1} \to X \otimes Y$ such that

$$X \simeq \mathbb{1} \otimes X \xrightarrow{\epsilon\otimes\mathrm{id}} (X \otimes Y) \otimes X = X \otimes (Y \otimes X) \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} X$$

and the same map with X and Y interchanged are identity maps. □

DEFINITION 1.10. A tensor functor $(F,c)\colon(\mathsf{C},\otimes) \to (\mathsf{C}',\otimes')$ is a ***tensor equivalence*** (or an ***equivalence of tensor categories)*** if $F\colon\mathsf{C} \to \mathsf{C}'$ is an equivalence of categories.

This definition is justified by the following proposition.

PROPOSITION 1.11. *Let $(F,c)\colon(\mathsf{C},\otimes) \to (\mathsf{C}',\otimes')$ be a tensor equivalence. Then there exists a tensor functor $(F',c')\colon\mathsf{C}' \to \mathsf{C}$ and isomorphisms of functors $F' \circ F \to \mathrm{id}_{\mathsf{C}}$ and $F \circ F' \to \mathrm{id}_{\mathsf{C}'}$ commuting with tensor products (that is, they are isomorphisms of tensor functors — see below).*

PROOF. Saavedra Rivano 1972, I, 4.4. □

A tensor functor $F\colon\mathsf{C} \to \mathsf{C}'$ of rigid tensor categories induces a morphism $F\colon\mathrm{End}(\mathbb{1}) \to \mathrm{End}(\mathbb{1}')$. The following formulas hold:

$$\begin{aligned} \mathrm{Tr}_{F(X)} F(f) &= F(\mathrm{Tr}_X(f)) \\ \mathrm{rank}(F(X)) &= F(\mathrm{rank}(X)). \end{aligned}$$

Morphisms of tensor functors

DEFINITION 1.12. Let (F,c) and (G,d) be tensor functors $\mathsf{C} \to \mathsf{C}'$; a ***morphism of tensor functors*** $(F,c) \to (G,d)$ is a morphism of functors $\lambda\colon F \to G$ such that, for all finite families $(X_i)_{i\in I}$ of objects in C, the diagram

$$\begin{tikzcd} \bigotimes_{i\in I} F(X_i) \arrow[r, "c"] \arrow[d, "\bigotimes_{i\in I}\lambda_{X_i}"] & F(\bigotimes_{i\in I} X_i) \arrow[d, "\lambda_{\otimes_{i\in I} X_i}"] \\ \bigotimes_{i\in I} G(X_i) \arrow[r, "c"] & G(\bigotimes_{i\in I} X_i) \end{tikzcd} \tag{1.12.1}$$

is commutative.

In fact, it suffices to require that the diagram (1.12.1) be commutative when I is $\{1,2\}$ or the empty set. For the empty set, (1.12.1) becomes

$$\begin{tikzcd} \mathbb{1}' \arrow[r, "\simeq"] \arrow[d, equal] & F(\mathbb{1}) \arrow[d, "\lambda_{\mathbb{1}}"] \\ \mathbb{1}' \arrow[r, "\simeq"] & G(\mathbb{1}) \end{tikzcd} \tag{1.12.2}$$

in which the horizontal maps are the unique isomorphisms compatible with the structures of $\mathbb{1}'$, $F(\mathbb{1})$, and $G(\mathbb{1})$ as identity objects of C'. In particular, when (1.12.2) commutes, $\lambda_{\mathbb{1}}$ is an isomorphism.

We write $\mathrm{Hom}^{\otimes}(F,G)$ for the set[2] of morphisms of tensor functors $(F,c) \to (G,d)$.

[2]Or, perhaps, the class…

PROPOSITION 1.13. *Let (F,c) and (G,d) be tensor functors $\mathsf{C} \to \mathsf{C}'$. If C and C' are rigid, then every morphism of tensor functors $\lambda\colon F \to G$ is an isomorphism.*

PROOF. The morphism $\mu\colon G \to F$ making the diagrams

$$\begin{array}{ccc} F(X^\vee) & \xrightarrow{\lambda_{X^\vee}} & G(X^\vee) \\ \downarrow\simeq & & \downarrow\simeq \\ F(X)^\vee & \xrightarrow{{}^t(\mu_X)} & G(X)^\vee \end{array}$$

commutative for all $X \in \mathrm{ob}(\mathsf{C})$ is an inverse for λ. [3] □

For any field k and k-algebra R, there is a canonical tensor functor $\phi_R\colon \mathsf{Vec}_k \to \mathsf{Mod}_R$, namely, $V \rightsquigarrow V \otimes_k R$. If (F,c) and (G,d) are tensor functors $\mathsf{C} \to \mathsf{Vec}_k$, then we define $\underline{\mathrm{Hom}}^{\otimes}(F,G)$ to be the functor of k-algebras such that

$$\underline{\mathrm{Hom}}^{\otimes}(F,G)(R) = \mathrm{Hom}^{\otimes}(\phi_R \circ F, \phi_R \circ G). \tag{1.13.1}$$

Tensor subcategories

DEFINITION 1.14. Let C' be a strictly full subcategory of a tensor category C. We say that C' is a ***tensor subcategory*** of C if it is closed under the formation of finite tensor products (equivalently, if it contains an identity object for C and if it contains $X \otimes Y$ whenever it contains X and Y). A tensor subcategory of a rigid tensor category is said to be a ***rigid tensor subcategory*** if it contains $X^\vee$ whenever it contains X.

A tensor subcategory becomes a tensor category under the induced tensor structure, and similarly for rigid tensor subcategories.

Abelian tensor categories; $\mathrm{End}(\mathbb{1})$

Our convention, that functors between additive categories are to be additive, forces the following definition.

DEFINITION 1.15. An ***additive*** (resp. ***abelian***) ***tensor category*** is a tensor category $(\mathsf{C},\otimes)$ such that C is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

When $(\mathsf{C},\otimes)$ is an abelian tensor category, we say that a family $(X_i)_{i\in I}$ of objects C is a ***tensor generating family*** for C if every object of C is isomorphic to a subquotient of $P(X_i)$ for some $P(t_i) \in \mathbb{N}[t_i]_{i\in I}$; in $P(X_i)$ multiplication is interpreted as $\otimes$ and addition as $\oplus$.

If $(\mathsf{C},\otimes)$ is an additive tensor category and $(\mathbb{1},e)$ is an identity object, then $R \overset{\text{def}}{=} \mathrm{End}(\mathbb{1})$ is a ring which acts, via $l_X\colon X \xrightarrow{\simeq} \mathbb{1} \otimes X$, on each object of X. The action of R on X commutes with endomorphisms of X and so, in particular, R is commutative. If $(\mathbb{1}',e')$ is a second identity object, the unique isomorphism $a\colon(\mathbb{1},e) \to (\mathbb{1}',e')$ (see 1.3(b)) defines an isomorphism $R \simeq \mathrm{End}(\mathbb{1}')$. Therefore C is R-linear in the sense that each Hom-set is endowed with the structure of an R-linear module and $\circ$ is R-bilinear, and the functor $\otimes$ is bilinear. When C is rigid, the trace morphism is an R-linear map $\mathrm{Tr}\colon \mathrm{End}(X) \to R$.

[3] One way to prove that μ is inverse to λ is to use the Yoneda lemma, which allows us to assume that we are working with categories of sets. For another, see the discussion in mathoverflow 116104.

PROPOSITION 1.16. *Let* $(\mathsf{C},\otimes)$ *be a rigid tensor category. If* C *is abelian, then* $\otimes$ *is bi-additive and commutes with direct and inverse limits in each variable; in particular, it is exact in each variable.*

PROOF. The functor $X \rightsquigarrow X\otimes Y$ has a right adjoint, namely, $Z \rightsquigarrow \underline{\mathrm{Hom}}(Y,Z)$, and therefore commutes with direct limits and is additive. By considering the opposite category $\mathsf{C}^{\mathrm{opp}}$, one deduces that it also commutes with inverse limits. (In fact, $Z \rightsquigarrow \underline{\mathrm{Hom}}(Y,Z)$ is also a left adjoint for $X \rightsquigarrow X\otimes Y$.) □

PROPOSITION 1.17. *Let* $(\mathsf{C},\otimes)$ *be a rigid abelian tensor category. If* U *is a subobject of* $\mathbb{1}$*, then* $\mathbb{1} = U\oplus U^{\perp}$ *where* $U^{\perp} = \mathrm{Ker}(\mathbb{1}\to U^{\vee})$*. Consequently,* $\mathbb{1}$ *is a simple object if* $\mathrm{End}(\mathbb{1})$ *is a field.*

PROOF. Let $V = \mathrm{Coker}(U\to\mathbb{1})$. On tensoring

$$0\to U\to\mathbb{1}\to V\to 0$$

with $U \hookrightarrow \mathbb{1}$, we obtain an exact commutative diagram

$$\begin{tikzcd} U \arrow[r, hook] & \mathbb{1} \arrow[r, two heads] & V \\ U\otimes U \arrow[r, hook] \arrow[u, hook] & U \arrow[r, two heads] \arrow[u, hook] \arrow[ur, dashed, "0"] & V\otimes U, \arrow[u, hook] \end{tikzcd}$$

from which it follows that $V\otimes U = 0$, and that $U\otimes U = U$ as a subobject of $\mathbb{1}\otimes\mathbb{1} = \mathbb{1}$.

For any object T, the map $T\otimes U\to T$ obtained from $U\hookrightarrow\mathbb{1}$ by tensoring with T, is injective. This proves the first equivalence in

$$T\otimes U = 0 \iff \text{the map } T\otimes U\to T \text{ is zero} \iff \text{the map } T\to U^{\vee}\otimes T \text{ is zero;}$$

the second equivalence follows from the canonical isomorphisms

$$\mathrm{Hom}(T\otimes U,T) \overset{1.6.5}{\simeq} \mathrm{Hom}(T\otimes U\otimes T^{\vee},\mathbf{1}) \overset{1.6.5}{\simeq} \mathrm{Hom}(T,U^{\vee}\otimes T).$$

Therefore, for any object X, the largest subobject T of X such that $T\otimes U = 0$ is the largest subobject T such that $T\to U^{\vee}\otimes X$ is zero; hence

$$T = \mathrm{Ker}(X\to U^{\vee}\otimes X)\simeq U^{\perp}\otimes X.$$

On applying this remark with $X = V$ and using that $V\otimes U = 0$, we find that $U^{\perp}\otimes V\simeq V$; on applying it with $X = U$ and using that $U\otimes U = U$, we find that $U^{\perp}\otimes U = 0$. From the exact sequence

$$0\to U^{\perp}\otimes U\to U^{\perp}\otimes\mathbb{1}\to U^{\perp}\otimes V\to 0$$

we deduce that $U^{\perp}\simeq V$, and that $\mathbb{1}\simeq U^{\perp}\oplus U$. □

REMARK 1.18. The proposition shows that there is a one-to-one correspondence between subobjects of $\mathbb{1}$ and idempotents in $\mathrm{End}(\mathbb{1})$. Such an idempotent e determines a decomposition of tensor categories $\mathsf{C} = \mathsf{C}'\times\mathsf{C}''$ in which an object is in C' (resp. C'') if e (resp. $1-e$) acts as the identity morphism on it.

PROPOSITION 1.19. *Let C and C' be rigid abelian tensor categories and let $\mathbb{1}$ and $\mathbb{1}'$ be identity objects of C and C' respectively. If $\mathrm{End}(\mathbb{1})$ is a field and $\mathbb{1}' \neq 0$, then every exact tensor functor $F{:}\,\mathsf{C} \to \mathsf{C}'$ is faithful.*

PROOF. The criterion in C,

$$X \neq 0 \iff X \otimes X^{\vee} \to \mathbb{1} \text{ is an epimorphism}$$

is respected by F. □

A criterion to be a rigid tensor category

PROPOSITION 1.20. [4]*Let C be a k-linear abelian category, where k is a field, and let $\otimes{:}\,\mathsf{C} \times \mathsf{C} \to \mathsf{C}$ be a k-bilinear functor. Suppose that there are given a faithful exact k-linear functor $F{:}\,\mathsf{C} \to \mathsf{Vec}_k$, a functorial isomorphism $\phi_{X,Y,Z}{:}\,X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, and a functorial isomorphism $\psi_{X,Y}{:}\,X \otimes Y \to Y \otimes X$ with the following properties*

(a) $F \circ \otimes = \otimes \circ (F \times F)$;
(b) *$F(\phi_{X,Y,Z})$ is the usual associativity isomorphism in Vec_k;*
(c) *$F(\psi_{X,Y})$ is the usual commutativity isomorphism in Vec_k;*
(d) *there exists an identity object U in C such that $k \to \mathrm{End}(U)$ is an isomorphism and $F(U)$ has dimension 1;*
(e) *if $F(L)$ has dimension 1, then there exists an object L^{-1} in C such that $L \otimes L^{-1} = U$.*

Then $(C, \otimes, \phi, \psi)$ is a rigid abelian tensor category.

PROOF. For a direct proof, see Milne 2017, 9.24. We indicate a more elegant approach in (2.18) below. □

Examples

EXAMPLE 1.21. The category Vec_k of finite-dimensional vector spaces over a field k is a rigid abelian tensor category and $\mathrm{End}(\mathbb{1}) = k$. All the above definitions take on a familiar meaning when applied to Vec_k. For example, $\mathrm{Tr}{:}\,\mathrm{End}(X) \to k$ is the usual trace map.

EXAMPLE 1.22. The category Mod_R of finitely generated modules over a commutative ring R is an abelian tensor category and $\mathrm{End}(\mathbb{1}) = R$. In general it will not be rigid because not all R-modules will be reflexive.

EXAMPLE 1.23. The category Proj_R of finitely generated projective modules over a commutative ring R is a rigid additive tensor category and $\mathrm{End}(\mathbb{1}) = R$, but, in general, it is not abelian. The rigidity follows easily from considering the objects of Proj_R as locally-free modules of finite rank on $\mathrm{Spec}(R)$. Alternatively, apply Bourbaki, Algèbre, II 4.4, II 2.7.

[4]In the original, it was not required that the U in (d) and (e) be an identity object. That this is necessary is shown by the following example of Deligne:

> Let C be the category of pairs (V,α) where V is a finite dimensional vector space over a field k and α is an endomorphism of V such that $\alpha^2 = \alpha$, and let F be the forgetful functor. Then (V,α) is a tensor category with identity object (k,id), but it is not rigid because internal Homs and duals don't always exist (in fact, C is the category of (unital) representations of the multiplicative monoid $\{1,0\}$). Let $U = (k,0)$. Then, (d) holds, and, for any L of dimension 1, $(L,\alpha) \otimes U \approx U$, and so (e) holds with $L^{-1} = U$.

EXAMPLE 1.24. Let G be an affine group scheme over a field k, and let $\mathsf{Rep}_k(G)$ be the category of finite-dimensional representations of G over k. Thus, an object of $\mathsf{Rep}_k(G)$ consists of a finite-dimensional vector space V over k and a homomorphism $g \mapsto g_V : G \to \mathrm{GL}_V$ of affine group schemes over k — we sometimes refer to the objects of $\mathsf{Rep}_k(G)$ as G-modules. Then $\mathsf{Rep}_k(G)$ is a rigid abelian tensor category and $\mathrm{End}(\mathbb{1}) = k$. These categories, and more generally the categories of representations of affine groupoids (see §3), are the main topic of study of this article.

EXAMPLE 1.25. (Vector spaces graded by $\mathbb{Z}/2\mathbb{Z}$).[5] Let C be the category whose objects are pairs (V^0, V^1) of finite-dimensional vector spaces over k . We give C the tensor structure whose commutativity constraint is determined by the Koszul rule of signs, i.e., that defined by the isomorphisms

$$v \otimes w \mapsto (-1)^{ij} w \otimes v : V^i \otimes W^j \to W^j \otimes V^i.$$

Then C is a rigid abelian tensor category and $\mathrm{End}(\mathbb{1}) = k$, but it is not of the form $\mathsf{Rep}_k(G)$ for any G because

$$\mathrm{rank}(V^0, V^1) = \dim(V^0) - \dim(V^1),$$

which need not be positive.

EXAMPLE 1.26. The rigid additive tensor category ***freely generated*** by an object T is a pair (C, T) comprising a rigid additive tensor category C such that $\mathrm{End}(\mathbb{1}) = \mathbb{Z}[t]$ and an object T having the property that

$$F \rightsquigarrow F(T) : \mathrm{Hom}^{\otimes}(\mathsf{C}, \mathsf{C}') \to \mathsf{C}'$$

is an equivalence of categories for all rigid additive tensor categories C' (t will turn out to be the rank of T). We show how to construct such a pair (C, T) — clearly it is unique up to a unique equivalence of tensor categories preserving T.

Let V be a free module of finite rank over a commutative ring k and let $T^{a,b}(V)$ be the space $V^{\otimes a} \otimes V^{\vee \otimes b}$ of tensors with covariant degree a and contravariant degree b . A morphism $f : T^{a,b}(V) \to T^{c,d}(V)$ can be identified with a tensor "f" in $T^{b+c,a+d}(V)$. When $a + d = b + c$, $T^{b+c,a+d}(V)$ contains a special element, namely, the $(a+d)$th tensor power of "id"$\in T^{1,1}(V)$, and other elements can be obtained by allowing an element of the symmetric group S_{a+d} to permute the contravariant components of this special element. We have therefore a map

$$\epsilon : S_{a+d} \to \mathrm{Hom}(T^{a,b}, T^{c,d}) \qquad (\text{when } a + d = b + c).$$

The induced map $k[S_{a+d}] \to \mathrm{Hom}(T^{a,b}, T^{c,d})$ is injective provided $\mathrm{rank}(V) \geq a+d$. One checks that the composite of two such maps $\epsilon(\sigma) : T^{a,b}(V) \to T^{c,d}(V)$ and $\epsilon(\tau) : T^{c,d}(V) \to T^{e,f}(V)$ is given by a universal formula

$$\epsilon(\tau) \cdot \epsilon(\sigma) = (\mathrm{rank}\, V)^N \cdot \epsilon(\rho) \tag{1.26.1}$$

with ρ and N depending only on a, b, c, d, e, f, σ, and τ.

We define C' to be the category having as objects symbols $T^{a,b}$ ($a, b \in \mathbb{N}$), and for which $\mathrm{Hom}(T^{a,b}, T^{c,d})$ is the free $\mathbb{Z}[t]$-module with basis S_{a+d} if $a + d = b + c$ and is

[5] Now called superspaces.

zero otherwise. Composition of morphisms is defined to be $\mathbb{Z}[t]$-bilinear and to agree on basis elements with the universal formula (1.26.1) with rank(V) replaced by the indeterminate t. The associativity law holds for this composition because it does whenever t is replaced by a large enough positive integer (it becomes the associativity law in a category of modules). Tensor products are defined by

$$T^{a,b} \otimes T^{c,d} = T^{a+c,b+d}$$

and by an obvious rule for morphisms. We define T to be $T^{1,0}$.

The category C is deduced from C' by formally adjoining direct sums of objects. Its universality follows from the fact that the formula (1.26.1) holds in any rigid additive category.

EXAMPLE 1.27. (GL_t) Let n be an integer, and use $t \mapsto n\colon \mathbb{Z}[t] \to \mathbb{C}$ to extend the scalars in the above example from $\mathbb{Z}[t]$ to $\mathbb{C}$. If V is an n-dimensional complex vector space and if $a+d \leq n$, then

$$\mathrm{Hom}(T^{a,b}, T^{c,d}) \otimes_{\mathbb{Z}[t]} \mathbb{C} \to \mathrm{Hom}_{\mathrm{GL}_V}(T^{a,b}(V), T^{c,d}(V))$$

is an isomorphism. For any sum T' of $T^{a,b}$s and large enough integer n, $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{C}$ is therefore a product of matrix algebras. This implies that $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{Q}[t]$ is a semisimple algebra.

After extending the scalars in C to $\mathbb{Q}(t)$, i.e., replacing $\mathrm{Hom}(T', T'')$ with $\mathrm{Hom}(T', T'') \otimes_{\mathbb{Z}[t]} \mathbb{Q}[t]$ and passing to the pseudo-abelian (Karoubian) envelope (formally adjoining images of idempotents), we obtain a semisimple rigid abelian tensor category GL_t. The rank of T in GL_t is $t \notin \mathbb{N}$ and so, although $\mathrm{End}(\mathbb{1}) = \mathbb{Q}(t)$ is a field, GL_t is not of the form $\mathsf{Rep}_k(G)$ for any group scheme (or gerbe) G.[6]

2. Neutral Tannakian categories

Throughout this section, k is a field. Unadorned tensor products are over k.

Affine group schemes

We review the basic theory of affine group schemes and their representations. For more details, see Waterhouse 1979, Chapters 1,3.[7]

Let $G = \operatorname{Spec} A$ be an affine group scheme over k. The maps

$$\text{mult}\colon G \times G \to G, \quad \text{identity}\colon \{1\} \to G, \quad \text{inverse}\colon G \to G$$

induce maps of k-algebras

$$\Delta\colon A \to A \otimes_k A, \quad \epsilon\colon A \to k, \quad S\colon A \to A$$

(the comultiplication, coidentity, and coinverse maps) such that

[6]For more on such categories, see: Deligne, P., La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. Algebraic groups and homogeneous spaces, 209–273, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007.

[7]Or Milne 2017, especially 3.7, 4.1, 4.7, 4.9, 9.8.

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \colon A \to A \otimes A \rightrightarrows A \otimes A \otimes A$$

(coassociativity axiom),

$$\mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta \colon A \to A \otimes A \to k \otimes A \simeq A$$

(coidentity axiom), and

$$\left(A \xrightarrow{\Delta} A \otimes A \xrightarrow{(S,\mathrm{id})} A\right) = \left(A \xrightarrow{\epsilon} k \hookrightarrow A\right)$$

(coinverse axiom). We define a ***bialgebra*** over k to be a k-algebra A together with maps Δ, ϵ, and S satisfying the three axioms.[8]

PROPOSITION 2.1. *The functor $A \rightsquigarrow \operatorname{Spec} A$ defines an equivalence of categories between the category of k-bialgebras and the category of affine group schemes over k.*

PROOF. Obvious. □

If A is finitely generated as a k-algebra we say that G is ***algebraic*** or that it is an ***algebraic group***.[9]

A ***coalgebra*** over k is a k-vector space C together with k-linear maps $\Delta\colon C \to C \otimes_k C$ and $\epsilon\colon C \to k$ satisfying the coassociativity and coidentity axioms. A ***comodule*** over a coalgebra C is a vector space V over k together with a k-linear map $\rho\colon V \to V \otimes_k C$ such that

$$V \xrightarrow{\rho} V \otimes C \xrightarrow{\mathrm{id} \otimes \epsilon} V \otimes k \simeq V$$

is the identity map and

$$(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho \colon V \to V \otimes C \otimes C.$$

For example, Δ defines an C-comodule structure on C.

PROPOSITION 2.2. *For any affine k-group scheme $G = \operatorname{Spec} A$ and k-vector space V, there is a canonical one-to-one correspondence between the A-comodule structures on V and the linear representations of G on V.*

PROOF. Let $r\colon G \to \mathrm{GL}_V$ be a representation. For the "universal" element $\mathrm{id}_G \in \mathrm{Mor}(G,G) = G(A)$, $r(\mathrm{id}_G)$ is an A-isomorphism $V \otimes A \to V \otimes A$ whose restriction to $V = V \otimes k \subset V \otimes A$ determines it and is an A-comodule structure ρ on V. Conversely, a comodule structure ρ on V determines a representation of G on V such that, for any k-algebra R and $g \in G(R) = \mathrm{Hom}_k(A, R)$, the restriction of $g_V\colon V \otimes R \to V \otimes R$ to $V \otimes k \subset V \otimes R$ is

$$(\mathrm{id}_V \otimes g) \circ \rho \colon V \to V \otimes A \to V \otimes R.$$

See Waterhouse 1979, 3.2, for the details. □

[8] More correctly, it is a commutative bialgebra admitting an antipode, or a commutative Hopf algebra.

[9] For us, an algebraic group will always mean an affine algebraic group scheme.

The representation of G on A defined by the A-comodule structure Δ is called the ***regular representation*** of G.

PROPOSITION 2.3. *Let C be a k-coalgebra and let (V,ρ) a comodule over C. Every finite subset of V is contained in a sub-comodule of V having finite dimension over k.*

PROOF. Let $\{c_i\}$ be a basis for C over k (possibly infinite). For v in the finite subset, write $\rho(v) = \sum v_i \otimes c_i$ (finite sum). The k-space generated by the v and the v_i is a sub-comodule over C (Waterhouse 1979, 3.3). □

COROLLARY 2.4. *Every linear representation of an affine group scheme is a directed union of finite-dimensional subrepresentations.*

PROOF. The set of all sub-comodules of a comodule V that are finite-dimensional over k is partially ordered by inclusion, directed (any two are contained in a third), and has union V (see 2.3). Now apply (2.2). □

COROLLARY 2.5. *An affine group scheme G is algebraic if and only if it has a faithful finite-dimensional representation over k.*

PROOF. The sufficiency is obvious. For the necessity, let V be the regular representation of G, and write it as a directed union $V = \bigcup_i V_i$ of finite-dimensional subrepresentations. Then $\bigcap_i \operatorname{Ker}(G \to \mathrm{GL}(V_i)) = \{1\}$ because V is a faithful representation, and it follows that $\operatorname{Ker}(G \to \mathrm{GL}(V_{i_0})) = \{1\}$ for some i_0 because G is Noetherian as a topological space. □

PROPOSITION 2.6. *Let A be a k-bialgebra. Every finite subset of A is contained in a sub-bialgebra that is finitely generated as a k-algebra.*

PROOF. According to (2.3), the finite subset is contained in a finite-dimensional k-subspace V of A such that $\Delta(V) \subset V \otimes_k A$. Let $\{v_j\}$ be a basis for V, and let $\Delta(v_j) = \sum v_i \otimes a_{ij}$. The subalgebra $k[v_j, a_{ij}, Sv_j, Sa_{ij}]$ of A is a sub-bialgebra (Waterhouse 1979, 3.3). □

COROLLARY 2.7. *Every affine k-group scheme G is a directed inverse limit $G = \varprojlim G_i$ of affine algebraic groups over k in which the transition maps $G_i \leftarrow G_j$, $i \leq j$, are faithfully flat.*

PROOF. Write A as a union $A = \bigcup A_i$ of finite-dimensional sub-bialgebras with $A_i \subset A_j$ for $i \leq j$. The functor Spec transforms the direct limit $A = \varinjlim A_i$ into an inverse limit $G = \varprojlim G_i$. The transition map $G_i \leftarrow G_j$ is faithfully flat because A_j is faithfully flat over its subalgebra A_i (Waterhouse 1979, 14.1). □

The converse to (2.7) is also true; in fact the inverse limit of any family of affine group schemes is again an affine group scheme.

Recovering an affine group scheme from its representations

Let G be an affine group scheme over k, and let ω (or ω^G) be the forgetful functor $\mathsf{Rep}_k(G) \to \mathsf{Vec}_k$. For R a k-algebra, $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ consists of the families (λ_X), $X \in$

$\mathrm{ob}(\mathsf{Rep}_k(G))$, where λ_X is an R-linear automorphism of $X \otimes R$ such that $\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}$, $\lambda_{1\!\!1}$ is the identity map (on R) , and

$$\lambda_Y \circ (\alpha \otimes 1) = (\alpha \otimes 1) \circ \lambda_X \colon X \otimes R \to Y \otimes R$$

for all G-equivariant maps $\alpha\colon X \to Y$ (see 1.12). Clearly, every $g \in G(R)$ defines an element of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$.

PROPOSITION 2.8. *The natural map $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an isomorphism of functors of k-algebras.*

PROOF. Let $X \in \mathsf{Rep}_k(G)$, and let C_X be the strictly full subcategory $\mathsf{Rep}_k(G)$ of objects isomorphic to a subquotient of $P(X, X^{\vee})$ for some $P \in \mathbb{N}[t,s]$ (cf. the discussion following 1.14). The map $\lambda \mapsto \lambda_X$ identifies $\underline{\mathrm{Aut}}^{\otimes}(\omega|\mathsf{C}_X)(R)$ with a subgroup of $\mathrm{GL}(X \otimes R)$. Let G_X be the image of G in GL_X; it is a closed algebraic subgroup of GL_X, and clearly

$$G_X(R) \subset \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathsf{C}_X)(R) \subset \mathrm{GL}(X \otimes R).$$

If $V \in \mathrm{ob}(\mathsf{C}_X)$ and $t \in V$ is fixed by G, then

$$a \mapsto at\colon k \xrightarrow{\alpha} V$$

is G-equivariant, and so

$$\lambda_V(t \otimes 1) = (\alpha \otimes 1)\lambda_{1\!\!1}(1) = t \otimes 1.$$

Thus $\underline{\mathrm{Aut}}^{\otimes}(\omega|\mathsf{C}_X)$ is the subgroup of GL_X fixing all tensors in representations of G_X fixed by G_X, which implies that $G_X = \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathsf{C}_X)$ (see Deligne 1982, 3.2).

If $X' = X \oplus Y$ for some representation Y of G, then $\mathsf{C}_X \subset \mathsf{C}_{X'}$, and there is a commutative diagram

$$\begin{array}{ccc} G_{X'} & \xrightarrow{\simeq} & \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathsf{C}_{X'}) \\ \downarrow & & \downarrow \\ G_X & \xrightarrow{\simeq} & \underline{\mathrm{Aut}}^{\otimes}(\omega|\mathsf{C}_X). \end{array}$$

It is clear from (2.5) and (2.7) and $G = \varprojlim G_X$, and so, on passing to the inverse limit over these diagrams, we obtain an isomorphism $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$. □

A homomorphism $f\colon G \to G'$ defines a tensor functor $\omega^f\colon \mathsf{Rep}_k(G') \to \mathsf{Rep}_k(G)$ such that $\omega^G \circ \omega^f = \omega^{G'}$, namely, $\omega^f(X, r_X) = (X, r_X \circ f)$. Our next result shows that all such functors arise in this fashion.

COROLLARY 2.9. *Let G and G' be affine k-group schemes, and let $F\colon \mathsf{Rep}_k(G') \to \mathsf{Rep}_k(G)$ be a tensor functor such that $\omega^G \circ F = \omega^{G'}$. Then there exists a unique homomorphism $f\colon G \to G'$ such that $F = \omega^f$.*

PROOF. Such an F defines a homomorphism (functorial in the k-algebra R)

$$F^*\colon \underline{\mathrm{Aut}}^{\otimes}(\omega^G)(R) \to \underline{\mathrm{Aut}}^{\otimes}(\omega^{G'})(R), \quad F^*(\lambda)_X = \lambda_{F(X)}.$$

Proposition 2.8 and the Yoneda lemma allow us to identify F^* with a homomorphism $G \to G'$. Obviously $F \mapsto F^*$ and $f \mapsto \omega^f$ are inverse maps. □

REMARK 2.10. Proposition 2.8 shows that G is determined by the triple $(\mathsf{Rep}_k(G), \otimes, \omega^G)$. In fact, the coalgebra of G is already determined by $(\mathsf{Rep}_k(G), \omega^G)$ (see the proof of Theorem 2.11 below).

The main theorem

THEOREM 2.11. *Let* $(\mathsf{C},\otimes)$ *be a rigid abelian tensor category such that* $k=\mathrm{End}(\mathbb{1})$, *and let* $\omega\colon\mathsf{C}\to\mathsf{Vec}_k$ *be an exact faithful* k*-linear tensor functor. Then,*

(a) *the functor* $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ *of* k*-algebras is represented by an affine group scheme* G*;*
(b) *the functor* $\mathsf{C}\to\mathsf{Rep}_k(G)$ *defined by* ω *is an equivalence of tensor categories.*

The proof will occupy the rest of this subsection. We first construct the coalgebra A of G without using the tensor structure on C. The tensor structure then enables us to define an algebra structure on A, and the rigidity of C implies that $\mathrm{Spec}\,A$ is a group scheme (rather than a monoid scheme). The following easy observation will allow us to work initially with algebras rather than coalgebras: for a finite-dimensional (not necessarily commutative) k-algebra A and its dual coalgebra $A^{\vee}\overset{\text{def}}{=}\mathrm{Hom}_{k\text{-lin}}(A,k)$, the bijections

$$\mathrm{Hom}_{k\text{-lin}}(V\otimes_k A,V)\simeq\mathrm{Hom}_{k\text{-lin}}(V,\mathrm{Hom}(A,V))\simeq\mathrm{Hom}_{k\text{-lin}}(V,V\otimes_k A^{\vee})$$

determine a one-to-one correspondence between the A-module structures on a vector space V and the $A^{\vee}$-comodule structures on V.

We begin with some constructions that are valid in any k-linear abelian category C.

Let Vec_k^s be the full subcategory Vec_k whose objects are the vector spaces k^n, and let ι be the inclusion functor. For each finite-dimensional vector space V over k, choose an isomorphism $\beta_V\colon k^{\dim V}\to V$. Then there is exactly one functor[10] $\gamma\colon\mathsf{Vec}_k\to\mathsf{Vec}_k^s$ such that $\gamma(V)=k^{\dim V}$ for all V and β is a natural isomorphism $\gamma\circ\iota\to\mathrm{id}_{\mathsf{Vec}}$.

We define a functor

$$\otimes\colon\mathsf{Vec}_k\times\mathsf{C}\to\mathsf{C}$$

such that

$$\mathrm{Hom}_{\mathsf{C}}(T,V\otimes X)\simeq V\otimes_k\mathrm{Hom}_{\mathsf{C}}(T,X),$$

(functorially in T). For $V=k^n$, we set $V\otimes X=X^n$ (direct sum of n-copies of X). For a general V, we set $V\otimes X=\gamma(V)\otimes X$. There is also an isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(V\otimes X,T)\simeq\mathrm{Hom}_{k\text{-lin}}(V,\mathrm{Hom}_{\mathsf{C}}(X,T)),$$

functorial in T. For any k-linear functor $F\colon\mathsf{C}\to\mathsf{C}'$, $F(V\otimes X)\cong V\otimes F(X)$.

We define $\underline{\mathrm{Hom}}(V,X)$ to be $V^{\vee}\otimes X$. If $W\subset V$ and $Y\subset X$, then the ***transporter of*** W ***to*** Y is

$$(Y\colon W)=\mathrm{Ker}(\underline{\mathrm{Hom}}(V,X)\to\underline{\mathrm{Hom}}(W,X/Y)).$$

For any k-linear functor F, $F(\underline{\mathrm{Hom}}(V,X))=\underline{\mathrm{Hom}}(V,FX)$, and if F is exact, then $F(Y\colon W)=(FY\colon W)$.

LEMMA 2.12. *Let* C *be a* k*-linear abelian category and let* $\omega\colon\mathsf{C}\to\mathsf{Vec}_k$ *be a* k*-linear exact faithful functor. Then, for any object* $X\in\mathrm{ob}(\mathsf{C})$, *the following two objects are equal:*

(a) *the largest subobject* P *of* $\underline{\mathrm{Hom}}(\omega(X),X))$ *whose image in* $\underline{\mathrm{Hom}}(\omega(X)^n,X^n)$ *(embedded diagonally) is contained in* $(Y\colon\omega(Y))$ *for all* $Y\subset X^n$*;*

[10] The category $\mathsf{Vec}(k)^s$ is a skeleton of $\mathsf{Vec}(k)$, and in γ we are choosing an adjoint to ι — see the discussion Mac Lane 1998, IV 4, p. 93. For a way of avoiding having to choose a γ, see the original article p. 131.

(b) *the smallest subobject* P' *of* $\underline{\mathrm{Hom}}(\omega(X), X)$ *such that the subspace*

$$\omega(P') \subset \mathrm{Hom}(\omega(X), \omega(X))$$

contains $\mathrm{id}_{\omega(X)}$.

PROOF. Clearly $\omega(X) = 0$ implies $\mathrm{End}(X) = 0$, which implies $X = 0$. Thus, if $X \subset Y$ and $\omega(X) = \omega(Y)$, then $X = Y$, and it follows that all objects of C are both Artinian and Noetherian. The objects P and P' therefore exist.

The functor ω maps $\underline{\mathrm{Hom}}(V, X)$ to $\mathrm{Hom}(V, \omega X)$ and $(Y : W)$ to $(\omega Y : W)$ for all $W \subset V \in \mathrm{ob}(\mathsf{Vec}_k)$ and $Y \subset X \in \mathrm{ob}(\mathsf{C})$. It therefore maps

$$P \stackrel{\text{def}}{=} \bigcap (\underline{\mathrm{Hom}}(\omega X, X) \cap (Y : \omega Y))$$

to

$$\bigcap (\mathrm{End}(\omega X) \cap (\omega Y : \omega Y)).$$

This means ωP is the largest subring of $\mathrm{End}(\omega X)$ stabilizing ωY for all $Y \subset X^n$. Hence $\mathrm{id}_{\omega X} \in \omega P$ and $P \supset P'$.

Let V be a finite-dimensional vector space over k. There is an obvious map

$$\underline{\mathrm{Hom}}(\omega X, X) \to \underline{\mathrm{Hom}}(\omega(V \otimes X), V \otimes X)$$

which, after the application of ω, becomes

$$f \mapsto \mathrm{id}_V \otimes f : \mathrm{End}(\omega X) \to \mathrm{End}(V \otimes \omega(X)).$$

By definition, $\omega P \subset \mathrm{End}(\omega X)$ stabilizes ωY for all $Y \subset V \otimes X$. On applying this remark to a subobject

$$Q \subset \underline{\mathrm{Hom}}(\omega X, X) = (\omega X)^\vee \otimes X,$$

we find that ωP, when acting by left multiplication on $\mathrm{End}(\omega X)$, stabilizes ωQ. Therefore, if ωQ contains $\mathrm{id}_{\omega X}$, then $\omega P \subset \omega Q$, and $P \subset Q$. On applying this statement with $Q = P'$, we find that $P \subset P'$. □

Let $P_X \subset \underline{\mathrm{Hom}}(\omega(X), X)$ be the subobject defined in (a) (equivalently (b)) of the lemma, and let $A_X = \omega(P_X)$ — it is the largest k-subalgebra of $\mathrm{End}(\omega(X))$ stabilizing $\omega(Y)$ for all $Y \subset X^n$. Let $\langle X \rangle$ be the strictly full subcategory of C whose objects are those isomorphic to a subquotient of X^n for some $n \in \mathbb{N}$. Then $\omega|\langle X \rangle : \langle X \rangle \to \mathsf{Vec}_k$ factors through Mod_{A_X}.

LEMMA 2.13. *Let* C, ω *be as in (2.12). There is a natural action of the ring* A_X *on* $\omega(Y)$, $Y \in \langle X \rangle$, *and* ω *defines an equivalence of categories* $\langle X \rangle \to \mathsf{Mod}(A_X)$ *carrying* $\omega|\langle X \rangle$ *to the forgetful functor. Moreover* $A_X = \mathrm{End}(\omega|\langle X \rangle)$.

PROOF. The right action $f \mapsto f \circ a$ of A_X on $\underline{\mathrm{Hom}}(\omega X, X)$ stabilizes P_X because obviously,

$$(Y : \omega Y)(\omega Y : \omega Y) \subset (Y : \omega Y).$$

If M is an A_X-module, we define

$$P_X \otimes_{A_X} M = \mathrm{Coker}(P_X \otimes A_X \otimes M \rightrightarrows P_X \otimes M).$$

Then
$$\omega(P_X \otimes_{A_X} M) \simeq \omega(P_X) \otimes_{A_X} M = A_X \otimes_{A_X} M \simeq M.$$
This shows that ω is essentially surjective. A similar argument shows that $\langle X\rangle \to \mathsf{Mod}(A_X)$ is full.

Clearly any element of A_X defines an endomorphism of $\omega|\langle X\rangle$. On the other hand an element λ of $\mathrm{End}(\omega|\langle X\rangle)$ is determined by $\lambda_X \in \mathrm{End}(\omega(X))$; thus $\mathrm{End}(\omega(X)) \supset \mathrm{End}(\omega|\langle X\rangle) \supset A_X$. But λ_X stabilizes $\omega(Y)$ for all $Y \subset X^n$, and so $\mathrm{End}(\omega|\langle X\rangle) \subset A_X$. This completes the proof of the lemma. □

Let $B_X = A_X^\vee$. The observation at the start of the proof, allows us to restate (2.13) as follows: ω defines an equivalence
$$(\langle X\rangle, \omega|\langle X\rangle) \to (\mathsf{Comod}_{B_X}, \text{forget})$$
where Comod_{B_X} is the category of B_X-comodules of finite dimension over k.

On passing to the inverse limit over X (cf. the proof of 2.8), we obtain the following result.

PROPOSITION 2.14. *Let (C,ω) be as in (2.12) and let $B = \varinjlim \mathrm{End}(\omega|\langle X\rangle)^\vee$. Then ω defines an equivalence of categories $\mathsf{C} \to \mathsf{Comod}_B$ carrying ω into the forgetful functor.*

EXAMPLE 2.15. Let A be a finite-dimensional k-algebra and let ω be the forgetful functor $\mathsf{Mod}_A \to \mathsf{Vec}_k$. For R a commutative k-algebra, let ϕ_R be the functor $R \otimes -\colon \mathsf{Vec}_k \to \mathsf{Mod}_R$. There is a canonical map $\alpha\colon R \otimes_k A \to \mathrm{End}(\phi_R \circ \omega)$, which we shall show to be an isomorphism by defining an inverse β. For $\lambda \in \mathrm{End}(\phi_R \circ \omega)$, set $\beta(\lambda) = \lambda_A(1)$. Clearly $\beta \circ \alpha = \mathrm{id}$, and so we only have to show $\alpha \circ \beta = \mathrm{id}$. For $M \in \mathrm{ob}(\mathsf{Mod}_A)$, let $M_0 = \omega(M)$. The A-module $A \otimes_k M_0$ is a direct sum of copies of A, and the additivity of λ shows that $\lambda_{A \otimes M_0} = \lambda_A \otimes \mathrm{id}_{M_0}$. The map $a \otimes m \mapsto am\colon A \otimes_k M_0 \to M$ is A-linear, and hence
$$\begin{array}{ccc} R\otimes A\otimes M_0 & \longrightarrow & R\otimes M \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ R\otimes A\otimes M_0 & \longrightarrow & R\otimes M \end{array}$$
is commutative. Therefore $\lambda_M(m) = \lambda_A(1)m = (\alpha \circ \beta(\lambda))_M(m)$ for $m \in R \otimes M$. In particular, $A \xrightarrow{\simeq} \mathrm{End}(\omega)$, and it follows that, if in (2.13) we take $\mathsf{C} = \mathsf{Mod}_A$, so that $\mathsf{C} = \langle A\rangle$, then the equivalence of categories obtained is the identity functor.

Let B be a coalgebra over k and let ω be the forgetful functor $\mathsf{Comod}_B \to \mathsf{Vec}_k$. The discussion in Example 2.15 shows that $B = \varinjlim \mathrm{End}(\omega|\langle X\rangle)^\vee$. We deduce, as in (2.9), that every functor $\mathsf{Comod}_B \to \mathsf{Comod}_{B'}$ carrying the forgetful functor into the forgetful functor arises from a unique homomorphism $B \to B'$.

Again, let B be a coalgebra over k. A homomorphism $u\colon B \otimes_k B \to B$ defines a functor
$$\phi^u\colon \mathsf{Comod}_B \times \mathsf{Comod}_B \to \mathsf{Comod}_B$$
sending (X,Y) to $X \otimes_k Y$ with the B-comodule structure
$$X \otimes Y \xrightarrow{\rho_X \otimes \rho_Y} X \otimes B \otimes Y \otimes B \xrightarrow{1 \otimes u} X \otimes Y \otimes B.$$

PROPOSITION 2.16. *The map $u \mapsto \phi^u$ defines a one-to-one correspondence between the set of homomorphisms $B \otimes_k B \to B$ and the set of functors $\phi : \mathsf{Comod}_B \times \mathsf{Comod}_B \to \mathsf{Comod}_B$ such that $\phi(X,Y) = X \otimes_k Y$ as k-vector spaces. The natural associativity and commutativity constraints on Vec_k induce similar contraints on $(\mathsf{Comod}_B, \phi^u)$ if and only if the multiplication defined by u on B is associative and commutative; there is an identity object in $(\mathsf{Comod}_B, \phi^u)$ with underlying vector space k if and only if B has an identity element.*

PROOF. The pair $(\mathsf{Comod}_B \times \mathsf{Comod}_B, \omega \otimes \omega)$, with $(\omega \otimes \omega)(X \otimes Y) = \omega(X) \otimes \omega(Y)$ (as a k-vector space), satisfies the conditions of (2.14), and $\varinjlim \mathrm{End}(\omega \otimes \omega | \langle (X,Y) \rangle)^\vee = B \otimes B$. Thus the first statement of the proposition follows from (2.15). The remaining statements are easy. □

Let (C, ω) and B be as in (2.14) except now assume that C is a tensor category and ω is a tensor functor. The tensor structure on C induces a similar structure on Comod_B, and hence, because of (2.16), the structure of an associative commutative k-algebra with identity element on B. Thus B lacks only a coinverse map S to be a bialgebra, and $G = \operatorname{Spec} B$ is an affine monoid scheme. Using (2.15) we find that, for any k-algebra R,

$$\underline{\mathrm{End}}(\omega)(R) \stackrel{\text{def}}{=} \mathrm{End}(\phi_R \circ \omega) = \varprojlim \mathrm{Hom}_{k\text{-linear}}(B_X, R) = \mathrm{Hom}_{k\text{-linear}}(B, R).$$

An element $\lambda \in \mathrm{Hom}_{k\text{-linear}}(B_X, R)$ corresponds to an element of $\underline{\mathrm{End}}(\omega)(R)$ commuting with the tensor structure if and only if λ is a k-algebra homomorphism; thus

$$\underline{\mathrm{End}}^{\otimes}(\omega)(R) = \mathrm{Hom}_{k\text{-algebra}}(B, R) = G(R).$$

We have shown that, if in the statement of (2.11) the rigidity condition is omitted, then one can conclude that $\underline{\mathrm{End}}^{\otimes}(\omega)$ is representable by an affine monoid scheme $G = \operatorname{Spec} B$ and ω defines an equivalence of tensor categories

$$\mathsf{C} \to \mathsf{Comod}_B \to \mathsf{Rep}_k(G).$$

If we now assume that $(\mathsf{C}, \otimes)$ is rigid, then (1.13) shows that $\underline{\mathrm{End}}^{\otimes}(\omega) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and the theorem follows.

REMARK 2.17. Let (C, ω) be $(\mathsf{Rep}_k(G), \omega^G)$. On following through the proof of (2.11) in this case one recovers (2.8): $\underline{\mathrm{Aut}}^{\otimes}(\omega^G)$ is represented by G.

REMARK 2.18. Let $(\mathsf{C}, \otimes, \phi, \psi, F)$ satisfy the conditions of (1.20). Then certainly $(\mathsf{C}, \otimes, \phi, \psi)$ is a tensor category, and the proof of (2.11) shows that F defines an equivalence of tensor categories $\mathsf{C} \to \mathsf{Rep}_k(G)$ where G is the affine monoid scheme representing $\mathrm{End}_k^{\otimes}(\omega)$. Thus, we may assume $\mathsf{C} = \mathsf{Rep}_k(G)$. Let U be as in (d). Because it is an identity object, ωU is isomorphic to k with the trivial action of G (i.e., each element of G acts as the identity on k; cf. 1.3b). Let $\lambda \in G(R)$. If L in $\mathsf{Rep}_k(G)$ has dimension 1, then $\lambda_L : R \otimes L \to R \otimes L$ is invertible, as follows from the existence of a G-isomorphism $L \otimes L^{-1} \to U$. It follows that λ_X is invertible for all X in $\mathsf{Rep}_k(G)$, because

$$\det(\lambda_X) \stackrel{\text{def}}{=} \bigwedge^d \lambda_X = \lambda_{\bigwedge^d X}, \quad d = \dim X,$$

is invertible. Thus, G is an affine group scheme.

DEFINITION 2.19. A rigid abelian tensor category C with $\mathrm{End}(\mathbb{1}) = k$ is a ***neutral Tannakian category*** over k if it admits an exact faithful k-linear tensor functor $\omega\colon \mathsf{C} \to \mathsf{Vec}_k$. Any such functor is said to be a ***fibre functor*** for C.

Thus (2.11) shows that every neutral Tannakian category is equivalent (in possibly many different ways) to the category of finite-dimensional representations of an affine group scheme.

Properties of G and of $\mathsf{Rep}(G)$

In view of the previous theorems, it is natural to ask how properties of G are reflected in $\mathsf{Rep}_k(G)$.

PROPOSITION 2.20. *Let G be an affine group scheme over k.*

(a) *G is finite if and only if there exists an object X of $\mathsf{Rep}_k(G)$ such that every object of $\mathsf{Rep}_k(G)$ is isomorphic to a subquotient of X^n for some $n \geq 0$.*
(b) *G is algebraic if and only if $\mathsf{Rep}_k(G)$ has a tensor generator X.*[11]

PROOF. (a) If G is finite, then the regular representation X of G is finite-dimensional and has the required property. Conversely if, with the notations of (2.11), $\mathsf{Rep}_k(G) = \langle X \rangle$, then $G = \mathrm{Spec}\, B$ where B is the linear dual of the finite k-algebra A_X.

(b) If G is algebraic, then it has a finite-dimensional faithful representation X (see 2.5), and one shows as in Deligne 1982, 3.1a, that $X \oplus X^\vee$ is a tensor generator for $\mathsf{Rep}_k(G)$. Conversely, if X is a tensor generator for $\mathsf{Rep}_k(G)$, then it is a faithful representation of G. □

PROPOSITION 2.21. *Let $f\colon G \to G'$ be a homomorphism of affine group schemes over k, and let ω^f be the corresponding functor $\mathsf{Rep}_k(G') \to \mathsf{Rep}_k(G)$.*

(a) *f is faithfully flat if and only if ω^f is fully faithful and every subobject of $\omega^f(X')$, for $X' \in \mathrm{ob}(\mathsf{Rep}_k(G'))$, is isomorphic to the image of a subobject of X'.*
(b) *f is a closed immersion if and only if every object of $\mathsf{Rep}_k(G)$ is isomorphic to a subquotient of an object of the form of $\omega^f(X')$, $X' \in \mathrm{ob}(\mathsf{Rep}_k(G'))$.*

PROOF. (a) If $G \xrightarrow{f} G'$ is faithfully flat, and therefore an epimorphism, then $\mathsf{Rep}_k(G')$ can be identified with the subcategory of $\mathsf{Rep}_k(G)$ of representations $G \to \mathrm{GL}(V)$ factoring through G'. It is therefore obvious that ω^f has the stated properties. Conversely, if ω^f is fully faithful, it defines an equivalence of $\mathsf{Rep}_k(G')$ with a full subcategory of $\mathsf{Rep}_k(G)$, and the second condition shows that, for $X' \in \mathrm{ob}(\mathsf{Rep}_k(G'))$, $\langle X' \rangle$ is equivalent to $\langle \omega^f(X') \rangle$. Let $G = \mathrm{Spec}\, B$ and $G' = \mathrm{Spec}\, B'$; then (2.15) shows that

$$B' = \varinjlim \mathrm{End}(\omega' | \langle X' \rangle)^\vee = \varinjlim \mathrm{End}(\omega | \langle \omega^f(X') \rangle)^\vee \subset \varinjlim \mathrm{End}(\omega | \langle X \rangle)^\vee = B,$$

and $B' \to B$ being injective implies that $G \to G'$ is faithfully flat (Waterhouse 1979, 14.1).

(b) Let C be the strictly full subcategory of $\mathsf{Rep}_k(G)$ whose objects are isomorphic to subquotients of objects of the form of $\omega^f(X')$. The functors

$$\mathsf{Rep}_k(G') \to \mathsf{C} \to \mathsf{Rep}_k(G)$$

[11] An object X of $\mathsf{Rep}_k(G)$ is a tensor generator if every object of $\mathsf{Rep}_k(G)$ is isomorphic to a subquotient of $P(X, X^\vee)$ for some $P \in \mathbb{N}[t,s]$.

correspond (see 2.14), 2.15) to homomorphisms of k-coalgebras

$$B' \to B'' \to B$$

where $G = \operatorname{Spec} B$ and $G' = \operatorname{Spec} B'$. An argument as in the above above proof shows that $B'' \to B$ is injective. Moreover, for $X' \in \mathrm{ob}(\mathsf{Rep}_k(G'))$, $\mathrm{End}(\omega|\langle\omega^f(X)\rangle) \to \mathrm{End}(\omega'|\langle X'\rangle)$ is injective, and so $B' \to B''$ is surjective. If f is a closed immersion, then $B' \to B$ is surjective and it follows that $B'' \xrightarrow{\approx} B$, and $\mathsf{C} = \mathsf{Rep}_k(G)$. Conversely, if $\mathsf{C} = \mathsf{Rep}_k(G)$, $B'' = B$ and $B' \to B$ is surjective. □

COROLLARY 2.22. *Assume that k has characteristic zero. Then G is connected if and only if, for every representation X of G on which G acts non-trivially, $\langle X\rangle$ is not stable under $\otimes$.*[12]

PROOF. The group G is connected if and only if there is no non-trivial epimorphism $G \to G'$ with G' finite. According to (2.21a) this is equivalent to $\mathsf{Rep}_k(G)$ having no non-trivial subcategory of the type described in (2.20a). □

PROPOSITION 2.23. *Let G be a connected affine group scheme over a field k of characteristic zero. The category $\mathsf{Rep}_k(G)$ is semisimple if and only if G is pro-reductive (i.e., a projective limit of reductive groups).*

This will proved as a consequence of a series of lemmas (for another exposition of the proof, see Milne 2017, 22.42). As every finite-dimensional representation $G \to \mathrm{GL}_V$ of G factors through an algebraic quotient of G, we can assume that G itself is an algebraic group. In the lemmas, G is assumed to be connected.

LEMMA 2.24. *Let X be a representation of G; a subspace $Y \subset X$ is stable under G if and only if it is stable under* $\mathrm{Lie}(G)$.

PROOF. Standard. □

LEMMA 2.25. *Let G be an affine group scheme over a field k of characteristic zero, and let $\bar{k}$ be an algebraic closure of k. Then $\mathsf{Rep}_k(G)$ is semisimple if and only if $\mathsf{Rep}_{\bar{k}}(G_{\bar{k}})$ is semisimple.*

PROOF. Let $U(G)$ be the universal enveloping algebra of $\mathrm{Lie}(G)$, and let X be a finite-dimensional representation of G. The last lemma shows that X is semisimple as a representation of G if and only if it is semisimple as a representation of $\mathrm{Lie}(G)$, or of $U(G)$. But X is a semisimple $U(G)$-module if and only if $\bar{k} \otimes X$ is a semisimple $\bar{k} \otimes U(G)$-module (Bourbaki Algèbre, VIII, 13.4). Since $\bar{k} \otimes U(G) = U(G_{\bar{k}})$, this shows that $\mathsf{Rep}_k(G)$ is semisimple then so is $\mathsf{Rep}_{\bar{k}}(G_{\bar{k}})$. For the converse, let $\bar{X}$ be an object of $\mathsf{Rep}_{\bar{k}}(G_{\bar{k}})$. There is a finite extension k' of k and a representation X' of $G_{k'}$ over k' giving $\bar{X}$ by extension of scalars. When we regard X' as a vector space over k, we obtain a k-representation X of G. By assumption, X is semisimple and, as was observed above, this implies that $\bar{k} \otimes_k X$ is semisimple. Since $\bar{X}$ is a quotient of $\bar{k} \otimes_k X$, $\bar{X}$ is semisimple. □

[12] Recall that $\langle X\rangle$ is the strictly full subcategory of $\mathsf{Rep}_k(G)$ whose objects are those isomorphic to a subquotient of X^n for some $n \in \mathbb{N}$.

LEMMA 2.26. *(Weyl). Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic zero. Every finite-dimensional representation of $\mathfrak{g}$ is semisimple.*

PROOF. For an algebraic proof, see, for example, Humphreys 1972, 6.3. Weyl's original proof was as follows: we can assume that $k = \mathbb{C}$; let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$, and let G_0 be a connected simply-connected real Lie group with Lie algebra L_0; as G_0 is compact, every finite-dimensional representation (V, r) of it carries a $\mathfrak{g}_0$-invariant positive-definite form, namely, $\langle x, y\rangle_0 = \int_{G_0} \langle x, y\rangle dg$ where $\langle\ ,\ \rangle$ is any positive-definite form on V, and therefore is semisimple; thus every finite-dimensional (real or complex) representation of G_0 is semisimple, but, for any complex vector space V, the restriction map is an isomorphism

$$\mathrm{Hom}(G, \mathrm{GL}_V) \simeq \mathrm{Hom}(G_0, \mathrm{GL}_V),$$

and so every complex representation of G is semisimple. □

For the remainder of the proof, we assume that k is algebraically closed.

LEMMA 2.27. *Let N be a normal closed subgroup of the affine group scheme G. If (X, ρ) is a semisimple representation of G, then $(X, \rho|N)$ is a semisimple representation of N.*

PROOF. We can assume that X is a simple G-module. Let Y be a nonzero simple N-submodule of X. For any $g \in G(k)$, gY is an N-module and it is simple because $g \mapsto g^{-1}S$ maps N-submodules of gY to N-submodules of Y. The sum $\sum gY$, $g \in G(k)$, is G-stable and nonzero, and therefore equals X. Thus X, being a sum of simple N-submodules, is semisimple. □

We now prove the proposition. If G is reductive, then $G = Z \cdot G'$ where Z is the centre of G and G' is the derived subgroup of G. Let $\rho: G \to \mathrm{GL}_X$ be a finite-dimensional representation of G. As Z is a torus, $\rho|Z$ is diagonalizable: $X = \bigoplus_i X_i$ as a Z-module, where each element z of Z acts on X_i as a scalar $\chi_i(z)$. Each X_i is G'-stable and, as G' is semisimple, is a direct sum of simple G'-modules. It is now clear that X is semisimple as a G-module.

Conversely, assume that $\mathsf{Rep}_k(G)$ is semisimple and choose a faithful representation X of G. Let N be the unipotent radical of G. Lemma 2.27 shows that X is semisimple as an N-module: $X = \bigoplus_i X_i$ where each X_i is a simple N-module. As N is solvable, the Lie-Kolchin theorem shows that each X_i has dimension one, and as N is unipotent, it has a fixed vector in each X_i. Therefore N acts trivially on each X_i, and on X, and, as X is faithful, this shows that $N = \{1\}$.

REMARK 2.28. The proposition can be strengthened as follows: assume that k has characteristic zero; then the identity component G° of G is pro-reductive if and only if $\mathsf{Rep}_k(G)$ is semisimple.

To prove this, we have to show that the category $\mathsf{Rep}_k(G)$ is semisimple if and only if $\mathsf{Rep}_k(G^\circ)$ is semisimple. As G° is a closed normal subgroup of G, the necessity follows from (2.27). For the sufficiency, let X be a representation of G. After replacing G with its image in GL_X, we may assume that G is algebraic. Let Y be a G-stable subspace of X. By assumption, there is a G°-equivariant map $p: X \to Y$ such that $p|Y = \mathrm{id}$. Define

$$q: \bar{k} \otimes X \to \bar{k} \otimes Y, \quad q = \frac{1}{n} \sum_g g_Y p g_X^{-1}$$

where $n = (G(\bar{k}) : G^{\circ}(\bar{k}))$ and g runs over a set of coset representatives for $G^{\circ}(\bar{k})$ in $G(\bar{k})$. One checks easily that q has the following properties:

(a) it is independent of the choice of the coset representatives;
(b) for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, $\sigma(q) = q$;
(c) for all $y \in \bar{k} \otimes Y$, $q(y) = q$;
(d) for all $g \in G(\bar{k})$, $g_Y \cdot q = q \cdot g_X$.

Thus q is defined over k, restricts to the identity map on Y, and is G-equivariant.

REMARK 2.29. When, as in the above remark, $\mathsf{Rep}_k(G)$ is semisimple, the second condition in (2.21a) is superfluous; thus $f: G \to G'$ is faithfully flat if and only if ω^f is fully faithful.

Examples

2.30. (Graded vector spaces) Let C be the category whose objects are families $(V^n)_{n \in \mathbb{Z}}$ of vector spaces over k with finite-dimensional sum $V = \bigoplus V^n$. There is an obvious rigid tensor structure on C for which $\mathrm{End}(1\!\!1) = k$ and $\omega: (V^n) \mapsto \bigoplus V^n$ is a fibre functor. Thus, according to (2.11), there is an equivalence of tensor categories $\mathsf{C} \to \mathsf{Rep}_k(G)$ for some affine k-group scheme G. This equivalence is easy to describe: take $G = \mathbb{G}_m$ and make (V^n) correspond to the representation of $\mathbb{G}_m$ on V for which $\mathbb{G}_m$ acts on V^n through the character $\lambda \mapsto \lambda^n$.

2.31. A real Hodge structure is a finite-dimensional vector space V over $\mathbb{R}$ together with a decomposition

$$V \otimes \mathbb{C} = \bigoplus\nolimits_{p,q} V^{p,q}$$

such that $V^{p,q}$and $V^{q,p}$ are conjugate complex subspaces of $V \otimes \mathbb{C}$. There is an obvious rigid tensor structure on the category $\mathsf{Hod}_{\mathbb{R}}$ of real Hodge structures, and

$$\omega: (V, (V^{p,q})) \rightsquigarrow V$$

is a fibre functor. The group corresponding to $\mathsf{Hod}_{\mathbb{R}}$ and ω is the real algebraic group $\mathbb{S}$ obtained from $\mathbb{G}_m$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$: $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.[13] The real Hodge structure $(V, (V^{p,q}))$ corresponds to the representation of $\mathbb{S}$ on V such that an element $\lambda \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ acts on $V^{p,q}$ as $\lambda^{-p}\bar{\lambda}^{-q}$. We can write $V = \bigoplus V^n$ where $V^n \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$. The functor $(V, (V^{p,q})) \mapsto (V^n)$ from $\mathsf{Hod}_{\mathbb{R}}$ to the category of graded real vector spaces corresponds to a homomorphism $\mathbb{G}_m \to \mathbb{S}$ which, on real points, is $t \mapsto t^{-1}: \mathbb{R}^{\times} \to \mathbb{C}^{\times}$.

2.32. The preceding examples have a common generalization. Let $\bar{k}$ be a separable algebraic closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Recall that an algebraic group G over k is of ***multiplicative type*** every representation of G becomes diagonalizable over $\bar{k}$. In characteristic zero, this is equivalent to the identity component of G being a torus. The character group $X(G) \stackrel{\text{def}}{=} \mathrm{Hom}(G_{\bar{k}}, \mathbb{G}_m)$ of such a G is a finitely generated abelian group on which Γ acts continuously. Let $M = X(G)$, and let $k' \subset \bar{k}$ be a Galois extension of k over which all elements of M are defined. For any finite-dimensional representation V of G,

$$V \otimes_k k' = \bigoplus_{m \in M} V^m, \quad V^m \stackrel{\text{def}}{=} \{v \in V \otimes_k k' \mid gv = m(g)v \text{ all } g \in G(k)\}.$$

[13] This is often called the Deligne torus.

A finite-dimensional vector space V over k together with a decomposition $k' \otimes V = \bigoplus_{m \in M} V^m$ arises from a representation of G if and only if $V^{\sigma(m)} = \sigma V^m$ for all $m \in M$ and $\sigma \in \Gamma$. Thus an object of $\mathsf{Rep}_k(G)$ can be identified with a finite-dimensional vector space V over k together with an M-grading on $V \otimes k'$ that is compatible with the action of Γ.

2.33. (Tannakian duality) Let K be a topological group. The category $\mathsf{Rep}_{\mathbb{R}}(K)$ of continuous representations of K on finite-dimensional real vector spaces is, in a natural way, a neutral Tannakian category with the forgetful functor as fibre functor. There is therefore a real affine algebraic group $\tilde{K}$ called the ***real algebraic envelope*** of K, for which there exists an equivalence $\mathsf{Rep}_{\mathbb{R}}(K) \to \mathsf{Rep}_{\mathbb{R}}(\tilde{K})$. This equivalence arises from a homomorphism $K \to \tilde{K}(\mathbb{R})$, which is an isomorphism if K is compact.[14]

In general, a real algebraic group G is said to be ***compact*** if $G(\mathbb{R})$ is compact and the natural functor $\mathsf{Rep}_{\mathbb{R}}(G(\mathbb{R})) \to \mathsf{Rep}_{\mathbb{R}}(G)$ is an equivalence. The second condition is equivalent to each connected component of $G(\mathbb{C})$ containing a real point (or to $G(\mathbb{R})$ being Zariski dense in G). We note for reference that Deligne 1972, 2.5, shows that a subgroup of a compact real group is compact.

2.34. (The true fundamental group.) Recall that a vector bundle E on a curve C is ***semi-stable*** if for every sub-bundle $E' \subset E$,

$$\frac{\deg(E')}{\operatorname{rank}(E')} \leq \frac{\deg(E)}{\operatorname{rank}(E)}.$$

Let X be a complete connected reduced k-scheme, where k is assumed to be perfect. A vector bundle E on X will be said to be ***semi-stable*** if for every nonconstant morphism $f : C \to X$ with C a projective smooth connected curve, f^*E is semi-stable of degree zero. Such a bundle E is ***finite*** if there exist polynomials $g, h \in \mathbb{N}[t]$, $g \neq h$, such that $g(E) \approx h(E)$. Let C denote the category of semi-stable vector bundles on X isomorphic to a subquotient of a finite vector bundle. If X has a k-rational point x, then C is a neutral Tannakian category over k with fibre functor $\omega : E \rightsquigarrow E_x$. The affine group scheme attached to (C, ω) is a pro-finite group scheme over k, called the ***true fundamental group*** $\pi_1(X, x)$ of X, which classifies all G-coverings of X with G a finite group scheme over k. In particular, the largest pro-étale quotient of $\pi_1(X, x)$ classifies the finite étale coverings of X together with a k-point lying over x; it coincides with the usual étale fundamental group of X when $k = \bar{k}$. See Nori 1976.

2.35. Let K be a field of characteristic zero, complete with respect to a discrete valuation, whose residue field is algebraically closed of characteristic $p \neq 0$. The Hodge-Tate modules for K form a neutral Tannakian category over $\mathbb{Q}_p$ (see Serre 1979).

3. Fibre functors; the general notion of a Tannakian category

Throughout this section, k denotes a field.

[14] See §5 of Serre, Jean-Pierre. Gèbres. Enseign. Math. (2) 39 (1993), no. 1-2, 33–85. For the original Tannakian duality, see Tannaka, Tadao, Über den Dualitätssatz der nichtkommutativen topologischen Gruppen. Tohoku Math. J. 45, 1-12 (1938).

Fibre Functors

Let G be an affine group scheme over k and $U = \mathrm{Spec}(R)$ an affine k-scheme. A ***G-torsor*** over U (for the fpqc topology) is an affine scheme T, faithfully flat over U, together with a morphism $T \times_U G \to T$ such that

$$(t,g) \mapsto (t,tg) \colon T \times_U G \to T \times_U T$$

is an isomorphism. Such a scheme T is determined by its points functor, $h_T = (R' \rightsquigarrow T(R'))$.

3.1. A non-vacuous set-valued functor h of R-algebras equipped with a functorial pairing $h(R') \times G(R') \to h(R')$ arises from a G-torsor if,

(a) for each R-algebra R' such that $h(R')$ is non-empty, $G(R')$ acts simply transitively on $h(R')$, and
(b) h is representable by an affine scheme faithfully flat over U.

Descent theory shows that (3.1b) can be replaced by the condition that h be a sheaf for the fpqc topology on U (see Waterhouse 1979, 18.4). There is an obvious notion of a morphism of G-torsors.

Assume now that C is a k-linear abelian tensor category. A ***fibre functor*** on C with values in a k-algebra R is a k-linear exact faithful tensor functor $\eta \colon \mathsf{C} \to \mathsf{Mod}_R$ that takes values in the subcategory Proj_R of Mod_R. Assume now that C is a neutral Tannakian category over k. There then exists a fibre functor ω with values in k and we proved in the last section that if we let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, then ω defines an equivalence $\mathsf{C} \to \mathsf{Rep}_k(G)$. For any fibre functor η with values in R, composition defines a pairing

$$\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta) \times \underline{\mathrm{Aut}}^{\otimes}(\omega) \to \underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$$

of functors of R-algebras. Proposition 1.13 shows that $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta) = \underline{\mathrm{Isom}}^{\otimes}(\omega,\eta)$, and therefore that $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ satisfies (3.1a).

THEOREM 3.2. *Let* C *be a neutral Tannakian category over* k.

(a) *For any fibre functor* η *on* C *with values in* R, $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ *is representable by an affine scheme faithfully flat over* $\mathrm{Spec}\, R$*; it is therefore a* G*-torsor.*
(b) *The functor* $\eta \rightsquigarrow \underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ *determines an equivalence between the category of fibre functors on* C *with values in* R *and the category of* G*-torsors over* R.

PROOF. Let $X \in \mathrm{ob}(\mathsf{C})$, and, with the notations of the proof of (2.11), define

$$A_X \subset \mathrm{End}(\omega X), \quad A_X = \bigcap_Y (\omega Y : \omega Y), \quad Y \subset X^n,$$
$$P_X \subset \mathrm{End}(\omega X, X), \quad P_X = \bigcap_Y (Y : \omega Y), \quad Y \subset X^n.$$

Then $\omega(P_X) = A_X$ and $P_X \in \mathrm{ob}(\langle X \rangle)$. For any R-algebra R', $\underline{\mathrm{Hom}}(\omega|\langle X\rangle, \eta|\langle X\rangle)(R')$ is the subspace of $\mathrm{Hom}(\omega(P_X) \otimes_k R', \eta(P_X) \otimes_R R')$ of maps respecting all $Y \subset X^n$; it therefore equals $\eta(P_X) \otimes R'$. Thus

$$\underline{\mathrm{Hom}}(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \xrightarrow{\simeq} \mathrm{Hom}_{R\text{-lin}}(\eta(P_X^{\vee}), R').$$

Let Q be the ind-object $(P_X^\vee)_X$, and let $B = \varinjlim A_X^\vee$. As we saw in the last section, the tensor structure on C defines an algebra structure on B; it also defines a ring structure on Q (i.e., a map $Q \otimes Q \to Q$ in $\mathrm{Ind}(\mathsf{C})$) making $\omega(Q) \to B$ into an isomorphism of k-algebras. We have

$$\begin{aligned}\underline{\mathrm{Hom}}(\omega,\eta)(R') &= \varprojlim \underline{\mathrm{Hom}}(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \\ &= \varprojlim \mathrm{Hom}_{R\text{-lin}}(\eta(P_X^\vee), R') \\ &= \mathrm{Hom}_{R\text{-lin}}(\eta(Q), R)\end{aligned}$$

where $\eta(Q) \overset{\text{def}}{=} \varinjlim \eta(P_X^\vee)$. Under this correspondence,

$$\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)(R') = \mathrm{Hom}_{R\text{-alg}}(\eta(Q), R'),$$

and so $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ is represnted by $\eta(Q)$. By definition, $\eta(P_X^\vee)$ is a projective R-module, and so $\eta(Q) = \varinjlim \eta(P_X^\vee)$ is flat over R. For each X, there is a surjection $P_X \twoheadrightarrow \mathbb{1}$, and the exact sequence

$$0 \to \mathbb{1} \to P_X^\vee \to P_X^\vee/\mathbb{1} \to 0$$

gives rise to an exact sequence

$$0 \to \eta(\mathbb{1}) \to \eta(P_X^\vee) \to \eta(P_X^\vee/\mathbb{1}) \to 0.$$

As $\eta(\mathbb{1}) = R$ and $\eta(P_X^\vee/\mathbb{1})$ is flat, this shows that $\eta(P_X^\vee)$ is a faithfully flat R-module. Hence $\eta(Q)$ is faithfully flat over R, which completes the proof that $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ is a G-torsor.

To show that $\eta \rightsquigarrow \underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ is an equivalence, we construct a quasi-inverse. Let T be a G-torsor over R. For a fixed X, define $R' \rightsquigarrow \eta_T(X)(R')$ to be the sheaf associated with

$$R' \rightsquigarrow (\omega(X) \otimes R') \times T(R')/G(R').$$

Then $X \rightsquigarrow \eta_T(X)$ is a fibre functor on C with values in R. □

REMARK 3.3. (a) Define

$$\underline{A}_X \subset \underline{\mathrm{Hom}}(X,X), \quad \underline{A}_X = \bigcap (Y{:}Y), \quad Y \subset X^n.$$

Then $\underline{A}_X$ is a ring in C such that $\omega(\underline{A}_X) = A_X$ (as k-algebras). Let B be the ind-object $(\underline{A}_X^\vee)$. Then

$$\begin{aligned}\underline{\mathrm{End}}^{\otimes}(\omega) &= \mathrm{Spec}\,\omega(B) = G \\ \underline{\mathrm{End}}^{\otimes}(\eta) &= \mathrm{Spec}\,\eta(B).\end{aligned}$$

(b) The proof of (3.2) can be made more concrete (but less canonical) by using (2.11) to replace (C,ω) with $(\mathsf{Rep}_k(G), \omega^G)$.

REMARK 3.4. The situation described in the theorem is analogous to the following. Let X be a connected topological space, and let C be the category of locally constant sheaves of $\mathbb{Q}$-vector spaces on X. For any $x \in X$, there is a fibre functor $\omega_x{:}\mathsf{C} \to \mathsf{Vec}_{\mathbb{Q}}$, and ω_x defines an equivalence of categories $\mathsf{C} \to \mathsf{Rep}_{\mathbb{Q}}(\pi_1(X,x))$. Let $\Pi_{x,y}$ be the set of homotopy classes of paths from x to y; then $\Pi_{x,y} \simeq \mathrm{Isom}(\omega_x, \omega_y)$, and $\Pi_{x,y}$ is a $\pi_1(X,x)$-torsor.

QUESTION 3.5. Let $(\mathsf{C},\otimes)$ be a rigid abelian tensor category whose objects are of finite length and which is such that $\mathrm{End}(\mathbb{1}) = k$ and $\otimes$ is exact. (Thus $(\mathsf{C},\otimes)$ lacks only a fibre functor with values in k to be a neutral Tannakian category). As in (3.3) one can define

$$\underline{A}_X \subset \underline{\mathrm{Hom}}(X,X), \quad \underline{A}_X = \bigcap (Y:Y), \quad Y \subset X^n$$

and hence obtain a bialgebra $\underline{B} = \text{``}\varinjlim\text{''}\underline{A}_X^\vee$ in $\mathrm{Ind}(\mathsf{C})$ which can be thought of as defining an affine group scheme G in $\mathrm{Ind}(\mathsf{C})$. Is it true that, for $X \subset X'$, the morphism $\underline{A}_{X'} \to \underline{A}_X$ is an epimorphism?

For any X in C, there is a morphism $X \xrightarrow{\rho} X \otimes \underline{B}$, which can be regarded as a representation of G. Define X^G, the subobject fixed by G, to be the largest subobject of X such that $X^G \to X \otimes B_X$ factors through $X^G \otimes \mathbb{1} \hookrightarrow X \otimes B_X$. Is it true that $\underline{\mathrm{Hom}}(\mathbb{1},X) \otimes_k \mathbb{1} \to X^G$ is an isomorphism?

If for all X in C there exists an integer $N \geq 0$ such that $\bigwedge^N X = 0$, does C admit a fibre functor (and so is Tannakian in the sense of Definition 3.7 below)?

Assume that k has characteristic zero. Then the answer to the last question is positive (Deligne 1990, 7.1); in particular, $(\mathsf{C},\otimes)$ admits a fibre functor with values in a nonzero k-algebra R. For such a fibre functor ω, $G(\omega) \overset{\text{def}}{=} \mathrm{Spec}(\omega(\underline{B}))$ is the affine group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ (Deligne 1989, §6), and it follows that the answer to the first two questions is also positive.[15]

The general notion of a Tannakian category

In this subsection, we need to use some terminology from non-abelian 2-cohomology, for which we refer the reader to the Appendix. In particular, Aff_S or Aff_k denotes the category of affine schemes over $S = \mathrm{Spec}(k)$ and PROJ is the stack over Aff_S such that $\mathrm{PROJ}_U = \mathsf{Proj}_R$ for $R = \Gamma(U,\mathcal{O}_U)$. For a gerbe G over Aff_k (for the fpqc topology), we let $\mathsf{Rep}_k(\mathsf{G})$ denote the category of cartesian functors $\mathsf{G} \to \mathrm{PROJ}$. Thus, an object ϕ of $\mathsf{Rep}_k(\mathsf{G})$ determines (and is determined by) functors $\phi_R \colon \mathsf{G}_R \to \mathsf{Proj}_R$, one for each k-algebra R, and functorial isomorphisms

$$\phi_{R'}(g^*Q) \leftrightarrow \phi_R(Q) \otimes_R R'$$

defined whenever $g\colon R \to R'$ is a homomorphism of k-algebras and $Q \in \mathrm{ob}(\mathsf{G}_R)$. There is an obvious rigid tensor structure on $\mathsf{Rep}_k(\mathsf{G})$, and $\mathrm{End}(\mathbb{1}) = k$.

EXAMPLE 3.6. Let G be an affine group scheme over k, and let $\mathrm{TORS}(G)$ be the gerbe over Aff_S such that $\mathrm{TORS}(G)_U$ is the category of G-torsors over U. Let G_r be G regarded as a right G-torsor, and let Φ be an object of $\mathsf{Rep}_k(\mathrm{TORS}(G))$. The isomorphism $G \xrightarrow{\simeq} \underline{\mathrm{Aut}}(G_r)$ defines a representation of G on the vector space $\Phi_k(G_r)$, and it is not difficult to show that $\Phi \rightsquigarrow \Phi_k(G_r)$ extends to an equivalence of categories

$$\mathsf{Rep}_k(\mathrm{TORS}(G)) \to \mathsf{Rep}_k(G).$$

Let C be a rigid abelian tensor category with $\mathrm{End}(\mathbb{1}) = k$. For any k-algebra R, the fibre functors on C with values in R form a fibred category $\mathrm{FIB}(\mathsf{C})_R$ over Aff_k. Descent theory for projective modules shows that $\mathrm{FIB}(\mathsf{C})$ is a stack, and (1.13) shows that its fibres are groupoids. There is a canonical k-linear tensor functor $\mathsf{C} \to \mathsf{Rep}_k(\mathrm{FIB}(C))$ attaching to $X \in \mathrm{ob}(\mathsf{C})$ the family of functors $\omega \mapsto \omega(X)\colon \mathrm{FIB}(\mathsf{C})_R \to \mathsf{Proj}_R$.

[15] This paragraph has been added, and the next subsection rewritten, to take account of Deligne 1990.

DEFINITION 3.7. A rigid abelian tensor category C with $\mathrm{End}(\mathbb{1}) = k$ is a ***Tannakian category*** over k if it admits a fibre functor with values in some nonzero k-algebra.

EXAMPLE 3.8. A Tannakian category C over k is said to be ***neutral*** if it admits a fibre functor with values in k.[16] Clearly this agrees with the definition in (2.19). Let C be a neutral Tannakian category over k. Theorem 3.2 shows that the choice of a fibre functor ω with values in k determines an equivalence of fibred categories $\mathrm{FIB}(\mathsf{C}) \to \mathrm{TORS}(G)$ where G represents $\underline{\mathrm{Aut}}^{\otimes}(\omega)$. This shows that $\mathrm{FIB}(\mathsf{C})$ is an affine gerbe, and the commutative diagram of functors

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[d, "\omega", "\sim"'] & \mathsf{Rep}_k(\mathrm{FIB}(\mathsf{C})) \arrow[d, "\sim"'] \\ \mathsf{Rep}_k(G) & \mathsf{Rep}_k(\mathrm{TORS}(G)) \arrow[l, "\sim"'] \end{tikzcd}$$

shows that $\mathsf{C} \to \mathsf{Rep}_k(\mathrm{FIB}(\mathsf{C}))$ is an equivalence of categories.

The following fundamental theorem, which was not available when the original article was written, justifies the above definition of Tannakian category.

THEOREM 3.9. *Let C be a Tannakian category over k. Then $\mathrm{FIB}(\mathsf{C})$ is an affine gerbe and the functor $\mathsf{C} \to \mathsf{Rep}_k(\mathrm{FIB}(\mathsf{C}))$ is an equivalence of categories.*

PROOF. Deligne 1990, 1.12, 1.13. □

To say that $\mathrm{FIB}(\mathsf{C})$ is a gerbe means that any two fibre functors are locally isomorphic for the fpqc topology. To say that it is affine means that, for any fibre functor with values in a k-algebra R, $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine group scheme over R.

REMARK 3.10. A Tannakian category C over k is said to be ***algebraic*** if $\mathrm{FIB}(\mathsf{C})$ is an algebraic gerbe. There then exists a finite field extension k' of k and a fibre functor ω with values in k' (Appendix, p. 70, Proposition), and the algebraicity of C means that $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an algebraic group over k'. As in the neutral case (2.20), a Tannakian category is algebraic if and only if it has a tensor generator. Consequently, every Tannakian category is a filtered union of algebraic Tannakian categories.

Tannakian categories neutralized by a finite extension

Let C be a k-linear category and A a commutative k-algebra. An ***A-module*** in C is a pair (X, α_X) with X an object of C and α_X a homomorphism $A \to \mathrm{End}(X)$. For example, an A-module in $\mathsf{Vec}_{k'}$, where $k' \supset k$, is simply an $A \otimes_k k'$-module that is of finite dimension over k'. With an obvious notion of morphism, the A-modules in C form an A-linear category $\mathsf{C}_{(A)}$. If C is abelian, then so also is $\mathsf{C}_{(A)}$, and if C has a tensor structure and its objects have finite length, then we define $(X, \alpha_X) \otimes (Y, \alpha_Y)$ to be the A-module in C with object the largest quotient of $X \otimes Y$ to which $\alpha_X(a) \otimes \mathrm{id}$ and $\mathrm{id} \otimes \alpha_Y(a)$ agree for all $a \in A$.

Now let C be a Tannakian category over k, and let k' be a finite field extension of k. As the tensor operation on C commutes with direct limits (1.16) , it extends to $\mathrm{Ind}(\mathsf{C})$, which is therefore an abelian tensor category. The functor $\mathsf{C} \to \mathrm{Ind}(\mathsf{C})$ defines an equivalence

[16] Every Tannakian category over an algebraically closed field is neutral (letter of Deligne, November 30, 2011).

between C and the strictly full subcategory C^e of Ind(C) of essentially constant ind-objects. In C^e it is possible to define external tensor products with objects of Vec_k (cf. the proof of 2.11) and hence a functor

$$X \rightsquigarrow i(X) = (k' \otimes_k X, a' \mapsto a' \otimes \mathrm{id}) : \mathsf{C}^e \to \mathsf{C}^e_{(k')}.$$

This functor is left adjoint to

$$(X,\alpha) \rightsquigarrow j(X,\alpha) = X : \mathsf{C}^e_{(k')} \to \mathsf{C}^e$$

and has the property that $k' \otimes_k \mathrm{Hom}(X,Y) \xrightarrow{\simeq} \mathrm{Hom}(i(X), i(Y))$. Let ω be a fibre functor on C^e (or C) with values in k'. For any $(X,\alpha) \in \mathrm{ob}(\mathsf{C}^e_{(k')})$, $(\omega(X), \omega(\alpha))$ is a k'-module in $\mathsf{Vec}_{k'}$, i.e., it is a $k' \otimes_k k'$-module. If we define

$$\omega'(X,\alpha) = k' \otimes_{k' \otimes k'} \omega(X) \tag{3.10.1}$$

then

$$\begin{tikzcd} \mathsf{C}^e \arrow[r] \arrow[dr, "\omega"'] & \mathsf{C}^e_{(k')} \arrow[d, dashed, "\omega'"] \\ & \mathsf{Vec}_{k'}. \end{tikzcd}$$

commutes up to a canonical isomorphism.

PROPOSITION 3.11. *Let C be a Tannakian category over k and let ω be a fibre functor on C with values in a finite field extension k' of k; extend ω' to $\mathsf{C}_{(k')}$ using the formula (3.10.1); then ω' defines an equivalence of tensor categories $\mathsf{C}_{(k')} \to \mathsf{Rep}_{k'}(G)$ where $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$. In particular, ω' is exact.*

PROOF. One has simply to compose the following functors:

$$\mathsf{C}_{(k')} \xrightarrow{\sim} \mathsf{Rep}_k(G)_{(k')}$$

arising from the equivalence $\mathsf{C} \xrightarrow{\sim} \mathsf{Rep}_k(\mathsf{G})$, $\mathsf{G} = \mathrm{FIB}(\mathsf{C})$), in (3.9);

$$\mathsf{Rep}_k(\mathsf{G})_{(k')} \xrightarrow{\sim} \mathsf{Rep}_{k'}(\mathsf{G}/k')$$

where G/k' denotes the restriction of G to $\mathsf{Aff}_{k'}$ (the functor sends $(\phi,\alpha) \in \mathrm{ob}(\mathsf{Rep}_k(\mathsf{G})_{(k')})$ to ϕ' where, for any k'-algebra R and $Q \in \mathsf{G}_R$, $\phi'_R(Q) = R \otimes_{k' \otimes F} \phi_R(Q)$;

$$\mathsf{Rep}_{k'}(\mathsf{G}/k') \xrightarrow{\sim} \mathsf{Rep}_{k'}(\mathrm{TORS}(G))$$

arising from $\mathrm{TORS}(G) \xrightarrow{\sim} G/k'$;

$$\mathsf{Rep}_{k'}(\mathrm{TORS}(G)) \xrightarrow{\sim} \mathsf{Rep}_{k'}(G)$$

(see 3.6). □

REMARK 3.12. Let $\mathsf{C} = \mathsf{Rep}_k(G)$ and let k' be a finite extension of k. Then $\mathsf{C}_{(k')} = \mathsf{Rep}_{k'}(G)$ and $i : \mathsf{C} \to \mathsf{C}_{(k')}$ is $X \rightsquigarrow k' \otimes_k X$. Let ω be the fibre functor

$$X \rightsquigarrow k' \otimes_k X : \mathsf{Rep}_k(G) \to \mathsf{Vec}_{k'}.$$

Then $G_{k'} = \mathrm{Aut}^{\otimes}(\omega)$ and the equivalence $\mathsf{C}_{(k')} \longrightarrow \mathsf{Rep}_{k'}(G_{k'})$ defined by the proposition is

$$X \rightsquigarrow k' \otimes_{k' \otimes k'} X : \mathsf{Rep}_{k'}(G) \to \mathsf{Rep}_{k'}(G_{k'}).$$

DESCENT OF TANNAKIAN CATEGORIES

Let k'/k be a finite Galois extension with Galois group Γ, and let C' be a Tannakian category over k'. A ***descent datum*** on C' relative to k'/k is

3.13. (a) a family $(\beta_\gamma)_{\gamma\in\Gamma}$ of equivalences of tensor categories $\beta_\gamma\colon\mathsf{C}'\to\mathsf{C}'$, β_γ being semi-linear relative to γ, together with

(b) a family $(\mu_{\gamma',\gamma})$ of isomorphisms of tensor functors $\mu_{\gamma',\gamma}\colon\beta_{\gamma'\gamma}\xrightarrow{\approx}\beta_{\gamma'}\circ\beta_\gamma$ such that

$$\begin{tikzcd}
\beta_{\gamma''\gamma'\gamma}(X) \arrow[r, "\mu_{\gamma'',\gamma'\gamma}(X)"] \arrow[d, "\mu_{\gamma''\gamma',\gamma}(X)"] & \beta_{\gamma''}(\beta_{\gamma'\gamma}(X)) \arrow[d, "\beta_{\gamma''}(\mu_{\gamma'\gamma}(X))"] \\
\beta_{\gamma''\gamma'}(\beta_\gamma(X)) \arrow[r, "\mu_{\gamma''\gamma'}(\beta_\gamma(X))"] & \beta_{\gamma''}(\beta_{\gamma'}(\beta_\gamma(X)))
\end{tikzcd}$$

commutes for all $X\in\mathrm{ob}(\mathsf{C})$.

A Tannakian category C over k gives rise to a Tannakian category $\mathsf{C}'=\mathsf{C}_{(k')}$ over k' together with a descent datum for which $\beta_\gamma(X,\alpha_X)=(X,\alpha_X\circ\gamma^{-1})$. Conversely, a Tannakian category C' over k' together with a descent datum relative to k'/k gives rise to a Tannakian category C over k whose objects are pairs $(X,(a_\gamma))$, where $X\in\mathrm{ob}(\mathsf{C}')$ and $(a_\gamma\colon X\to\beta_\gamma(X))_{\gamma\in\Gamma}$ is such that $(\mu_{\gamma',\gamma})_X\circ a_{\gamma'\gamma}=\beta_{\gamma'}(a_\gamma)\circ a_{\gamma'}$, and whose morphisms are morphisms in C' commuting with the a_γ. These two operations are quasi-inverse, so that to give a Tannakian category over k (up to a tensor equivalence, unique up to a unique isomorphism) is the same as giving a Tannakian category over k' together with a descent datum relative to k'/k (Saavedra Rivano 1972, III, 1.2). On combining this statement with (3.11) we see that to give a Tannakian category over k together with a fibre functor with values in k' is the same as giving an affine group scheme G over k' together with a descent datum on the Tannakian category $\mathsf{Rep}_{k'}(G)$.

QUESTIONS

3.14. Let G be an affine gerbe over k. There is a morphism of gerbes

$$\mathsf{G}\to\mathrm{F{\scriptsize IB}}(\mathsf{Rep}_k(\mathsf{G})) \tag{3.14.1}$$

which, to an object Q of G over $S=\operatorname{Spec}R$, attaches the fibre functor $F\rightsquigarrow F(Q)$ with values in R. Is (3.14.1) an equivalence of gerbes? If G is algebraic, or if the band of G is defined by an affine group scheme over k, then it is (Saavedra Rivano 1972, III 3.2.5) but the general question is open. A positive answer would provide the following classification of Tannakian categories: the maps $\mathsf{C}\mapsto\mathrm{F{\scriptsize IB}}(\mathsf{C})$ and $\mathsf{G}\mapsto\mathsf{Rep}_k(\mathsf{G})$ determine a one-to-one correspondence between the set of tensor equivalence classes of Tannakian categories over k and the set of equivalence classes of affine gerbes over k; the affine gerbs banded[17] by a given band B are classified by $H^2(S,B)$, and $H^2(S,B)$ is a pseudo-torsor over $H^2(S,Z)$ where Z is the centre of B.

3.15. Saavedra (1972, III 3.2.1) defines a Tannakian category over k to be a k-linear rigid abelian tensor category C for which there exists a fibre functor with values in a field $k'\supset k$.

[17] The original says *bound*, but *banded* seems to have become more common.

He then claims to prove (ibid. 3.2.3.1) that C satisfies Theorem 3.9. This is false. For example, $\mathsf{Vec}_{k'}$ for k' a field containing k is a Tannakian category over k according to his definition, but the fibre functors $V \rightsquigarrow \sigma V \overset{\text{def}}{=} V \otimes_{k',\sigma} k'$ for $\sigma \in \mathrm{Aut}(k'/k)$ are not locally isomorphic for the fpqc topology on $\mathrm{Spec}\, k'$. There is an error in the proof (ibid. p. 197, line 7) where it is asserted that "par définition" the objects of G_S are locally isomorphic.

At the time the original article was written, Theorem 3.9 had not been proved. The essential point at the time was the following: let C be a rigid abelian tensor category with $\mathrm{End}(\mathbb{1}) = k$ and let ω be a fibre functor with values in a finite field extension k' of k; is the functor ω',

$$X \rightsquigarrow k' \otimes_{k' \otimes k'} \omega(X) \colon \mathsf{C}_{(k')} \to \mathsf{Vec}_{k'}$$

exact? (See Saavedra Rivano 1972, p. 195; the proof there that ω' is faithful is valid.) The answer is yes if $\mathsf{C} = \mathsf{Rep}_k(G)$, G an affine group scheme over k, but we know of no proof simpler than to say that ω' is defined by a G-torsor on k', and $\mathsf{C}_{(k')} = \mathsf{Rep}_{k'}(G)$.

4. Polarizations

Throughout this section C will be an algebraic Tannakian category over $\mathbb{R}$ and C' will be its extension to $\mathbb{C}$: $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Complex conjugation on $\mathbb{C}$ is denoted by ι or by $z \mapsto \bar{z}$.

Tannakian categories over $\mathbb{R}$

4.1. *According to (3.13) and the paragraph following it, to give* C *is the same as giving the following data:*

(a) *an algebraic Tannakian category* C' *over* $\mathbb{C}$*;*
(b) *a semilinear*[18] *tensor functor* $X \rightsquigarrow \bar{X} \colon \mathsf{C}' \to \mathsf{C}'$*;*[19] *and*
(c) *a functorial tensor isomorphism* $\mu_X \colon X \to \bar{\bar{X}}$ *such that* $\mu_{\bar{X}} = \overline{\mu_X}$.

An object of C can be identified with an object X over C' together with a descent datum (an isomorphism $a \colon X \to \bar{\bar{X}}$ such that $\bar{a} \circ a = \mu_X$). Note that C' is automatically neutral (3.10).

EXAMPLE 4.2. Let G be an affine group scheme over $\mathbb{C}$ and let $\sigma \colon G \to G$ be a semi-linear isomorphism (meaning that $f \mapsto \sigma \circ f \colon \Gamma(G, \mathcal{O}_G) \to \Gamma(G, \mathcal{O}_G)$ is a semi-linear isomorphism). Assume that there is given a $c \in G(\mathbb{C})$ such that

$$\sigma^2 = \mathrm{ad}(c), \quad \sigma(c) = c. \tag{4.2.1}$$

From (G, σ, c) we can construct data as in (4.1):

(a) define C' to be $\mathsf{Rep}_{\mathbb{C}}(G)$;
(b) for any vector space V over $\mathbb{C}$, there is an (essentially) unique vector space $\bar{V}$ and semi-linear isomorphism $v \mapsto \bar{v} \colon V \to \bar{V}$; if V is a G-representation, we define a representation of G on $\bar{V}$ by the rule $\overline{gv} = \sigma(g)\bar{v}$;
(c) define μ_V to be the map $cv \mapsto \bar{\bar{v}} \colon V \xrightarrow{\simeq} \bar{\bar{V}}$.

[18]An additive map $f \colon V \to W$ of $\mathbb{C}$-vector spaces is ***semilinear*** if $f(zv) = \bar{z} f(v)$ for $z \in \mathbb{C}$ and $v \in V$. An additive functor $F \colon \mathsf{C}_1 \to \mathsf{C}_2$ of $\mathbb{C}$-linear categories is ***semilinear*** if $F(z_X) = \bar{z}_{FX}$, where z_X denotes the action of $z \in \mathbb{C}$ on X. A morphism of $\mathbb{C}$-schemes $\alpha \colon T \to S$ is ***semilinear*** if $f \mapsto f \circ \alpha \colon \Gamma(S, \mathcal{O}_S) \to \Gamma(T, \mathcal{O}_T)$ is semilinear as a map of $\mathbb{C}$-vector spaces.

[19]so $\overline{z_X} = \bar{z}_X$, $z \in \mathbb{C}$

Let $m \in G(\mathbb{C})$. Then $\sigma' = \sigma \circ \mathrm{ad}(m)$ and $c' = \sigma(m)cm$ again satisfiy (4.2.1). The element m defines an isomorphism of the functor $V \rightsquigarrow \bar{V}$ (rel. to (σ, c)) with the functor $V \mapsto \bar{V}$ (rel. to (σ', c')) by

$$\overline{mv} \mapsto \bar{v} \colon \bar{V} \text{ (rel. to } (\sigma, c)) \to \bar{V} \text{ (rel. to } (\sigma', c')).$$

This isomorphism carries μ_V (rel. to (σ, c)) to μ_V (rel. to (σ', c')), and hence defines an equivalence C (rel. to (σ, c)) with C (rel. to (σ', c')).

PROPOSITION 4.3. *Let* C *be an algebraic Tannakian category over* $\mathbb{R}$*, and let* $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$*. Choose a fibre functor* ω *on* C' *with values in* $\mathbb{C}$*, and let* $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$*.*

(a) There exists a pair (σ, c) *satisfying (4.2.1) and such that under the equivalence* $\mathsf{C}' \to \mathsf{Rep}_{\mathbb{C}}(G)$ *defined by* ω*, the functor* $X \rightsquigarrow \bar{X}$ *corresponds to* $V \rightsquigarrow \bar{V}$ *and* $\omega(\mu_X) = \mu_{\omega(X)}$*.*

(b) The pair (σ, c) *in (a) is uniquely determined up to replacement by a pair* (σ', c') *with* $\sigma' = \sigma \circ \mathrm{ad}(m)$ *and* $c' = \sigma(m)cm$*, some* $m \in G(\mathbb{C})$*.*

PROOF. (a) Let $\bar{\omega}$ be the fibre functor $X \rightsquigarrow \overline{\omega(\bar{X})}$ and let $T = \underline{\mathrm{Hom}}^{\otimes}(\omega, \bar{\omega})$. According to (3.2), T is a G-torsor, and the Nullstellensatz shows that it is trivial. The choice of a trivialization provides us with a functorial isomorphism $\omega(X) \to \bar{\omega}(X)$ and therefore with a semi-linear functorial isomorphism $\lambda_X \colon \omega(X) \to \omega(\bar{X})$. Define σ by the condition that $\sigma(g)_{\bar{X}} = \lambda_X \circ g_X \circ \lambda_X^{-1}$ for all $g \in G(\mathbb{C})$, and let c be such that $c_X = \omega(\mu_X)^{-1} \circ \lambda_{\bar{X}} \circ \lambda_X$.

(b) The choice of a different trivialization of T replaces λ_X with $\lambda_X \circ m_X$ for some $m \in G(\mathbb{C})$, σ with $\sigma \circ \mathrm{ad}(m)$, and c with $\sigma(m)cm$. □

Sesquilinear forms

Let C be Tannakian category over $\mathbb{R}$, and let $(\mathsf{C}', X \mapsto \bar{X}, \mu_X)$ be the associated triple (3.13).

Let $(\mathbb{1}, e)$, $e \colon \mathbb{1} \otimes \mathbb{1} \xrightarrow{\simeq} \mathbb{1}$, be an identity object for C'. Then $(\bar{\mathbb{1}}, \bar{e})$ is again an identity object, and the unique isomorphism of identity objects $a \colon (\mathbb{1}, e) \to (\bar{\mathbb{1}}, \bar{e})$ is a descent datum. It will be used to identify $\bar{\mathbb{1}}$ with $\mathbb{1}$.

A ***sesquilinear form*** on an object X of C' is a morphism

$$\phi \colon X \otimes \bar{X} \to \mathbb{1}.$$

On applying $-$, we obtain a morphism $\bar{X} \otimes \bar{\bar{X}} \to \bar{\mathbb{1}}$, which can be identified (using μ_X) with a morphism

$$\bar{\phi} \colon \bar{X} \otimes X \to \mathbb{1}.$$

There are associated with ϕ two morphisms $\phi^{\sim}, {}^{\sim}\phi \colon X \to \bar{X}^{\vee}$ determined by[20]

$$\begin{array}{rcl} \phi^{\sim}(x)(y) & = & \phi(x \otimes y) \\ {}^{\sim}\phi(x)(y) & = & \bar{\phi}(y \otimes x) \end{array} \qquad (4.3.1)$$

[20]Take $\phi^{\sim}$ to be the morphism corresponding to ϕ under the canonical isomorphisms

$$\mathrm{Hom}(X \otimes \bar{X}, \mathbb{1}) \simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(\bar{X}, \mathbb{1})) = \mathrm{Hom}(X, \bar{X}^{\vee}).$$

The form ϕ is said to be ***nondegenerate*** if $\phi^{\sim}$ (equivalently ${}^{\sim}\phi$) is an isomorphism. The ***parity*** of a nondegenerate sesquilinear form ϕ is the unique morphism $\varepsilon_\phi\colon X \to X$ such that

$$\phi^{\sim} = {}^{\sim}\phi \circ \varepsilon_\phi; \quad \phi(x,y) = \bar{\phi}(y, \varepsilon_\phi x) \tag{4.3.2}$$

Note that

$$\phi \circ (\varepsilon_\phi \otimes \bar{\varepsilon}_\phi) = \phi; \quad \phi(\varepsilon_\phi x, \bar{\varepsilon}_\phi y) = \phi(x,y) \tag{4.3.3}$$

The ***transpose*** u^ϕ of $u \in \mathrm{End}(X)$ relative to ϕ is determined by

$$\phi \circ (u \otimes \mathrm{id}_{\bar{X}}) = \phi \circ (\mathrm{id}_X \otimes \overline{u^\phi}); \quad \phi(ux,y) = \phi(x, \overline{u^\phi} y). \tag{4.3.4}$$

There are formulas

$$(uv)^\phi = v^\phi u^\phi, \quad (\mathrm{id}_X)^\phi = \mathrm{id}_X, \quad (u^\phi)^\phi = \varepsilon_\phi u \varepsilon_\phi^{-1}, \quad (\varepsilon_\phi)^\phi = \varepsilon_\phi^{-1} \tag{4.3.5}$$

and $u \mapsto u^\phi$ is a semilinear bijection $\mathrm{End}(X) \to \mathrm{End}(X)$.

If ϕ is a nondegenerate sesquilinear form on X, then any other nondegenerate sesquilinear form can be written

$$\phi_\alpha = \phi \circ (\alpha \otimes \mathrm{id}), \quad \phi_\alpha(x,y) = \phi(\alpha x, y) = \phi(x, \overline{\alpha^\phi} y) \tag{4.3.6}$$

for a uniquely determined automorphism α of X. There are the formulas

$$u^{\phi_\alpha} = (\alpha u \alpha^{-1})^\phi, \quad \varepsilon_{\phi_\alpha} = (\alpha^\phi)^{-1} \varepsilon_\phi \alpha. \tag{4.3.7}$$

Therefore, when ε_ϕ is in the centre of $\mathrm{End}(X)$, ϕ_α has the same parity as ϕ if and only if $\alpha^\phi = \alpha$.

REMARK 4.4. There is also a notion of a ***bilinear form*** on an object X of a tensor category: it is a morphism $X \otimes X \to \mathbb{1}$. Most of the notions associated with bilinear forms on vector spaces make sense in the context of Tannakian categories; see Saavedra Rivano 1972, V 2.1.

Weil forms

A nondegenerate sesquilinear form ϕ on X is a ***Weil form*** if its parity ε_ϕ is in the centre of $\mathrm{End}(X)$ and if for all nonzero u in $\mathrm{End}(X)$, $\mathrm{Tr}_X(u \circ u^\phi) > 0$.

PROPOSITION 4.5. *Let ϕ be a Weil form on X.*

(a) *The map $u \mapsto u^\phi$ is an involution of* $\mathrm{End}(X)$ *inducing complex conjugation on $\mathbb{C} = \mathbb{C} \cdot \mathrm{id}_X$, and $(u,v) \mapsto \mathrm{Tr}_X(uv^\phi)$ is a positive-definite Hermitian form on* $\mathrm{End}(X)$.

(b) $\mathrm{End}(X)$ *is a semisimple $\mathbb{C}$-algebra.*

(c) *Any commutative sub-$\mathbb{R}$-algebra A of* $\mathrm{End}(X)$ *composed of symmetric elements (i.e., elements such that $u^\phi = u$) is a product of copies of $\mathbb{R}$.*

PROOF. (a) is obvious.

(b) Let I be a nilpotent ideal in $\mathrm{End}(X)$. We have to show that $I = 0$. Suppose on the contrary that there is a $u \neq 0$ in I. Then $v = uu^\phi \in I$ and is nonzero because $\mathrm{Tr}_X(v) > 0$. As $v = v^\phi$, we have that $\mathrm{Tr}_X(v^2) > 0$, $\mathrm{Tr}_X(v^4) > 0, \ldots$ contradicting the nilpotence of I.

(c) The argument used in (b) shows that A is semisimple and is therefore a product of fields. Moreover, for any $u \in A$, $\mathrm{Tr}_X(u^2) = \mathrm{Tr}_X(uu^\phi) > 0$. If $\mathbb{C}$ occurs as a factor of A, then $\mathrm{Tr}_X|\mathbb{C}$ is a multiple of the identity map, which contradicts $\mathrm{Tr}_X(u^2) > 0$. □

Two Weil forms, ϕ on X and ψ on Y, are said to be ***compatible*** if the sesquilinear form $\phi \oplus \psi$ on $X \oplus Y$ is again a Weil form.

Let ϕ and ψ be Weil forms on X and Y respectively. Then ϕ and ψ define isomorphisms

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X \otimes Y, 1\!\!1) \leftarrow \mathrm{Hom}(Y,X).$$

Let $u \in \mathrm{Hom}(X,Y)$, and let u' be the corresponding element in $\mathrm{Hom}(Y,X)$. Then ϕ and ψ are compatible if and only if, for all $u \neq 0$, $\mathrm{Tr}_Y(u \circ u') > 0$. In particular, if $\mathrm{Hom}(X,Y) = 0$, then ϕ and ψ are automatically compatible.

PROPOSITION 4.6. *Let ϕ be a Weil form on X, and let $\phi_\alpha = \phi \circ (\alpha \otimes \mathrm{id}_X)$ for some $\alpha \in \mathrm{Aut}(X)$.*

(a) *The form ϕ_α has the same parity as ϕ if and only if α is symmetric, i.e., $\alpha^\phi = \alpha$.*
(b) *Assume α is symmetric. Then ϕ_α is a Weil form if and only if α is a square in $\mathbb{R}[\alpha] \subset \mathrm{End}(X)$.*
(c) *If ϕ_α is a Weil form with the same parity as ϕ, then ϕ_α is compatible with ϕ.*
(d) *For any Weil form ϕ on X, the map $\alpha \mapsto \phi_\alpha$ defines a one-to-one correspondence between the set of totally positive symmetric endomorphisms of X and the set of Weil forms on X that have the same parity as ϕ and are compatible with ϕ.*

PROOF. (a) According to (4.3.7), the parity of ϕ_α is $(\alpha^\phi)^{-1}\epsilon_\phi\alpha$. As ϵ_ϕ is in the centre of $\mathrm{End}(X)$, this equals ϵ_ϕ if and only if $\alpha^\phi = \alpha$.

(b) As $\alpha = \alpha^\phi$, (4.3.7) and (4.3.5) show that $u^{\phi_\alpha} = \alpha^{-1} \cdot u^\phi \cdot \alpha$. Thus, ϕ_α is a Weil form if and only if

$$\mathrm{Tr}_X(u \cdot \alpha^{-1} \cdot u^\phi \cdot \alpha) > 0, \text{ all } u \neq 0,\ u \in \mathrm{End}(X).$$

If $\alpha = \beta^2$ with $\beta \in \mathbb{R}[\alpha]$, then

$$\begin{aligned}
\mathrm{Tr}_X(u\alpha^{-1}u^\phi\alpha) &= \mathrm{Tr}_X((u\beta^{-1})\beta^{-1}u^\phi\alpha^{-1}) \\
&= \mathrm{Tr}_X(\beta^{-1}u^\phi\alpha^{-1}(u\beta^{-1})) \qquad (\mathrm{Tr}_X(vw) = \mathrm{Tr}_X(wv)) \\
&= \mathrm{Tr}_X((\beta u\beta^{-1})^\phi(\beta^{-1}u\beta)) > 0
\end{aligned}$$

for $u \neq 0$. Conversely, if ϕ_α is a Weil form, then $\mathrm{Tr}_X(u^2\alpha) > 0$ for all $u \neq 0$ in $\mathbb{R}[\alpha]$, which implies that α is a square in $\mathbb{R}[\alpha]$.

(c) Let u be a nonzero endomorphism of X. Then $u' = u^{\phi_\alpha}$, and so ϕ and ϕ_α are compatible if and only if $\mathrm{Tr}_X(u \cdot u^{\phi_\alpha}) > 0$ for all $u \neq 0$, but this is implied by ϕ_α's being a Weil form.

(d) According to (4.3.6), every nondegenerate sesquilinear form on X is of the form ϕ_α for a unique automorphism α of X. Thus, the proposition is an immediate consequence of the preceding statements. □

An element of a semisimple $\mathbb{R}$-algebra B of finite degree is said to be ***totally positive*** if the roots of its characteristic polynomial P_α are all > 0. This condition is equivalent to α being invertible in B and a square in $\mathbb{R}[\alpha]$.

The relation of compatibility on the set of Weil forms on X is obviously reflexive and symmetric, and the next corollary implies that it is also transitive on any set of Weil forms on X having a fixed parity.

COROLLARY 4.7. *Let ϕ and ϕ' be compatible Weil forms on X with the same parity, and let ψ be a Weil form on Y. If ϕ is compatible with ψ, then so also is ϕ'.*

PROOF. This follows easily from writing $\phi' = \phi_\alpha$. □

EXAMPLE 4.8. Let X be a simple object in C', so that $\mathrm{End}(X) = \mathbb{C}$, and let $\varepsilon \in \mathrm{End}(X)$. If $\bar{X}$ is isomorphic to $X^\vee$, so that there exists a nondegenerate sesquilinear form on X, then (4.3.6) shows that the sesquilinear forms on X are parametrized by $\mathbb{C}$; moreover, (4.3.7) shows that if there is a nonzero such form with parity ε, then the set of sesquilinear forms on X with parity ε is parametrized by $\mathbb{R}$; finally, (4.6) shows that if there is a Weil form with parity ε, then the set of such forms falls into two compatibility classes, each parametrized by $\mathbb{R}_{>0}$.

REMARK 4.9. Let X_0 be an object in C and let ϕ_0 be a nondegenerate bilinear form $\phi_0\colon X_0 \otimes X_0 \to \mathbb{1}$. The ***parity*** ε_{ϕ_0} of ϕ_0 is defined by the equation

$$\phi_0(x,y) = \phi_0(y, \varepsilon_{\phi_0} x).$$

The form ϕ_0 is said to be a ***Weil form*** on X_0 if ε_{ϕ_0} is in the centre of $\mathrm{End}(X_0)$ and if for all nonzero $u \in \mathrm{End}(X_0)$, $\mathrm{Tr}_{X_0}(u \circ u^{\phi_0}) > 0$. Two Weil forms ϕ_0 and ψ_0 are said to be ***compatible*** if $\phi_0 \oplus \psi_0$ is also a Weil form.

Let X_0 correspond to the pair (X,a) with $X \in \mathrm{ob}(\mathsf{C}')$. Then ϕ_0 defines a bilinear form ϕ on X, and

$$\psi \stackrel{\text{def}}{=} (X \otimes \bar{X} \xrightarrow{1\otimes a^{-1}} X \otimes X \xrightarrow{\phi} \mathbb{1})$$

is a nondegenerate sesquilinear form on X. If ϕ_0 is a Weil form, then ψ is a Weil form on X which is compatible with its conjugate $\bar{\psi}$, and every such ψ arises from a ϕ_0; moreover, $\varepsilon_\psi = \varepsilon_{\phi_0}$.

Polarizations

Let Z be the centre of the band associated with C (see the appendix). Thus Z is a commutative algebraic group over $\mathbb{R}$ such that

$$Z(\mathbb{C}) \simeq \mathrm{Centre}(\mathrm{Aut}^\otimes(\omega))$$

for every $\mathbb{C}$-valued fibre functor on C'. Moreover, Z represents $\underline{\mathrm{Aut}}^\otimes(\mathrm{id}_{\mathsf{C}})$.

DEFINITION 4.10. Let $\varepsilon \in Z(\mathbb{R})$ and, for each $X \in \mathrm{ob}(\mathsf{C}')$, let $\pi(X)$ be an equivalence class (for the relation of compatibility) of Weil forms on X with parity ε. Then π is a ***(homogeneous) polarization*** on C if

(a) for all X, $\bar{\phi} \in \pi(X)$ whenever $\phi \in \pi(\bar{X})$, and
(b) for all X and Y, $\phi \oplus \psi \in \pi(X \oplus Y)$ and $\phi \otimes \psi \in \pi(X \otimes Y)$ whenever $\phi \in \pi(X)$ and $\psi \in \pi(Y)$.

We call ε the ***parity*** of π and say that ϕ is ***positive*** for π if $\phi \in \pi(X)$. Thus the conditions require that $\bar{\phi}$, $\phi \oplus \psi$, and $\phi \otimes \psi$ be positive for π whenever ϕ and ψ are.

PROPOSITION 4.11. *Let π be a polarization on C.*

(a) *The categories C and C' are semisimple.*
(b) *If $\phi \in \pi(X)$ and $Y \subset X$, then $X = Y \oplus Y^\perp$ and the restriction ϕ_Y of ϕ to Y is in $\pi(Y)$.*

PROOF. (a) Let X be an object of $\mathsf{C}'$and let $u\colon Y \hookrightarrow X$ be a nonzero simple subobject of X. Choose $\phi \in \pi(Y)$ and $\psi \in \pi(X)$. Consider

$$v = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} : X \oplus Y \to X \oplus Y$$

and let $u'\colon X \to Y$ be such that

$$v^{\psi \oplus \phi} = \begin{pmatrix} 0 & 0 \\ u' & 0 \end{pmatrix}.$$

Then $\mathrm{Tr}_Y(u'u) = \mathrm{Tr}_{Y \oplus X}(v^{\psi\oplus\phi} \circ v) > 0$, and so $u'u$ is an automorphism w of Y. The map $p = w^{-1} \circ u'$ projects X onto Y, which shows that Y is a direct summand of X. We have shown that X is semisimple.

The same argument, using the bilinear forms (4.9) shows that C is semisimple.

(b) Let $Y' = Y \cap Y^{\perp}$, where $Y^{\perp}$ is the largest subobject of X such that ϕ is zero on $Y \otimes \bar{Y}^{\perp}$, and let $p\colon X \to X$ be the projection of X onto Y' (by which we mean that $p(X) \subset Y'$ and $p|Y' = \mathrm{id}_{Y'}$). As ϕ is zero on $Y' \otimes \overline{Y'}$,

$$0 = \phi \circ (p \otimes \bar{p}) = \phi \circ (\mathrm{id} \otimes \overline{p^{\phi} p}),$$

and so $p^{\phi} p = 0$. Therefore, $\mathrm{Tr}_X(p^{\phi} p) = 0$, and so p, and Y', are zero. Thus $X = Y \oplus Y^{\perp}$ and $\phi = \phi_Y \oplus \phi_Y^{\perp}$. Let $\phi_1 \in \pi(Y)$ and $\phi_2 \in \pi(Y^{\perp})$. Then $\phi_1 \oplus \phi_2$ is compatible with ϕ, and this implies that ϕ_1 is compatible with ϕ_Y. □

REMARK 4.12. Suppose C is defined by a triple (G, σ, c), as in (4.1), so that $\mathsf{C}' = \mathsf{Rep}_{\mathbb{C}}(G)$. A sesquilinear form $\phi\colon X \otimes \bar{X} \to \mathbb{1}$ defines a sesquilinear form ϕ' on X in the usual, vector space, sense by the formula

$$\phi'(x, y) = \phi(x \otimes \bar{y}), \quad x, y \in X. \tag{4.12.1}$$

The conditions that ϕ be a G-morphism and have parity $\varepsilon \in Z(\mathbb{R})$ become respectively

$$\begin{array}{rcll} \phi'(x,y) & = & \phi'(gx, \sigma^{-1}(g)y), & g \in G(\mathbb{C}), \\ \overline{\phi'(y,x)} & = & \phi'(x, \varepsilon c^{-1} y). & \end{array} \tag{4.12.2}$$

When G acts trivially on X, then the last equation becomes

$$\overline{\phi'(y,x)} = \phi'(x,y),$$

and so ϕ' is a Hermitian form in the usual sense on X. When X is one-dimensional, ϕ' is positive-definite (for otherwise $\phi \otimes \phi \notin \pi(X)$). Now (4.11b) shows that the same is true for any X on which G acts trivially, and (4.6) shows that $\{\phi' \mid \phi \in \pi(X)\}$ is the complete set of positive-definite Hermitian forms on X. In particular, $\mathsf{Vec}_{\mathbb{R}}$ has a unique polarization.

REMARK 4.13. A polarization π on C with parity ε defines, for each simple object X of C', an orientation of the real line of sesquilinear forms on X with parity ε (see 4.8), and π is obviously determined by this family of orientations. Choose a fibre functor ω for C', and choose for each simple object X_i a $\phi_i \in \pi(X_i)$. Then

$$\pi(X_i) = \{r\phi_i \mid r \in \mathbb{R}_{>0}\}.$$

If X is isotypic of type X_i, so that $\omega(X) = W \otimes \omega(X_i)$ where $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ acts trivially on W, then

$$\{\omega(\phi)' \mid \phi \in \pi(X)\} = \{\psi \otimes \omega(\phi_i)' \mid \psi \text{ Hermitian } \psi > 0\}.$$

If $X = \bigoplus X^{(i)}$, where the $X^{(i)}$ are the isotypic components of X, then

$$\pi(X) = \bigoplus \pi(X^{(i)}).$$

REMARK 4.14. Let $\varepsilon \in Z(\mathbb{R})$ and, for each $X_0 \in \mathrm{ob}(\mathsf{C})$, let $\pi(X_0)$ be a nonempty compatibility class of bilinear Weil forms on X_0 with parity ε (see 4.9). One says that π is a ***homogeneous polarization*** on C if $\phi_0 \oplus \psi_0 \in \pi(X \oplus Y)$ and $\phi_0 \otimes \psi_0 \in \pi(X \otimes Y)$ whenever $\phi_0 \in \pi(X)$ and $\psi_0 \in \pi(Y)$. As $\{X \mid (X,a) \in \mathrm{ob}(\mathsf{C})\}$ generates C', the relation between bilinear and sesquilinear forms noted in (4.9) establishes a one-to-one correspondence between polarizations in this bilinear sense and in the sesquilinear sense of (4.10).

In the situation of (4.12), a bilinear form ϕ_0 on X_0 defines a sesquilinear form ψ' on $X = \mathbb{C} \otimes X_0$ (in the usual vector space sense) by the formula:

$$\psi'(z_1 v_1, z_2 v_2) = z_1 \bar{z}_2 \phi_0(v_1, v_2), \quad v_1, v_2 \in X_0, \quad z_1, z_2 \in \mathbb{C}.$$

Description of the polarizations

Let C be defined by a triple (G,σ,c) satisfying (4.2.1), and let K be a maximal compact subgroup of $G(\mathbb{C})$. As all maximal compact subgroups of $G(\mathbb{C})$ are conjugate (Hochschild 1965, XV, 3.1), there exists an $m \in G(\mathbb{C})$ such that $\sigma^{-1}(K) = mKm^{-1}$. Therefore, after replacing σ with $\sigma \circ \mathrm{ad}(m)$, we can assume that $\sigma(K) = K$. Subject to this constraint, (σ, c) is determined up to modification by an element m in the normalizer of K.

Assume that C is polarizable. Then (4.11a) and (2.28) show that G° is reductive, and it follows that K is a compact real form of G, i.e., that K has the structure of a compact real algebraic group G in the sense of (2.33) and $K_{\mathbb{C}} = G$ (see Springer 1979, 5.6). Let σ_K be the semilinear automorphism of G such that, for $g \in G(\mathbb{C})$, $\sigma_K(g)$ is the conjugate of g relative to the real structure on G defined by K; note that σ_K determines K. The normalizer of K is $K \cdot Z(\mathbb{C})$, and so $c \in K \cdot Z(\mathbb{C})$.

Fix a polarization π on C with parity ε. Let X be an irreducible representation of G, and let ψ be a positive-definite K-invariant Hermitian form on X. For any $\phi \in \pi(X)$, the associated form $\phi'(x,y) \stackrel{\text{def}}{=} \phi(x \otimes \bar{y})$ can be expressed

$$\phi'(x,y) = \psi(x, \beta y)$$

for some $\beta \in \mathrm{Aut}(X)$. The equations (4.12.2) can be re-written as

$$\begin{array}{rcll} \beta \cdot g_X & = & \sigma(g)_X \cdot \beta & \quad g \in K(\mathbb{R}) \\ \beta^* & = & \beta \cdot \varepsilon_X \cdot c_X^{-1} & \end{array} \tag{4.14.1}$$

where β^* is the adjoint of β relative to ψ:

$$\psi(\beta x, y) = \psi(x, \beta^* y).$$

As $K(\mathbb{R})$ is Zariski dense in $K(\mathbb{C})$, X is also irreducible as a representation of $K(\mathbb{R})$, and so the set $c(X,\pi)$ of such β is parametrized by $\mathbb{R}_{>0}$. An arbitrary finite-dimensional representation X of G can be written

$$X = \bigoplus\nolimits_i W_i \otimes X_i$$

where the sum is over the non-isomorphic irreducible representations X_i of G and G acts trivially on each W_i. Let ψ_i' and ψ_i respectively be K-invariant positive-definite Hermitian forms on W_i and X_i, and let $\psi = \oplus\psi_i' \otimes \psi_i$. Then for any $\phi \in \pi(X)$,

$$\phi'(x,y) = \psi(x,\beta y), \quad \beta \in \mathrm{Aut}(X),$$

where $\beta = \oplus\beta_i' \otimes \beta_i$ with $\beta_i \in c(X_i,\pi)$ and β_i' is positive-definite and Hermitian relative to ψ_i'. We again let $c(X,\pi)$ denote the set of β as ϕ runs through $\pi(X)$. The condition (4.10b) that

$$\pi(X_1) \otimes \pi(X_2) \subset \pi(X_1 \otimes X_2)$$

becomes

$$c(X_1,\pi) \otimes c(X_2,\pi) \subset c(X_1 \otimes X_2,\pi).$$

LEMMA 4.15. *There exists a $b \in K$ with the following properties:*

(a) $b_X \in c(X,\pi)$ *for all irreducible* X*;*

(b) $\sigma = \sigma_K \circ \mathrm{ad}(b)$*, where* σ_K *denotes complex conjugation on* G *relative to* K*;*

(c) $\varepsilon^{-1}c = \sigma b \cdot b = b^2$.

PROOF. Let $a = \varepsilon c^{-1} \in G(\mathbb{C})$. When X is irreducible, the first equality in (4.14.1) applied twice shows that

$$\beta^2 \cdot g \cdot x = \sigma^2(g) \cdot \beta^2 \cdot x = c \cdot g \cdot c^{-1} \cdot \beta^2 \cdot x$$

for $\beta \in c(X,\pi)$, $g \in K$, and $x \in X$; therefore

$$(c^{-1}\beta^2)gx = g(c^{-1}\beta^2)x,$$

and so $c^{-1}\beta^2$ acts as a scalar on X. Hence $a\beta^2 = \varepsilon c^{-1}\beta^2$ also acts as a scalar. Moreover, $\beta^2 a = \beta\beta^*$ (by the second equation in 4.14.1) and so

$$\mathrm{Tr}_X(a\beta^2) = \mathrm{Tr}_X(\beta^2 a) > 0;$$

we conclude that $a_X\beta^2 \in \mathbb{R}_{>0}$. It follows that there is a unique $\beta \in c(X,\pi)$ such that $a_X = \beta^{-2}$, $\beta g_X = \sigma(g)_X\beta$ ($g \in K$), and $\beta^* = \beta^{-1}$ (i.e., β is unitary).

For an arbitrary X, we write $X = \bigoplus W_i \otimes X_i$ as before, and set $\beta = \bigoplus \mathrm{id} \otimes \beta_i$, where β_i is the canonical element of $c(X_i,\pi)$ just defined. We still have $a_X = \beta^{-2}$, $\beta g_X = \sigma(g)_X\beta$ ($g \in K$), and $\beta \in c(X,\pi)$. Moreover, these conditions characterize β: if $\beta' \in c(X,\pi)$ has the same properties, then $\beta' = \sum \gamma_i \otimes \beta_i$ (this expresses that $\beta' g_X = \sigma(g)_X\beta'$, $g \in K$) with $\gamma_i^2 = 1$ (as $\beta'^2 = a_X^{-1}$) and γ_i positive-definite and Hermitian. Hence $\gamma_i = 1$.

The conditions are compatible with tensor products, and so the canonical β are compatible with tensor products: they therefore define an element $b \in G(\mathbb{C})$. As b is unitary on all irreducible representations, it lies in K. The equations $\beta^2 = a_X^{-1}$ show that $b^2 = a^{-1} = \varepsilon^{-1}c$. Finally, $\beta g_X = \sigma(g)_X\beta$ implies that $\sigma(g) = \mathrm{ad}(b(g))$ for all $g \in K$; therefore $\sigma \circ \mathrm{ad}(b)^{-1}$ fixes K, and as it has order 2, it must equal σ_K. □

THEOREM 4.16. *Let* C *be a Tannakian category over* $\mathbb{R}$*, and let* $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ *where* ω *is a fibre functor on* C *with values in* $\mathbb{C}$*; let* π *be a polarization on* C *with parity* ε*. For any compact real form* K *of* G*, the pair* (σ_K,ε) *satisfies (4.2.1), and the equivalence* $\mathsf{C}' \to \mathsf{Rep}_{\mathbb{C}}(G)$ *defined by* ω *carries the descent datum on* C' *defined by* C *into that on* $\mathsf{Rep}_{\mathbb{C}}(G)$ *defined by* (σ_K,ε)*:*

$$\omega(\bar{X}) = \overline{\omega(X)}, \quad \omega(\mu_X) = \mu_{\omega(X)}.$$

For any simple X in C',

$$\{\omega(\phi)' \mid \phi \in \pi(X)\}$$

is the set of K-invariant positive-definite Hermitian forms on $\omega(X)$.

PROOF. Let (C,ω) correspond to the triple (G,σ_1,c_1) (see 4.3a), and let $b \in K$ be the element constructed in the lemma. Then $\sigma_1 = \sigma_K \circ \mathrm{ad}(b)$ and $c = \varepsilon \cdot \sigma b \cdot b = \sigma b \cdot \varepsilon \cdot b$. Therefore, (σ_K,ε) has the same property as (σ_1,c_1) (see 4.3b), which proves the first assertion. The second assertion follows from the fact that $b \in c(\omega(X),\pi)$ for any simple X. □

Classification of polarized Tannakian categories

THEOREM 4.17. *(a) An algebraic Tannakian category C over $\mathbb{R}$ is polarizable if and only if its band is defined by a compact real algebraic group K.*

(b) For any compact real algebraic group K and $\varepsilon \in Z(\mathbb{R})$, where Z is the centre of K, there exists a Tannakian category C over $\mathbb{R}$ whose gerbe is banded by the band $B(K)$ of K and a polarization π on C with parity ε.

(c) Let (C_1,π_1) and (C_2,π_2) be polarized algebraic Tannakian categories over $\mathbb{R}$ with isomorphic bands B_1 and B_2. If there exists an isomorphism $B_2 \to B_1$ sending $\varepsilon(\pi_1)$ to $\varepsilon(\pi_2)$ (as elements of $Z(B_i)(\mathbb{R})$), then there is a tensor equivalence $\mathsf{C}_1 \to \mathsf{C}_2$ respecting the polarizations and the actions of B_1 and B_2 (i.e., such that $\mathrm{FIB}(\mathsf{C}_2) \to \mathrm{FIB}(\mathsf{C}_1)$ is a banded by $B_2 \to B_1$), and this equivalence is unique up to isomorphism.

PROOF. We have already seen that if C is polarizable, then C' is semisimple, and so, for any fibre functor ω with values in $\mathbb{C}$, (the identity component of) $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is reductive, and has a compact real form K. This proves half of (a). Part (b) is proved in the first lemma below, and the sufficiency in (a) follows from (b) and the second lemma below. Part (c) follows from (4.16). □

LEMMA 4.18. *Let K and ε be as in (b) of the theorem, and let $G = K_{\mathbb{C}}$. Then K corresponds to a Cartan involution σ' of G, and we let $\sigma(g) = \sigma'(\bar{g})$. The pair (σ,ε) then satisfies (4.2.1) and the Tannakian category C defined by (G,σ,ε) has a polarization with parity ε.*

PROOF. Since $\sigma^2 = \mathrm{id}$ and σ fixes all elements of K, (4.2.1) is obvious. There exists a polarization π on C such that, for all simple X, $\{\phi' \mid \phi \in \pi(X)\}$ is the set of positive-definite K-invariant Hermitian forms on X. (In the notation of (4.15), $b = 1$.) This polarization has parity ε. □

Let C correspond to $(\mathsf{C}', X \mapsto \bar{X}, \mu)$. For any $z \in Z(\mathbb{R})$, where Z is the centre of the band B of C, $(\mathsf{C}', X \mapsto \bar{X}, \mu \circ z)$ defines a new Tannakian category ${}^{z}\mathsf{C}$ over $\mathbb{R}$.

LEMMA 4.19. *Every Tannakian category over $\mathbb{R}$ whose gerbe is banded by B is of the form ${}^{z}\mathsf{C}$ for some $z \in Z(\mathbb{R})$. There is a tensor equivalence ${}^{z}\mathsf{C} \to {}^{z'}\mathsf{C}$ respecting the action of B if and only if $z'z^{-1} \in Z(\mathbb{R})^2$.*

PROOF. Let ω be a fibre functor on C, and let (C,ω) correspond to (G,σ,c). We can assume that the second category C_1 corresponds to (G,σ_1,c_1). Let γ and γ_1 be the functors $V \mapsto \bar{V}$ defined by (σ,c) and (σ_1,c_1) respectively. Then $\gamma_1^{-1} \circ \gamma$ defines a tensor automorphism of ω, and so corresponds to an element $m \in G(\mathbb{C})$. We have $\sigma = \sigma_1 \circ \mathrm{ad}(m)$, and so we can modify (σ_1, c_1) in order to get $\sigma_1 = \sigma$. Let μ and μ_1 be the functorial isomorphisms $V \to \bar{\bar{V}}$ defined by (σ, c) and (σ, c_1) respectively. Then $\mu_1^{-1} \circ \mu$ defines a tensor automorphism of id_{C}, and so $\mu_1^{-1} \circ \mu = z^{-1}$, $z \in Z(\mathbb{R})$. We have $\mu_1 = \mu \circ z$.

The second part of the lemma is obvious. □

REMARK 4.20. Some of the above results can be given a more cohomological interpretation. Let B be the band defined by a compact real algebraic group K, and let Z be the centre of B; let C be a Tannakian category whose band is B.

(a) As Z is a subgroup of a compact real algebraic group, it is also compact (see 2.33). It is easy to compute its cohomology. One finds that

$$\begin{aligned} H^1(\mathbb{R}, Z) &= {}_2Z(\mathbb{R}) \stackrel{\text{def}}{=} \mathrm{Ker}(2\colon Z(\mathbb{R}) \to Z(\mathbb{R})) \\ H^2(\mathbb{R}, Z) &= Z(\mathbb{R})/Z(\mathbb{R})^2. \end{aligned}$$

(b) The general theory (Saavedra Rivano 1972, III 2.3.4.2, p. 184) shows that there is an isomorphism $H^1(\mathbb{R}, Z) \to \mathrm{Aut}_B(\mathsf{C})$, which can be described explicitly as the map sending $z \in {}_2Z(\mathbb{R})$ to the automorphism w_z

$$\begin{cases} (X, a_X) & \mapsto & (X, a_X z_X) \\ f & \mapsto & f \end{cases}$$

(c) The Tannakian categories banded by B are classified, up to B-equivalence, by $H^2(\mathbb{R}, B)$, and $H^2(\mathbb{R}, B)$, if nonempty, is an $H^2(\mathbb{R}, Z)$-torsor. The action of $H^2(\mathbb{R}, Z) = Z(\mathbb{R})/Z(\mathbb{R})^2$ on set of B-equivalence classes is made explicit in (4.19).

(d) Let $\mathrm{Pol}(\mathsf{C})$ denote the set of polarizations on C. For $\pi \in \mathrm{Pol}(\mathsf{C})$ and $z \in Z(\mathbb{R})$ we define $z\pi$ to be the polarization such that

$$\phi(x, y) \in z\pi(X) \iff \phi(x, zy) \in \pi(X);$$

it has parity $\varepsilon(z\pi) = z^2\varepsilon(\pi)$. The pairing

$$(z, \pi) \mapsto z\pi\colon Z(\mathbb{R}) \times \mathrm{Pol}(\mathsf{C}) \to \mathrm{Pol}(\mathsf{C})$$

makes $\mathrm{Pol}(\mathsf{C})$ into a $Z(\mathbb{R})$-torsor.

(e) Let $\pi \in \mathrm{Pol}(\mathsf{C})$ and let $\varepsilon = \varepsilon(\pi)$; then C has a polarization with parity $\varepsilon' \in Z(\mathbb{R})$ if and only if $\varepsilon' = \varepsilon z^2$ for some $z \in Z(\mathbb{R})$.

REMARK 4.21. In Saavedra Rivano 1972, V, 1, there is a table of Tannakian categories whose bands are simple, from which it is possible to read off those that are polarizable (loc. cit. V, 2.8.3).

Neutral polarized categories

The above results can be made more explicit when C has a fibre functor with values in $\mathbb{R}$.

Let G be an algebraic group over $\mathbb{R}$, and let $C \in G(\mathbb{R})$. A G-invariant sesquilinear form $\psi\colon V \times V \to \mathbb{C}$ on $V \in \mathrm{ob}(\mathsf{Rep}_{\mathbb{C}}(G))$ is said to be a ***C-polarization*** if

$$\psi^C(x, y) \stackrel{\text{def}}{=} \psi(x, Cy)$$

is a positive-definite Hermitian form on V. When every object of $\mathsf{Rep}_{\mathbb{C}}(G)$ has a C-polarization, C is called a ***Hodge element***.

PROPOSITION 4.22. *Assume that $G(\mathbb{R})$ contains a Hodge element C.*

(a) *There is a polarization π_C on $\mathsf{Rep}_{\mathbb{R}}(G)$ for which the positive forms are exactly the C-polarizations; the parity of π_C is C^2.*

(b) *For any $g \in G(\mathbb{R})$ and $z \in Z(\mathbb{R})$, where Z is the centre of G, $C' = zgCg^{-1}$ is also a Hodge element and $\pi_{C'} = z\pi_C$.*

(c) *Every polarization on $\mathsf{Rep}_{\mathbb{R}}(G)$ is of the form $\pi_{C'}$ for some Hodge element C'.*

PROOF. Let ψ be a C-polarization on $V \in \mathrm{ob}(\mathsf{Rep}_{\mathbb{C}}(C))$; then

$$\psi(x,y) = \psi(Cx,Cy)$$

because ψ is G-invariant, and

$$\psi(Cx,Cy) = \psi^C(Cx,y) = \overline{\psi^C(y,Cx)} = \overline{\psi(y,C^2x)}.$$

This shows that ψ has parity C^2. For any V and $g \in G(\mathbb{R})$,

$$\begin{aligned} \overline{\psi(y,C^2x)} &= \psi(x,y) \\ &= \psi(gx,gy) \\ &= \overline{\psi(gy,C^2gx)} \\ &= \overline{\psi(y,g^{-1}C^2gx)}. \end{aligned}$$

This shows that $C^2 \in Z(\mathbb{R})$. For any $u \in \mathrm{End}(V)$, $u^{\psi} = u^{\psi^C}$, and so $\mathrm{Tr}(uu^{\psi}) > 0$ if $u \neq 0$. This shows that ψ is a Weil form with parity C^2. Statement (a) is now easy to check. Statement (b) is straightforward to prove, and statement (c) follows from it and (4.19). □

PROPOSITION 4.23. *The following conditions on G are equivalent:*

(a) *there exists a Hodge element in $G(\mathbb{R})$;*

(b) *the category $\mathsf{Rep}_{\mathbb{R}}(G)$ is polarizable;*

(c) *G is an inner form of a compact real algebraic group K.*

PROOF. (a)⇒(b). This is proved in (4.22).

(b)⇒(c). To say that G is an inner form of K is the same as to say that G and K define the same band; this implication therefore follows from (4.17a).

(c)⇒(a). Let Z be the centre of K (and therefore also of G) and let $K^{\mathrm{ad}} = K/Z$. That G is an ***inner*** form of K means that its cohomology class is in the image of

$$H^1(\mathbb{R}, K^{\mathrm{ad}}) \to H^1(\mathbb{R}, \underline{\mathrm{Aut}}(K)).$$

More explicitly, this means that there is an isomorphism $\gamma\colon K_{\mathbb{C}} \to G_{\mathbb{C}}$ such that

$$\bar{\gamma} = \gamma \circ c, \quad \text{some } c \in K^{\mathrm{ad}}(\mathbb{C}).$$

According to Serre 1964, III, Thm 6, $H^1(\mathbb{R}, K^{\mathrm{ad}}) \simeq H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), K^{\mathrm{ad}}(\mathbb{R}))$, which is equal to the set of conjugacy classes in $K^{\mathrm{ad}}(\mathbb{R})$ consisting of elements of order 2. Thus, we can assume that $c \in K(\mathbb{R})$ and $c^2 = 1$. Consider the cohomology sequence

$$K(\mathbb{R}) \quad\to\quad K^{\mathrm{ad}}(\mathbb{R}) \quad\to\quad H^1(\mathbb{R}, Z) \quad\to\quad H^1(\mathbb{R}, K).$$

The last map is injective, and so $K(\mathbb{R}) \to K^{\mathrm{ad}}(\mathbb{R})$ is surjective. Thus $c = \mathrm{ad}(C')$ for some $C' \in K(\mathbb{R})$ whose square is in $Z(\mathbb{R})$. Let $C = \gamma(C')$; then $\bar{C} = \bar{\gamma}(C') = \gamma(C') = C$ and $\bar{\gamma}^{-1} \circ \mathrm{ad}(C) = \gamma^{-1}$. This shows that $C \in G(\mathbb{R})$ and that K is the form of G defined by C; the next lemma completes the proof. □

LEMMA 4.24. *An element $C \in G(\mathbb{R})$ such that $C^2 \in Z(\mathbb{R})$ is a Hodge element if and only if the real form K of G defined by C is a compact real group.*

PROOF. Identify $K_{\mathbb{C}}$ with $G_{\mathbb{C}}$ and let $\bar{g}$ and g^* be the complex conjugates of $g \in G(\mathbb{C})$ relative to the real forms K and G. Then

$$g^* = \mathrm{ad}(C^{-1})(\bar{g}) = C^{-1}\bar{g}C.$$

Let ψ be a sesquilinear form on $V \in \mathrm{ob}(\mathsf{Rep}_{\mathbb{C}}(G))$. Then ψ is G-invariant if and only if

$$\psi(gx, \bar{g}y) = \psi(x, y), \quad g \in G(\mathbb{C}).$$

On the other hand, ψ^C is K-invariant if and only if

$$\psi^C(gx, g^*y) = \psi^C(x, y), \quad g \in G(\mathbb{C}).$$

These conditions are equivalent. Therefore, V has a C-polarization if and only if V has a K-invariant positive-definite Hermitian form. Thus C is a Hodge element if and only if, for every complex representation V of K, the image of K in $\mathrm{Aut}(V)$ is contained in the unitary group of a positive-definite Hermitian form; this last condition is implied by K being compact and implies that K is contained in a compact real group, and so is compact (see 2.33). □

REMARK 4.25. (a) The centralizer of a Hodge element C of G is a maximal compact subgroup of G, and is the only maximal compact subgroup of G containing C; in particular, if G is compact, then C is a Hodge element if and only if it is in the centre of G (Saavedra Rivano 1972, V, 2.7.3.5).

(b) If C and C' are Hodge elements of G, then there exists a $g \in G(\mathbb{R})$ and a unique $z \in Z(\mathbb{R})$ such that $C' = zgCg^{-1}$ (Saavedra Rivano 1972, V, 2.7.4). As $\pi_{C'} = z\pi_C$, this shows that $\pi_{C'} = \pi_C$ if and only if C and C' are conjugate in $G(\mathbb{R})$.

REMARK 4.26. It would perhaps have been more natural to express the above results in terms of bilinear forms (see 4.4, 4.9, 4.14): a G-invariant bilinear form $\phi\colon V_0 \times V_0 \to \mathbb{R}$ on $V_0 \in \mathrm{ob}(\mathsf{Rep}_{\mathbb{R}}(G))$ is a ***C-polarization*** if $\phi^C(x,y) \overset{\text{def}}{=} \phi(x, Cy)$ is a positive-definite symmetric form on V_0; C is a ***Hodge element*** if every object of $\mathsf{Rep}_{\mathbb{R}}(G)$ has a C-polarization; the positive forms for the (bilinear) polarization defined by C are precisely the C-polarizations.

Symmetric polarizations

A polarization is said to be ***symmetric*** if its parity is 1.

Let K be a compact real algebraic group. As 1 is a Hodge element (4.24), $\mathsf{Rep}_{\mathbb{R}}(K)$ has a symmetric polarization π for which $\pi(X_0)$, $X_0 \in \mathrm{ob}(\mathsf{Rep}_{\mathbb{R}}(K))$, consists of the K-invariant positive-definite symmetric bilinear forms on X_0 (and $\pi(X)$, $X \in \mathrm{ob}(\mathsf{Rep}_{\mathbb{C}}(K))$, consists of the K-invariant positive-definite Hermitian forms on X).

THEOREM 4.27. *Let C be an algebraic Tannakian category over $\mathbb{R}$, and let π be a symmetric polarization on C. Then C has a unique (up to isomorphism) fibre functor ω with values in $\mathbb{R}$ transforming positive bilinear forms for π into positive-definite symmetric bilinear forms; ω defines a tensor equivalence $\mathsf{C} \to \mathsf{Rep}_{\mathbb{R}}(K)$, where $K = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is a compact real algebraic group.*

PROOF. Let ω_1 be a fibre functor with values in $\mathbb{C}$, and let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega_1)$. Because C is polarizable, G has a compact real form K. According to (4.16), $\omega_1' \colon \mathsf{C}' \to \mathsf{Rep}_{\mathbb{C}}(G)$ carries the descent datum on C' defined by C into that on $\mathsf{Rep}_{\mathbb{C}}(G)$ defined by $(\sigma_K, 1)$. It therefore defines a tensor equivalence $\omega\colon \mathsf{C} \to \mathsf{Rep}_{\mathbb{R}}(K)$ transforming π into the polarization on $\mathsf{Rep}_{\mathbb{R}}(K)$ defined by the Hodge element 1. The rest of the proof is now obvious. Briefly, let ω_1 and ω_2 be two such fibre functors. □

REMARK 4.28. Let π be a polarization on C. It follows from (4.20d) that C has a symmetric polarization if and only if $\varepsilon(\pi) \in Z(\mathbb{R})^2$.

Polarizations with parity ε of order 2

For $u = \pm 1$, define a ***real u-space*** to be a complex vector space V together with a semilinear automorphism σ such that $\sigma^2 = u$. A bilinear form ϕ on a real u-space is ***u-symmetric*** if $\phi(x,y) = u\phi(y,x)$ — thus a 1-symmetric form is a symmetric form, and a -1-symmetric form is a skew-symmetric form. A u-symmetric form is ***positive-definite*** if $\phi(x,\sigma x) > 0$ for all $x \neq 0$.

Let V_0 be the category whose objects are pairs (V,σ) where $V = V^0 \oplus V^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{C}$ and $\sigma\colon V \to V$ is a semilinear automorphism such that $\sigma^2 x = (-1)^{\deg(x)} x$. With the obvious tensor structure, V_0 becomes a Tannakian category over $\mathbb{R}$ with $\mathbb{C}$-valued fibre functor $(V,\sigma) \mapsto V$. There is a polarization $\pi = \pi_{\mathrm{can}}$ on V_0 such that, if V is homogeneous of degree m, then $\pi(V,\sigma)$ consists of the $(-1)^m$-symmetric positive-definite forms on V.

THEOREM 4.29. *Let C be an algebraic Tannakian category over $\mathbb{R}$, and let π be a polarization on C with parity ε where $\varepsilon^2 = 1$, $\varepsilon \neq 1$. There exists a unique (up to isomorphism) exact faithful functor $\omega\colon \mathsf{C} \to \mathsf{V}_0$ such that*

(a) *ω carries the grading on C defined by ε into the grading on V_0, i.e., $\omega(\varepsilon)$ acts as $(-1)^m$ on $\omega(V)^m$;*
(b) *ω carries π into π_{can}, i.e., $\phi \in \pi(X)$ if and only if $\omega(\phi) \in \pi_{\mathrm{can}}(\omega(X))$.*

PROOF. Note that V_0 is defined by the triple $(\mu_2, \sigma_0, \varepsilon_0)$ where σ_0 is the unique semilinear automorphism of μ_2 and ε_0 is the unique element of $\mu_2(\mathbb{R})$ of order 2. We can assume (by 4.3) that C corresponds to a triple (G,σ,ε). Let G_0 be the subgroup of G generated by ε; then $(G_0, \sigma|G_0, \varepsilon) \approx (\mu_2, \sigma_0, \varepsilon_0)$, and so the inclusion $(G_0, \sigma|G_0, \varepsilon) \hookrightarrow (G,\sigma,\varepsilon)$ induces a functor $\mathsf{C} \to \mathsf{V}_0$ having the required properties.

Let ω and ω' be two functors $\mathsf{C} \to \mathsf{V}_0$ satisfying (a) and (b). It is clear from (3.2a) that there exists an isomorphism $\lambda\colon \omega \to \omega'$ from ω to ω' regarded as $\mathbb{C}$-valued fibre functors. As $\lambda_X\colon \omega(X) \to \omega'(X)$ commutes with action of ε, it preserves the gradings; as λ commutes with $\omega(\phi)$, any $\phi \in \pi(X)$, it also commutes with σ; it follows that λ is an isomorphism from ω to ω' as functors to V_0. □

5. Graded Tannakian categories

Throughout this section, k will be a field of characteristic zero.

Gradings

Let M be a set. An ***M*-grading**[21] on an object X of an additive category is a decomposition $X = \bigoplus_{m \in M} X^m$; an ***M*-grading** on an additive functor $u: \mathsf{C} \to \mathsf{C}'$ is an M-grading on each $u(X)$, $X \in \mathrm{ob}(\mathsf{C})$, that depends functorially on X.

Suppose now that M is an abelian group, and let D be the algebraic group of multiplicative type over k whose character group is M (with the trivial Galois action; see (2.32)). The cases of most interest to us are $M = \mathbb{Z}$, $D = \mathbb{G}_m$ and $M = \mathbb{Z}/2\mathbb{Z}$, $D = \mu_2$ $(= \mathbb{Z}/2\mathbb{Z})$.

DEFINITION 5.1. An ***M*-grading** on a Tannakian category C over k can be variously described as follows:

(a) an M-grading, $X = \bigoplus X^m$, on each object X of C that depends functorially on X and is compatible with tensor products in the sense that $(X \otimes Y)^m = \bigoplus_{r+s=m} X^r \otimes Y^s$;
(b) an M-grading on the identity functor id_{C} of C that is compatible with tensor products;
(c) a homomorphism $D \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}})$;
(d) a central homomorphism $D \to G$, $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, for one (or every) fibre functor ω.

Definitions (a) and (b) are obviously equivalent. By a central homomorphism in (d), we mean a homomorphism from D into the centre of G defined over k. Although G need not be defined over k, its centre is, and equals $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}})$, from which follows the equivalence of (c) and (d). Finally, a homomorphism $w: D \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}})$ corresponds to a family of gradings $X = \bigoplus X^m$ for which $w(d)$ acts on $X^m \subset X$ as $m(d) \in k$.

Tate triples

A ***Tate triple*** T over k is a triple (C, w, T) comprising a Tannakian category C over k, a $\mathbb{Z}$-grading $w: \mathbb{G}_m \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}})$ on C (called the ***weight grading***), and an invertible object T (called the ***Tate object***) of weight -2. For any $X \in \mathrm{ob}(\mathsf{C})$ and $n \in \mathbb{Z}$, we write $X(n) = X \otimes T^{\otimes n}$. A ***fibre functor*** on T with values in R is a fibre functor $\omega: \mathsf{C} \to \mathsf{Mod}_R$ together with an isomorphism $\omega(T) \to \omega(T^{\otimes 2})$, i.e., the structure of an identity object on $\omega(T)$. If T has a fibre functor with values in k, then T is said to be ***neutral***. A ***morphism*** of Tate triples $(\mathsf{C}_1, w_1, T_1) \to (\mathsf{C}_2, w_2, T_2)$ is an exact tensor functor $\eta: \mathsf{C}_1 \to \mathsf{C}_2$ preserving the gradings together with an isomorphism $\eta(T_1) \to T_2$.

EXAMPLE 5.2. (a) The triple $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ in which

- $\mathsf{Hod}_{\mathbb{R}}$ is the category of real Hodge structures (see 2.31),
- w is the weight grading on $\mathsf{Hod}_{\mathbb{R}}$, and
- $\mathbb{R}(1)$ is the unique real Hodge structure with weight -2 and underlying vector space $2\pi i\mathbb{R}$,

is a neutral Tate triple over $\mathbb{R}$.

(b) The category of $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}$, together with the object $T = \mathbb{Q}_B(1)$, forms a neutral Tate triple T_B over $\mathbb{Q}$. The category of $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}_l$, together with the object $T = \mathbb{Q}_l(1)$, forms a neutral Tate triple T_l over $\mathbb{Q}_l$. The category of $\mathbb{Z}$-graded vector spaces over k, together with the object $T = k_{\mathrm{dR}}(1)$, forms a neutral Tate triple T_{dR} over k. (See Deligne 1982, § 1 for the terminology.)

[21] *Gradation* and *graduation* are also used. The Wikipedia prefers the former, and Bourbaki the latter.

EXAMPLE 5.3. Let V be the category of $\mathbb{Z}$-graded $\mathbb{C}$-vector spaces V with a semilinear automorphism a such that $a^2 v = (-1)^n v$ if $v \in V^n$. With the obvious tensor structure, V becomes a Tannakian category over $\mathbb{R}$, and $\omega\colon (V,a) \mapsto V$ is a fibre functor with values in $\mathbb{C}$. Clearly $\mathbb{G}_m = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and V corresponds (as in 4.3a) to the pair $(g \mapsto \bar{g}, -1)$. Let $w\colon \mathbb{G}_m \to \mathbb{G}_m$ be the identity map, and let $T = (V,a)$ where V is $\mathbb{C}$ regarded as a homogeneous vector space of weight -2 and a is $z \mapsto \bar{z}$. Then (V, w, T) is a non-neutral Tate triple over $\mathbb{R}$.

EXAMPLE 5.4. Let G be an algebraic group scheme over k and let $w\colon \mathbb{G}_m \to G$ be a central homomorphism and $t\colon G \to \mathbb{G}_m$ a homomorphism such that $t \circ w = -2$ ($\overset{\text{def}}{=} s \mapsto s^{-2}$). Let T be the representation of G on k such that g acts as multiplication by $t(g)$. Then $(\mathsf{Rep}_k(G), w, T)$ is a neutral Tate triple over k.

The next proposition is obvious.

PROPOSITION 5.5. *Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over k, and let ω be a fibre functor on T with values in k. Let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, so that w is a homomorphism $\mathbb{G}_m \to Z(G) \subset G$. There is a homomorphism $t\colon G \to \mathbb{G}_m$ such that g acts on T as multiplication by $t(g)$, and $t \circ w = -2$. The equivalence $\mathsf{C} \to \mathsf{Rep}_k(G)$ carries w and T into the weight grading and Tate object defined by t and w.*

More generally, a Tate triple T defines a band B, a homomorphism $w\colon \mathbb{G}_m \to Z$ into the centre Z of B, and a homomorphism $t\colon G \to \mathbb{G}_m$ such that $t \circ w = -2$. We say that T is ***banded*** by (B, w, t).

Let G, w, and t be as in (5.4). Let $G_0 = \mathrm{Ker}(t\colon G \to \mathbb{G}_m)$, and let $\varepsilon\colon \mu_2 \to G_0$ be the restriction of w to μ_2. We often identify ε with $\varepsilon(-1) = w(-1) \in Z(G_0)(k)$. Note that ε defines a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathsf{C}_0 = \mathsf{Rep}_k(G_0)$.

5.6. The inclusion $G_0 \hookrightarrow G$ defines a tensor functor $Q\colon \mathsf{C} \to \mathsf{C}_0$ with the following properties:

(a) if X is homogeneous of weight n, then $Q(X)$ is homogeneous of weight n (mod 2);
(b) $Q(T) = \mathbb{1}$;
(c) if X and Y are homogeneous of the same weight, then
$$\mathrm{Hom}(X,Y) \xrightarrow{\simeq} \mathrm{Hom}(Q(X), Q(Y));$$
(d) if X and Y are homogeneous with weights m and n respectively and $Q(X) \approx Q(Y)$, then $m - n$ is an even integer $2k$ and $X(k) \approx Y$;
(e) Q is essentially surjective.

The first four of these statements are obvious. For the last, note that
$$G = (G \times \mathbb{G}_m)/\mu_2,$$
and so we only have to show that every representation of μ_2 extends to a representation of $\mathbb{G}_m$, but this is obvious.

REMARK 5.7. (a) The identity component of G_0 is reductive if and only if the identity component of G is reductive; if G_0 is connected, so also is G, but the converse statement is false (e.g., $G_0 = \mu_2$, $G = \mathbb{G}_m$).

(b) It is possible to reconstruct (C, w, T) from $(\mathsf{C}_0, \varepsilon)$ — the following diagram makes it clear how to reconstruct (G, w, t) from (G_0, ε):

$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, "\varepsilon"] & \mathbb{G}_m \arrow[r, "-2"] \arrow[d, "w"] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & G_0 \arrow[r] & G \arrow[r, "t"] & \mathbb{G}_m \arrow[r] & 1. \end{tikzcd}$$

PROPOSITION 5.8. *Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over k with C algebraic. There exists a Tannakian category C_0 over k, an element ε in $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}_0})$ with $\varepsilon^2 = 1$, and a functor $Q\colon \mathsf{C} \to \mathsf{C}_0$ having the properties (5.6).*[22]

PROOF. For any fibre functor ω on C with values in a k-algebra R, $\underline{\mathrm{Isom}}(R, \omega(T))$, regarded as a sheaf on Spec R, is a torsor for $\mathbb{G}_m$. This association gives rise to a morphism of gerbes

$$\mathrm{F{\small IB}}(\mathsf{C}) \xrightarrow{t} \mathrm{T{\small ORS}}(\mathbb{G}_m),$$

and we define G_0 to be the gerbe of liftings of the canonical section of $\mathrm{T{\small ORS}}(\mathbb{G}_m)$, i.e., G_0 is the gerbe of pairs (ω, ξ) where ω is a fibre functor on C and ξ is an isomorphism $t(\omega) \to \mathbb{G}_m$ (Giraud 1971, IV, 3.2.1). Let C_0 be the category $\mathsf{Rep}_k(\mathsf{G}_0)$ which (see 3.14) is Tannakian. If $Z = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}})$ and $Z_0 = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}_0})$, then the homomorphism

$$\alpha \mapsto \alpha_T \colon Z \to \underline{\mathrm{Aut}}(T) = \mathbb{G}_m,$$

determined by t has kernel Z_0, and the composite $t \circ w = -2$. We let $\varepsilon = w(-1) \in Z_0$.

There is an obvious (restriction) functor $Q\colon \mathsf{C} \to \mathsf{C}_0$. In showing that Q has the properties (5.6), we can make a finite field extension $k \to k'$. We can therefore assume that T is neutral, but this case is covered by (5.5) and (5.6). □

EXAMPLE 5.9. Let (V, w, T) be the Tate triple defined in (5.3); then $(\mathsf{V}_0, \varepsilon)$ is the pair defined in the paragraph preceding (4.29).

EXAMPLE 5.10. Let $\mathsf{T} = (\mathsf{C}, w, T)$ be a Tate triple over $\mathbb{R}$, and let ω be a fibre functor on T with values in $\mathbb{C}$. On combining (4.3) with (5.5) we find that (T, ω) corresponds to a quintuple (G, σ, c, w, t) in which

(a) G is an algebraic group scheme over $\mathbb{C}$;
(b) (σ, c) satisfies (4.2.1);
(c) $w\colon \mathbb{G}_m \to G$ is a central homomorphism; that the grading is defined over $\mathbb{R}$ means that w is defined over $\mathbb{R}$, i.e., $\sigma(w(g)) = w(\bar{g})$;
(d) $t\colon G \to \mathbb{G}_m$ is such that $t \circ w = -2$; that T is defined over $\mathbb{R}$ means that $t(\sigma(g)) = \overline{t(g)}$ and there exists an $a \in \mathbb{G}_m(\mathbb{C})$ such that $t(c) = \sigma(a) \cdot a$.

Let $G_0 = \mathrm{Ker}(t)$, and let $m \in G(\mathbb{C})$ be such that $t(m) = a^{-1}$. After replacing (σ, c) with $(\sigma \circ \mathrm{ad}(m), \sigma(m) \cdot c \cdot m)$ we find that the new c is in G_0. The pair $(\mathsf{C}_0, \omega|\mathsf{C}_0)$ corresponds to $(G_0, \sigma|G_0, c)$.

[22] The Tannakian category C_0 is the quotient of C by the subcategory generated by T (see Milne, J. S., Quotients of Tannakian categories. Theory Appl. Categ. 18 (2007), No. 21, 654–664).

REMARK 5.11. (a) The functor $\omega \mapsto \omega|\mathsf{C}_0$ defines an equivalence from the gerbe of fibre functors on the Tate triple T to the gerbe of fibre functors on C_0.

(b) As in the neutral case, T can be reconstructed from $(\mathsf{C}_0, \varepsilon)$. This can be proved by substituting bands for group schemes in the argument used in the neutral case (Saavedra Rivano 1972, V, 3.14.1), or by using descent theory to deduce it from the neutral case.

There is a stronger result: $\mathsf{T} \mapsto (\mathsf{C}_0, \varepsilon)$ defines an equivalence between the 2-category of Tate triples and that of $\mathbb{Z}/2\mathbb{Z}$-graded Tannakian categories (ibid. V, 3.1.4).

Graded polarizations

For the remainder of this section, $\mathsf{T} = (\mathsf{C}, w, T)$ will be a Tate triple over $\mathbb{R}$ with C algebraic. We use the notations of §4; in particular $\mathsf{C}' = \mathsf{C}_{(\mathbb{C})}$. Let U be an invertible object of C' that is defined over $\mathbb{R}$, i.e., U is endowed with an identification $U \simeq \bar{U}$; then in the definitions and results of §4 concerning sesquilinear forms and Weil forms, it is possible to replace $\mathbb{1}$ with U.

DEFINITION 5.12. For each object $X \in \mathrm{ob}(\mathsf{C}')$ that is homogeneous of degree n, let $\pi(X)$ be an equivalence class of Weil forms $X \otimes \bar{X} \to \mathbb{1}(-n)$ of parity $(-1)^n$; we say that π is a (graded) ***polarization*** on T if

(a) for all X, $\bar{\phi} \in \pi(X)$ whenever $\phi \in \pi(\bar{X})$;
(b) for all X and Y that are homogeneous of the same degree, $\phi \oplus \psi \in \pi(X \oplus Y)$ whenever $\phi \in \pi(X)$ and $\psi \in \pi(Y)$;
(c) for all homogeneous X and Y, $\phi \otimes \psi \in \pi(X \otimes Y)$ whenever $\phi \in \pi(X)$ and $\psi \in \pi(Y)$;
(d) the map $T \otimes \bar{T} \to T^{\otimes 2} = \mathbb{1}(2)$, defined by $T \simeq \bar{T}$, is in $\pi(T)$.

PROPOSITION 5.13. *Let $(\mathsf{C}_0, \varepsilon)$ be the pair associated with T by (5.8). There is a canonical bijection*

$$Q \colon Pol(\mathsf{T}) \to Pol_{\varepsilon}(\mathsf{C}_0)$$

from the set of polarizations on T to the set of polarizations on C_0 of parity ε.

PROOF. For any $X \in \mathrm{ob}(\mathsf{C}')$ that is homogeneous of degree n, (5.6b) and (5.6c) give an isomorphism

$$Q \colon \mathrm{Hom}(X \otimes \bar{X}, \mathbb{1}(-n)) \to \mathrm{Hom}(Q(X) \otimes \overline{Q(X)}, \mathbb{1}).$$

We define $Q\pi$ to be the polarization such that, for any homogeneous X,

$$Q\pi(QX) = \{Q\phi \mid \phi \in \pi(X)\}.$$

It is clear that $\pi \mapsto Q\pi$ is a bijection. □

COROLLARY 5.14. *The Tate triple T is polarizable if and only if C_0 has a polarization π with parity $\varepsilon(\pi) \equiv \varepsilon$ (mod $Z_0(\mathbb{R})^2$).*

PROOF. See (4.20e) □

COROLLARY 5.15. *For each $z \in {}_2Z_0(\mathbb{R})$ and polarization π on T, there is a polarization $z\pi$ on T defined by the condition*

$$\phi(x, y) \in z\pi(X) \iff \phi(x, zy) \in \pi(X).$$

The map

$$(z,\pi) \mapsto z\pi\colon {}_2Z_0(\mathbb{R}) \times \mathrm{Pol}(\mathsf{T}) \to \mathrm{Pol}(\mathsf{T})$$

makes $\mathrm{Pol}(\mathsf{T})$ *into a pseudo-torsor for* ${}_2Z_0(\mathbb{R})$.

PROOF. See (4.20d). □

THEOREM 5.16. *Let π be a polarization on T, and let ω be a fibre functor on C' with values in $\mathbb{C}$. Let (G,w,t) correspond to $(\mathsf{T}_{(\mathbb{C})},\omega)$. For any real form K of G such that $K_0 = \mathrm{Ker}(t)$ is compact, the pair (σ_K,ε) where $\varepsilon = w(-1)$ satisfies (4.2.1), and ω defines an equivalence between T and the Tate triple defined by $(G,\sigma_K,\varepsilon,w,t)$. For any simple X in C',*

$$\{\omega(\phi)' \mid \phi \in \pi(X)\}$$

is the set of K_0-invariant positive-definite Hermitian forms on $\omega(X)$.

PROOF. See (4.16). □

REMARK 5.17. From (4.17) one can deduce the following: a triple (B,w,t), where B is an affine algebraic band over $\mathbb{R}$ and $t \circ w = -2$, bounds a polarizable Tate triple if and only if $B_0 = \mathrm{Ker}(t\colon B \to \mathbb{G}_m)$ is the band defined by a compact real algebraic group; when this condition holds, the polarizable Tate triple banded by (B,w,t) is unique up to a tensor equivalence preserving the action of B and the polarization, and the equivalence is unique up to isomorphism. The Tate triple is neutral if and only if $\varepsilon \overset{\text{def}}{=} w(-1) \in Z_0(\mathbb{R})^2$.

Let (G,w,t) be a triple as in (5.4) defined over $\mathbb{R}$, and let $G_0 = \mathrm{Ker}(t)$ and $\varepsilon = w(-1)$. A Hodge element $C \in G_0(\mathbb{R})$ is said to be a ***Hodge element*** for (G,w,t) if $C^2 = \varepsilon$. A G-invariant sesquilinear form $\psi\colon V \times V \to \mathbb{1}(-n)$ on a homogeneous complex representation V of G of degree n is said to be a ***C-polarization*** if

$$\psi^C(x,y) \overset{\text{def}}{=} \psi(x,Cy)$$

is a positive-definite Hermitian form on V. When C is a Hodge element for (G,w,t) there is a polarization π_C on the Tate triple defined by (G,w,t) for which the positive forms are exactly the C-polarizations.

PROPOSITION 5.18. *Every polarization on the Tate triple defined by (G,w,t) is of the form π_C for some Hodge element C.*

PROOF. See (4.22) and (4.23). □

PROPOSITION 5.19. *Assume that $w(-1) = 1$. Then there is a unique (up to isomorphism) fibre functor ω on T with values in $\mathbb{R}$ transforming positive bilinear forms for π into positive-definite symmetric bilinear forms.*

PROOF. See (4.27). □

PROPOSITION 5.20. *Let (V,w,T) be the Tate triple defined in (5.3), and let π_{can} be the polarization on V such that, if $(V,a) \in \mathrm{ob}(\mathsf{V})$ is homogeneous, then $\pi(V,a)$ comprises the $(-1)^{\deg V}$-symmetric positive-definite forms on V. If $w(-1) \neq 1$ for T and π is a polarization on T, then there exists a unique (up to isomorphism) exact faithful functor $\omega\colon \mathsf{C} \to \mathsf{V}$ preserving the Tate-triple structure and carrying π into π_{can}.*

PROOF. Combine (4.29) and (5.9). □

EXAMPLE 5.21. Let T be the Tate triple $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ defined in (5.2). A ***polarization*** on a real Hodge structure V of weight n is a bilinear form $\phi: V \times V \to \mathbb{R}(-n)$ such that the real-valued form $(x, y) \mapsto (2\pi i)^n \phi(x, Cy)$, where C denotes the element $i \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ is positive-definite and symmetric. These polarizations are the positive (bilinear) forms for a polarization π on the Tate triple T. The functor $\omega: \mathsf{Hod}_{\mathbb{R}} \to \mathsf{V}$ provided by the last proposition is $V \mapsto (V \otimes \mathbb{C}, v \mapsto C\bar{v})$. (Note that $(\mathsf{Hod}_{\mathbb{R}}, w, \mathbb{R}(1))$ is not quite the Tate triple associated, as in (5.4), with $(\mathbb{S}, w, t)$ because we have chosen a different Tate object; this difference explains the occurrence of $(2\pi i)^n$ in the above formula; π is essentially the polarization defined by the canonical Hodge element C.)

Filtered Tannakian categories

For this topic, we refer the reader to Saavedra Rivano 1972, IV, 2.

6. Motives for absolute Hodge cycles

Throughout this section, k will denote a field of characteristic zero with algebraic closure $\bar{k}$ and Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. All varieties will be projective and smooth, and, for X a variety (or motive) over k, $\bar{X}$ denotes $X \otimes_k \bar{k}$. We shall freely use the notations of Deligne 1982. For example, if $k = \mathbb{C}$, then $H_{\mathrm{B}}(X)$ denotes the graded vector space $\bigoplus H^i_{\mathrm{B}}(X)$.

Complements on absolute Hodge cycles

For X a variety over k, $C^p_{\mathrm{AH}}(X)$ denotes the $\mathbb{Q}$-vector space of absolute Hodge cycles on X (see Deligne 1982, §2). When X has pure dimension n, we write

$$\mathrm{Mor}^p_{\mathrm{AH}}(X, Y) = C^{n+p}_{\mathrm{AH}}(X \times Y).$$

Then

$$\begin{aligned}
\mathrm{Mor}^p_{\mathrm{AH}}(X, Y) &\subset H^{2n+2p}(X \times Y)(p+n) \\
&= \bigoplus_{r+s=2n+2p} H^r(X) \otimes H^s(Y)(p+n) \\
&= \bigoplus_{s=r+2p} H^r(X)^\vee \otimes H^s(Y)(p) \\
&= \bigoplus_r \mathrm{Hom}(H^r(X), H^{r+2p}(Y)(p)).
\end{aligned}$$

The next proposition is obvious from this and the definition of an absolute Hodge cycle.

PROPOSITION 6.1. *An element f of $\mathrm{Mor}^p_{\mathrm{AH}}(X, Y)$ gives rise to*

(a) *for each prime ℓ, a homomorphism $f_\ell: H_\ell(\bar{X}) \to H_\ell(\bar{Y})(p)$ of graded vector spaces (meaning that f_ℓ is a family of homomorphisms $f^r_\ell: H^r_\ell(\bar{X}) \to H^{r+2p}_\ell(\bar{Y})(p)$);*
(b) *a homomorphism $f_{\mathrm{dR}}: H_{\mathrm{dR}}(X) \to H_{\mathrm{dR}}(Y)(p)$ of graded vector spaces;*
(c) *for each $\sigma: k \hookrightarrow \mathbb{C}$, a homomorphism $f_\sigma: H_\sigma(X) \to H_\sigma(Y)(p)$ of graded vector spaces.*

These maps satisfy the following conditions

(d) *for all* $\gamma \in \Gamma$ *and primes* ℓ, $\gamma f_\ell = f_\ell$;

(e) f_{dR} *is compatible with the Hodge filtrations on each homogeneous factor;*

(f) *for each* $\sigma\colon k \hookrightarrow \mathbb{C}$, *the maps* f_σ, f_ℓ, *and* f_{dR} *correspond under the comparison isomorphisms (§1).*

Conversely, assume that k *is embeddable in* $\mathbb{C}$*; then a family of maps* f_ℓ, f_{dR} *as in (a), (b) arises from an* $f \in \mathrm{Mor}^p_{\mathrm{AH}}(X,Y)$ *provided* (f_ℓ) *and* f_{dR} *satisfy (d) and (e) respectively and for every* $\sigma\colon k \hookrightarrow \mathbb{C}$ *there exists an* f_σ *such that* (f_ℓ), f_{dR}, *and* f_σ *satisfy condition (f); moreover,* f *is unique.*

Similarly, a $\psi \in C^{2n-r}_{\mathrm{AH}}(X \times X)$ gives rise to pairings

$$\psi^s\colon H^s(X) \times H^{2r-s}(X) \to \mathbb{Q}(-r).$$

PROPOSITION 6.2. *On every variety* X *there exists a* $\psi \in C^{2\dim X - r}_{\mathrm{AH}}(X \times X)$ *such that, for every* $\sigma\colon k \hookrightarrow \mathbb{C}$,

$$\psi^r_\sigma\colon H^r_\sigma(X,\mathbb{R}) \times H^r_\sigma(X,\mathbb{R}) \to \mathbb{R}(-r)$$

is a polarization of real Hodge structures (in the sense of 5.21).

PROOF. Let $n = \dim X$. Choose a projective embedding of X, and let L be a hyperplane section of X. Let ℓ be the class of L in $H^2(X)(1)$, and write ℓ also for the map $H(X) \to H(X)(1)$ sending a class to its cup-product with ℓ. Assume that X is connected, and define the ***primitive cohomology*** of X by

$$H^r(X)_{\mathrm{prim}} = \mathrm{Ker}(\ell^{n-r+1}\colon H^r(X) \to H^{2n-r+2}(X)(n-r+1)).$$

The hard Lefschetz theorem states that

$$\ell^{n-r}\colon H^r(X) \to H^{2n-r}(X)(n-r)$$

is an isomorphism for $r \leq n$; it implies that

$$H^r(X) = \bigoplus_{s \geq r-n,\, s \geq 0} \ell^s H^{r-2s}(X)(-s)_{\mathrm{prim}}.$$

Thus, every $x \in H^r(X)$ can be written uniquely $x = \sum \ell^s(x_s)$ with $x_s \in H^{r-2s}(X)(-s)_{\mathrm{prim}}$. Define

$$^*x = \sum (-1)^{(r-2s)(r-2s+1)/2} \ell^{n-r+s} x_s \in H^{2n-r}(X)(n-r).$$

Then $x \mapsto {}^*x\colon H^r(X) \to H^{2n-r}(X)(n-r)$ is a well-defined map for each of the three cohomology theories, ℓ-adic, de Rham, and Betti. Proposition 6.1 shows that it is defined by an absolute Hodge cycle (rather, the map $H(X) \to H(X)(n-r)$ that is $x \mapsto {}^*x$ on H^r and zero elsewhere is so defined). We take ψ^r to be

$$H^r(X) \otimes H^r(X) \xrightarrow{\mathrm{id}\otimes *} H^r(X) \otimes H^{2n-r}(X)(n-r) \to H^{2n}(X)(n-r) \xrightarrow{\mathrm{Tr}} \mathbb{Q}(-r).$$

Clearly it is defined by an absolute Hodge cycle, and the Hodge-Riemann bilinear relations (see Wells 1980, 5.3) show that it defines a polarization on the real Hodge structure $H^r_\sigma(X,\mathbb{R})$ for each $\sigma\colon k \hookrightarrow \mathbb{C}$. □

PROPOSITION 6.3. *For any* $u \in \mathrm{Mor}^0_{\mathrm{AH}}(Y, X)$*, there exists a unique* $u' \in \mathrm{Mor}^0_{\mathrm{AH}}(X, Y)$ *such that*

$$\psi_X(uy, x) = \psi_Y(y, u'x), \quad x \in H^r(X), \quad y \in H^r(Y)$$

where ψ_X *and* ψ_Y *are the forms defined in (6.2); moreover,*

$$\mathrm{Tr}(u \circ u') = \mathrm{Tr}(u' \circ u) \in \mathbb{Q}$$
$$\mathrm{Tr}(u \circ u') > 0 \quad \text{if } u \neq 0.$$

PROOF. The first part is obvious, and the last assertion follows from the fact that the ψ_X and ψ_Y are positive forms for a polarization in $\mathsf{Hod}_{\mathbb{R}}$ (the Tannakian category of real Hodge structures). □

Note that the proposition show that $\mathrm{Mor}^0_{\mathrm{AH}}(X, X)$ is a semisimple $\mathbb{Q}$-algebra (see 4.5).

Construction of the category of motives

Let V_k be the category of (smooth projective, not necessarily connected) varieties over k. The category CV_k is defined to have as objects symbols $h(X)$, one for each object $X \in \mathrm{ob}(\mathsf{V}_k)$, and as morphisms

$$\mathrm{Hom}(h(X), h(Y)) = \mathrm{Mor}^0_{\mathrm{AH}}(X, Y).$$

There is a map

$$\mathrm{Hom}(Y, X) \to \mathrm{Hom}(h(X), h(Y))$$

sending a homomorphism to the cohomology class of its graph which makes h into a contravariant functor $\mathsf{V}_k \to \mathsf{CV}_k$.

Clearly CV_k is a $\mathbb{Q}$-linear category, and $h(X \sqcup Y) = h(X) \oplus h(Y)$. There is a $\mathbb{Q}$-linear tensor structure on CV_k for which

- ◇ $h(X) \otimes h(Y) = h(X \times Y)$,
- ◇ the associativity constraint is induced by $(X \times Y) \times Z \to X \times (Y \times Z)$,
- ◇ the commutativity constraint is induced by $Y \times X \to X \times Y$, and
- ◇ the identity object is $h(\text{point})$.

The false category of ***effective*** (or ***positive***) ***motives*** $\dot{\mathsf{M}}_k^+$ is defined to be the pseudo-abelian (Karoubian) envelope of CV_k. Thus, an object of $\dot{\mathsf{M}}_k^+$ is a pair (M, p) with $M \in \mathsf{CV}_k$ and p an idempotent in $\mathrm{End}(M)$, and

$$\mathrm{Hom}((M, p), (N, q)) = \{f : M \to N \mid f \circ p = q \circ f / \sim\} \tag{6.3.1}$$

where $f \sim 0$ if $f \circ p = 0 = q \circ f$. The rule

$$(M, p) \otimes (N, q) = (M \otimes N, p \otimes q)$$

defines a $\mathbb{Q}$-linear tensor structure on $\dot{\mathsf{M}}_k^+$, and $M \mapsto (M, \mathrm{id}) : \mathsf{CV}_k \to \dot{\mathsf{M}}_k^+$ is a fully faithful functor which we use to identify CV_k with a subcategory of $\dot{\mathsf{M}}_k^+$. With this identification, (M, p) becomes the image of $p : M \to M$. The category $\dot{\mathsf{M}}_k^+$ is pseudo-abelian: any decomposition of id_M into a sum of pairwise orthogonal idempotents

$$\mathrm{id}_M = e_1 + \cdots + e_m$$

corresponds to a decomposition

$$M = M_1 \oplus \cdots \oplus M_m$$

with $e_i | M_i = \mathrm{id}_{M_i}$. The functor $\mathsf{CV}_k \to \dot{\mathsf{M}}_k^+$ is universal for functors from CV_k to pseudo-abelian categories.

For any $X \in \mathrm{ob}(\mathsf{V}_k)$, the projection maps $p^r : H(X) \to H^r(X)$ define an element of $\mathrm{Mor}^0_{\mathrm{AH}}(X,X) = \mathrm{End}(h(X))$. Corresponding to the decomposition

$$\mathrm{id}_{h(X)} = p^0 + p^1 + p^2 + \cdots$$

there is a decompostion (in $\dot{\mathsf{M}}_k^+$)

$$h(X) = h^0(X) + h^1(X) + h^2(X) + \cdots.$$

This grading of objects of CV_k extends in an obvious way to objects of $\dot{\mathsf{M}}_k^+$, and the Künneth formulas show that these gradings are compatible with tensor products (and therefore satisfy 5.1a).

Let L be the Lefschetz motive $h^2(\mathbb{P}^1)$. With the notations of Deligne 1982, §1, $H(L) = \mathbb{Q}(-1)$, whence it follows that

$$\mathrm{Hom}(M,N) \xrightarrow{\approx} \mathrm{Hom}(M \otimes L, N \otimes L)$$

for any effective motives M and N. This means that $V \rightsquigarrow V \otimes L$ is a fully faithful functor and allows us to invert L.

DEFINITION 6.4. The ***false category*** $\dot{\mathsf{M}}_k$ ***of motives*** is defined as follows:

(a) an object of $\dot{\mathsf{M}}_k$ is a pair (M,m) with $M \in \mathrm{ob}(\dot{\mathsf{M}}_k^+)$ and $m \in \mathbb{Z}$;
(b) $\mathrm{Hom}((M,m),(N,n)) = \mathrm{Hom}(M \otimes L^{r-m}, N \otimes L^{r-n})$, $\quad r \geq m,n$ (for different r, these groups are canonically isomorphic);
(c) composition of morphisms is induced by that in $\dot{\mathsf{M}}_k^+$.

This category of motives is $\mathbb{Q}$-linear and pseudo-abelian and has a tensor structure

$$(M,m) \otimes (N,n) = (M \otimes N, m+n)$$

and grading

$$(M,m)^r = M^{r-2m}.$$

We identify $\dot{\mathsf{M}}_k^+$ with a subcategory of $\dot{\mathsf{M}}_k$ by means to $M \rightsquigarrow (M,0)$. The ***Tate motive*** T is $L^{-1} = (\mathbb{1},1)$. We abbreviate $M \otimes T^{\otimes m} = (M,m)$ by $M(m)$.

We shall see shortly that $\dot{\mathsf{M}}_k$ is a rigid abelian tensor category, and $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$. It is not however a Tannakian category because, for $X \in \mathrm{ob}(\mathsf{V}_k)$, $\mathrm{rank}(h(X))$ is the Euler-Poincaré characteristic, $\sum(-1)^r \dim H^r(X)$, of X, which is not necessarily positive. To remedy this we modify the commutativity constraint as follows: let

$$\dot{\psi} : M \otimes N \to N \otimes M, \quad \dot{\psi} = \oplus \dot{\psi}^{r,s}, \quad \dot{\psi}^{r,s} : M^r \otimes N^s \to N^s \otimes M^r$$

be the commutativity constraint on $\dot{\mathsf{M}}_k$; define a new commutativity constraint by

$$\psi : M \otimes N \to N \otimes M, \quad \psi = \oplus \psi^{r,s}, \quad \psi^{r,s} = (-1)^{rs} \dot{\psi}^{r,s}. \tag{6.4.1}$$

Then M_k, with $\dot{\psi}$ replaced by ψ, is the ***true category*** M_k ***of motives.***

PROPOSITION 6.5. *The category M_k is a semisimple Tannakian category over $\mathbb{Q}$.*

PROOF. As we observed above, Proposition 6.3 implies that the endomorphism rings of the objects of M_k are semisimple. Because they are also finite dimensional over $\mathbb{Q}$, we may apply the next lemma.[23] □

LEMMA 6.6. *Let C be a $\mathbb{Q}$-linear pseudo-abelian category such that, for all objects X,Y, $\mathrm{Hom}(X,Y)$ is finite dimensional and $\mathrm{End}(X)$ is semisimple. Then C is semisimple (and hence every additive functor from C to an abelian category is exact).*

PROOF. This is Lemma 2 of Jannsen 1992. □

The following theorem summarizes what we have (essentially) proved about M_k.

THEOREM 6.7. *(a) Let w be the grading on M_k; then (M_k,w,T) is a Tate triple over $\mathbb{Q}$.*

(b) There is a contravariant functor $h\colon\mathsf{V}_k\to\mathsf{M}_k$; every effective motive is the image $(h(X),p)$ of an idempotent $p\in\mathrm{End}(h(X))$ for some $X\in ob(\mathsf{V}_k)$; every motive is of the form $M(n)$ for some effective M and some $n\in\mathbb{Z}$.

(c) For all varieties X,Y with X of pure dimension m,

$$C_{\mathrm{AH}}^{m+s-r}(X\times Y)=\mathrm{Hom}(h(X)(r),h(Y)(s));$$

in particular,

$$C_{\mathrm{AH}}^{m}(X\times Y)=\mathrm{Hom}(h(X),h(Y));$$

morphisms of motives can be expressed in terms of absolute Hodge cycles on varieties by means of (6.3.1) and (6.4b).

(d) The constraints on M_k have an obvious definition, except that the obvious commutativity constraint has to be modified by (6.4.1).

(e) For varieties X and Y,

$$\begin{aligned}h(X\sqcup Y)&=h(X)\oplus h(Y)\\ h(X\times Y)&=h(X)\otimes h(Y)\\ h(X)^{\vee}&=h(X)(m)\text{ if }X\text{ is of pure dimension }n.\end{aligned}$$

(f) The fibre functors H_ℓ, H_{dR}, and H_σ define fibre functors on M_k; these fibre functors define morphisms of Tate triples $\mathsf{M}_k\to\mathsf{T}_\ell,\mathsf{T}_{dR},\mathsf{T}_B$ (see 5.2b); in particular, $H(T)=\mathbb{Q}(1)$.

(g) When k is embeddable in $\mathbb{C}$, $\mathrm{Hom}(M,N)$ is the vector space of families of maps

$$\begin{aligned}f_\ell\colon H_\ell(\bar{M})&\to H_\ell(\bar{N})\\ f_{dR}\colon H_{dR}(M)&\to H_{dR}(N)\end{aligned}$$

such that f_{dR} preserves the Hodge filtration, $\gamma f_\ell=f_\ell$ for all $\gamma\in\Gamma$, and for every $\sigma\colon k\hookrightarrow\mathbb{C}$ there exists a map $f_\sigma\colon H_\sigma(M)\to H_\sigma(N)$ agreeing with f_ℓ and f_{dR} under the comparison isomorphisms.

(h) The category M_k is semisimple.

(i) There exists a polarization on M_k for which $\pi(h^r(X))$ consists of the forms defined in (6.2).

[23] The original followed Saavedra 1972 in deducing Proposition 6.5 from the following statement:

> Let C be a $\mathbb{Q}$-linear pseudo-abelian category, and let $\omega\colon\mathsf{C}\to\mathsf{Vec}_{\mathbb{Q}}$ be a faithful $\mathbb{Q}$-linear functor. If every indecomposable object of C is simple, then C is a semisimple abelian category and ω is exact.

As Jannsen (1992, p. 451) points out, this statement is false.

Some calculations

According to (6.7g), to define a map $M \to N$ of motives it suffices to give a procedure for defining a map of cohomology groups $H(M) \to H(N)$ that works (compatibly) for all three theories: Betti, de Rham, and ℓ-adic. The map will be an isomorphism if its realization in one theory is an isomorphism.

Let G be a finite group acting on a variety. The group algebra $\mathbb{Q}[G]$ acts on $h(X)$, and we define $h(X)^G$ to be the motive $(h(X), p)$ with p equal to the idempotent

$$\frac{\sum_{g \in G} g}{(G:1)}.$$

Note that $H(h(X)^G) = H(X)^G$ in each of the standard cohomology theories.

PROPOSITION 6.8. *Assume that the finite group G acts freely on X, so that X/G is also smooth; then $h(X/G) = h(X)^G$.*

PROOF. Since cohomology is functorial, there exists a map $H(X/G) \to H(X)$ whose image lies in $H(X)^G = H(h(X)^G)$. The Hochschild-Serre spectral sequence

$$H^r(G, H^s(X)) \Rightarrow H^{r+s}(X/G)$$

shows that the map $H(X/G) \to H(X)^G$ is an isomorphism for, say, the ℓ-adic cohomology, because $H^r(G, V) = 0$, $r > 0$, if V is a vector space over a field of characteristic zero. □

REMARK 6.9. More generally, if $f: Y \to X$ is a map of finite (generic) degree n between connected varieties of the same dimension, then the composite

$$H(X) \xrightarrow{f^*} H(Y) \xrightarrow{f_*} H(X)$$

is multiplication by n; there therefore exist maps

$$h(X) \to h(Y) \to h(X)$$

with composite n, and $h(X)$ is a direct summand of $h(Y)$.

PROPOSITION 6.10. *Let E be a vector bundle of rank $m+1$ over a variety X, and let $p: \mathbb{P}(E) \to X$ be the associated projective bundle; then*

$$h(\mathbb{P}(E)) = h(X) \oplus h(X)(-1) \oplus \cdots \oplus h(X)(-m).$$

PROOF. Let γ be the class in $H^2(\mathbb{P}(E))(1)$ of the canonical line bundle on $\mathbb{P}(E)$, and let $p^*: H(X) \to H(\mathbb{P}(E))$ be the map induced by p. The map

$$(c_0, \ldots, c_m) \mapsto \sum p^*(c_i)\gamma^i : H(X) \oplus \cdots \oplus H(X)(-m) \to H(\mathbb{P}(E))$$

has the requisite properties. □

PROPOSITION 6.11. *Let Y be a smooth closed subvariety of codimension c in the variety X, and let X' be the variety obtained from X by blowing up Y; then there is an exact sequence*

$$0 \to h(Y)(-c) \to h(X) \oplus h(Y')(-1) \to h(X') \to 0$$

where Y' is the inverse image of Y.

PROOF. From the Gysin sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^{r-2c}(Y)(-c) & \longrightarrow & H^r(X) & \longrightarrow & H^r(X \setminus Y) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \| & & \\
\cdots & \longrightarrow & H^{r-2c}(Y')(-1) & \longrightarrow & H^r(X') & \longrightarrow & H^r(X' \setminus Y') & \longrightarrow & \cdots
\end{array}$$

we obtain a long exact sequence

$$\cdots \to H^{r-2c}(Y)(-c) \to H^r(X) \oplus H^{r-2}(Y')(-1) \to H^r(X') \to \cdots.$$

But Y' is a projective bundle over Y, and so $H^{r-2c}(Y)(-c) \to H^{r-2}(Y')(-1)$ is injective. Therefore, there are exact sequences

$$0 \to H^{r-2c}(Y)(-c) \to H^r(X) \oplus H^{r-2}(Y')(-1) \to H^r(X') \to 0,$$

which can be rewritten as

$$0 \to H(Y)(-c) \to H(X) \oplus H(Y')(-1) \to H(X') \to 0$$

We have constructed a sequence of motives, which is exact because the cohomology functors are faithful and exact. □

COROLLARY 6.12. *With the notations of the proposition,*

$$h(X') = h(X) \oplus \bigoplus_{r=1}^{c-1} h(Y)(-r).$$

PROOF. Proposition 6.10 shows that $h(Y') = \bigoplus_{r=1}^{c-1} h(Y)(r)$. □

PROPOSITION 6.13. *If X is an abelian variety, then $h(X) = \bigwedge(h^1(X))$.*

PROOF. Cup-product defines a map $\bigwedge(H^1(X)) \to H(X)$ which, for the Betti cohomology, say, is known to be an isomorphism. (See Mumford 1970, I.1.) □

PROPOSITION 6.14. *If X is a curve with Jacobian J, then*

$$h(X) = \mathbb{1} \oplus h^1(J) \oplus L.$$

PROOF. The map $X \to J$ (well-defined up to translation) defines an isomorphism $H^1(J) \to H^1(X)$. □

PROPOSITION 6.15. *Let X be a unirational variety of dimension $d \leq 3$ over an algebraically closed field; then*

$$\begin{aligned}
&(d=1) \quad h(X) = \mathbb{1} \oplus L; \\
&(d=2) \quad h(X) = \mathbb{1} \oplus rL \oplus L^2, \text{ some } r \in \mathbb{N}; \\
&(d=3) \quad h(X) = \mathbb{1} \oplus rL \oplus h^1(A)(-1) \oplus rL^2 \oplus L^3, \text{ some } r \in \mathbb{N},
\end{aligned}$$

where A is an abelian variety.

PROOF. We prove the proposition only for $d = 3$. According to the resolution theorem of Abhyankar 1966, there exist maps

$$\mathbb{P}^3 \xleftarrow{u} X' \xrightarrow{v} X$$

with v surjective of finite degree and u a composite of blowing-ups. We know

$$h(\mathbb{P}^3) = \mathbb{1} \oplus L \oplus L^2 \oplus L^3$$

(special case of (6.10)). When a point is blown up, a motive $L \oplus L^2$ is added, and when a curve Y is blown up, a motive $L \oplus h^1(Y)(-1) \oplus L^2$ is added. Therefore,

$$h(X') \approx \mathbb{1} \oplus sL \oplus M(-1) \oplus sL^2 \oplus L^3$$

where M is a sum of motives of the form $h^1(Y)$, Y a curve. A direct summand of such an M is of the form $h^1(A)$ for A an abelian variety (see 6.21 below). As $h(X)$ is a direct summand of $h(X')$ (see 6.9) and Poincaré duality shows that the multiples of L^2 and L^3 occurring in $h(X)$ are the same as those of L and $\mathbb{1}$ respectively, the proof is complete. □

PROPOSITION 6.16. *Let X_d^n denote the Fermat hypersurface of dimension n and degree d:*

$$T_0^d + T_1^d + \cdots + T_{n+1}^d = 0.$$

Then,

$$h^n(X_d^n) \oplus d h^n(\mathbb{P}^n) = h^n(X_d^{n-1} \times X_d^1)^{\mu_d} \oplus (d-1) h^{n-2}(X_d^{n-2})(-1)$$

where μ_d, the group of dth roots of 1, acts on $X_d^{n-1} \times X_d^1$ according to

$$\zeta(t_0 : \ldots : t_n ; s_0 : s_1 : s_2) = (t_0 : \ldots : \zeta t_n ; s_0 : s_1 : \zeta s_2)$$

PROOF. See Shioda and Katsura 1979, 2.5. □

Artin Motives

Let V_k^0 be the category of zero-dimensional varieties over k, and let CV_k^0 be the image of V_k^0 in M_k. The Tannakian subcategory M_k^0 of M_k generated by the objects of CV_k^0 is called the category of ***(Emil) Artin motives.***

For any X in $\mathrm{ob}(\mathsf{V}_k^0)$, $X(\bar{k})$ is a finite set on which Γ acts continuously. Thus, $\mathbb{Q}^{X(\bar{k})}$ is a finite-dimensional continuous representation of Γ. When we regard Γ, in an obvious way, as a (constant, pro-finite) affine group scheme over k, $\mathbb{Q}^{X(\bar{k})} \in \mathsf{Rep}_{\mathbb{Q}}(\Gamma)$. For $X, Y \in \mathrm{ob}(\mathsf{V}_k^0)$,

$$\begin{aligned} \mathrm{Hom}(h(X), h(Y)) &\overset{\text{def}}{=} C^0_{\mathrm{AH}}(X \times Y) \\ &= (\mathbb{Q}^{X(\bar{k}) \times Y(\bar{k})})^{\Gamma} \\ &= \mathrm{Hom}_{\Gamma}\left(\mathbb{Q}^{X(\bar{k})}, \mathbb{Q}^{Y(\bar{k})}\right). \end{aligned}$$

Thus,

$$h(X) \rightsquigarrow \mathbb{Q}^{X(\bar{k})} : \mathsf{CV}_k^0 \to \mathsf{Rep}_{\mathbb{Q}}(\Gamma)$$

is fully faithful, and Grothendieck's formulation of Galois theory shows that it is essentially surjective. Therefore, CV_k^0 is abelian and $\mathsf{M}_k^0 = \mathsf{CV}_k^0$. We have shown:

PROPOSITION 6.17. *The category of Artin motives* $\mathsf{M}_k^0 = \mathsf{CV}_k^0$. *The functor* $h(X) \rightsquigarrow \mathbb{Q}^{X(\bar{k})}$ *defines an equivalence of tensor categories* $\mathsf{M}_k^0 \xrightarrow{\sim} \mathsf{Rep}_{\mathbb{Q}}(\Gamma)$.

REMARK 6.18. Let M be an Artin motive, and regard M as an object of $\mathsf{Rep}_{\mathbb{Q}}(\Gamma)$. Then

$$\begin{aligned} H_\sigma(M) &= M \text{ (underlying vector space) for any } \sigma\colon k \hookrightarrow \mathbb{C};\\ H_\ell(\bar{M}) &= M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell, \text{ as a } \Gamma\text{-module};\\ H_{\mathrm{dR}}(M) &= (M \otimes_{\mathbb{Q}} \bar{k})^\Gamma. \end{aligned}$$

Note that, if $M = h(X)$ where $X = \operatorname{Spec}(A)$, then

$$H_{\mathrm{dR}}(M) = (\mathbb{Q}^{X(\bar{k})} \otimes_{\mathbb{Q}} \bar{k})^\Gamma = (A \otimes_k \bar{k})^\Gamma = A.$$

REMARK 6.19. The proposition shows that the category of Artin motives over k is equivalent to the category of sheaves of finite-dimensional $\mathbb{Q}$-vector spaces with finite-dimensional stalk[24] on the étale site $\operatorname{Spec}(k)_{\mathrm{et}}$.

Effective motives of degree 1

A ***$\mathbb{Q}$-rational Hodge structure*** is a finite-dimensional vector space V over $\mathbb{Q}$ together with a real Hodge structure on $V \otimes \mathbb{R}$ whose weight decomposition is defined over $\mathbb{Q}$. Let $\mathsf{Hod}_{\mathbb{Q}}$ be the category of $\mathbb{Q}$-rational Hodge structures. A ***polarization*** on an object V of $\mathsf{Hod}_{\mathbb{Q}}$ is a bilinear pairing $\psi\colon V \otimes V \to \mathbb{Q}(-n)$ such that $\psi \otimes \mathbb{R}$ is a polarization on the real Hodge structure $V \otimes \mathbb{R}$.

Let Isab_k be the category of abelian varieties up to isogeny over k. The following theorem summarizes part of the theory of abelian varieties.

THEOREM 6.20 (RIEMANN). *The functor* $H_B^1\colon \mathsf{Isab}_{\mathbb{C}} \to \mathsf{Hod}_{\mathbb{Q}}$ *is fully faithful; the essential image consists of polarizable Hodge structures of weight* 1.

Let M_k^{+1} be the pseudo-abelian subcategory of M_k generated by motives of the form $h^1(X)$ for X a geometrically connected curve; according to (6.14), M_k^{+1} can also be described as the category generated by motives of the form $h^1(J)$ for J a Jacobian.

PROPOSITION 6.21. *(a) The functor* $h^1\colon \mathsf{Isab}_k \to \mathsf{M}_k$ *factors through* M_k^{+1} *and defines an equivalence of categories,*

$$\mathsf{Isab}_k \xrightarrow{\sim} \mathsf{M}_k^{+1}.$$

(b) The functor $H^1\colon \mathsf{M}_{\mathbb{C}}^{+1} \to \mathsf{Hod}_{\mathbb{Q}}$ *is fully faithful; its essential image consists of polarizable Hodge structures of weight* 1.

PROOF. Every object of Isab_k is a direct summand of a Jacobian, which shows that h^1 factors through M_k^{+1}. Assume, for simplicity, that k is algebraically closed. Then, for any $A, B \in \operatorname{ob}(\mathsf{Isab}_k)$,

$$\operatorname{Hom}(B, A) \subset \operatorname{Hom}(h^1(A), h^1(B)) \subset \operatorname{Hom}(H_\sigma(A), H_\sigma(B)),$$

and (6.20) shows that $\operatorname{Hom}(B, A) = \operatorname{Hom}(H_\sigma(A), H_\sigma(B))$. Thus h^1 is fully faithful and (as Isab_k is abelian) essentially surjective. This proves (a), and (b) follows from (a) and (6.20). □

[24] This condition was omitted in the original.

The motivic Galois group

Let k be a field that is embeddable in $\mathbb{C}$. For any $\sigma: k \hookrightarrow \mathbb{C}$, we define $G(\sigma) = \underline{\mathrm{Aut}}^{\otimes}(H_\sigma)$. Thus, $G(\sigma)$ is an affine group scheme over $\mathbb{Q}$, and H_σ defines an equivalence of categories $\mathsf{M}_k \xrightarrow{\sim} \mathsf{Rep}_{\mathbb{Q}}(G(\sigma))$. Because $G(\sigma)$ plays the same role for M_k as $\Gamma = \mathrm{Gal}(\bar{k}/k)$ plays for M_k^0, it is called the ***motivic Galois group.***

PROPOSITION 6.22. [25] *(a) The group $G(\sigma)$ is a pro-reductive (not necessarily connected) affine group scheme over $\mathbb{Q}$, and it is connected if k is algebraically closed and all Hodge cycles are absolutely Hodge.*

(b) Let $k \subset k'$ be algebraically closed fields, let $\sigma': k' \hookrightarrow \mathbb{C}$, and let $\sigma = \sigma'|k$. The homomorphism $G(\sigma') \to G(\sigma)$ induced by $\mathsf{M}_k \to \mathsf{M}_{k'}$ is faithfully flat.

PROOF. (a) Let $X \in \mathrm{ob}(\mathsf{M}_k)$, and let C_X be the abelian tensor subcategory of M_k generated by X, $X^\vee$, T, and $T^\vee$. Let $G_X = \underline{\mathrm{Aut}}^{\otimes}(H_\sigma|\mathsf{C}_X)$. As C_X is semisimple (see (6.5)), G_X is a reductive group (2.23), and so $G = \varprojlim G_X$ is pro-reductive. If k is algebraically closed and all Hodge cycles are absolutely Hodge, then (cf. 3.4) G_X is the smallest subgroup of $\mathrm{Aut}(H_\sigma(X)) \times \mathbb{G}_m$ such that $(G_X)_{\mathbb{C}}$ contains the image of the homomorphism $\mu: \mathbb{G}_{m\mathbb{C}} \to \mathrm{Aut}(H_\sigma(X,\mathbb{C})) \times \mathbb{G}_{m\mathbb{C}}$ defined by the Hodge structure on $H_\sigma(X)$. As $\mathrm{Im}(\mu)$ is connected, so also is G_X.

(b) According to (2.9), $\mathsf{M}_k \to \mathsf{M}_{k'}$ is fully faithful, and so (2.29) shows that $G(\sigma') \to G(\sigma)$ is faithfully flat. □

Now let k be arbitrary, and fix an embedding $\sigma: \bar{k} \hookrightarrow \mathbb{C}$. The inclusion $\mathsf{M}_k^0 \to \mathsf{M}_k$ defines a homomorphism $\pi: G(\sigma) \to \Gamma$ because $\Gamma = \underline{\mathrm{Aut}}^{\otimes}(H_\sigma|\mathsf{M}_k^0)$ (see 6.17), and the functor $\mathsf{M}_k \to \mathsf{M}_{\bar{k}}$ defines a homomorphism $i: G^\circ(\sigma) \to G(\sigma)$ where $G^\circ(\sigma) \overset{\mathrm{df}}{=} \underline{\mathrm{Aut}}^{\otimes}(H_\sigma|\mathsf{M}_{\bar{k}})$.

PROPOSITION 6.23. *(a) The sequence*

$$1 \to G^\circ(\sigma) \xrightarrow{i} G(\sigma) \xrightarrow{\pi} \Gamma \to 1$$

is exact.

(b) If all Hodge cycles are absolutely Hodge, then the identity component of $G(\sigma)$ is $G^\circ(\sigma)$.

(c) For any $\tau \in \Gamma$, $\pi^{-1}(\tau) = \mathrm{Hom}^{\otimes}(H_\sigma, H_{\sigma\tau})$, regarding H_σ and H_τ as functors on $M_{\bar{k}}$.

(d) For any prime ℓ, there is a canonical continuous homomorphism $sp_\ell: \Gamma \to G(\sigma)(\mathbb{Q}_\ell)$ such that $\pi \circ sp_\ell = \mathrm{id}$.

PROOF. (a) As $\mathsf{M}_k^\circ \to \mathsf{M}_k$ is fully faithful, π is faithfully flat (2.29). To show that i is injective, it suffices to show that every motive $h(X)$, $X \in \mathsf{V}_{\bar{k}}$, is a subquotient of a motive $h(\bar{X}')$ for some $X' \in \mathsf{V}_k$; but X has a model X_0 over a finite extension k' of k, and we can take $X' = \mathrm{Res}_{k'/k} X_0$. The exactness at $G(\sigma)$ is a special case of (c).

(b) This is an immediate consequence of (6.22a) and (a).

[25] In the original, the hypothesis in 6.22 (a) and 6.23 (b) that all Hodge cycles are absolutely Hodge (for the varieties concerned) was omitted. In (b) it was claimed that if k has infinite transcendence degree over $\mathbb{Q}$, then $G(\sigma') \to G(\sigma)$ is an isomorphism. This is obviously false — the motive defined by an elliptic curve E over k' will arise from a motive over k if and only if $j(E) \in k$.

(c) Let $M,N\in \mathrm{ob}(\mathsf{M}_k)$. Then $\mathrm{Hom}(\bar{M},\bar{N})\in \mathrm{ob}(\mathsf{Rep}_{\mathbb{Q}}(\Gamma))$, and so we can regard it as an Artin motive over k. There is a canonical map of motives $\mathrm{Hom}(\bar{M},\bar{N})\hookrightarrow \underline{\mathrm{Hom}}(M,N)$ giving rise to

$$H_\sigma(\mathrm{Hom}(\bar{M},\bar{N}))=\mathrm{Hom}(\bar{M},\bar{N})\overset{H_\sigma}{\to}\mathrm{Hom}(H_\sigma(\bar{M}),H_\sigma(\bar{N}))=H_\sigma(\underline{\mathrm{Hom}}(M,N))$$

Let $\tau\in\Gamma$; then

$$H_\sigma(\bar{M})=H_\sigma(M)=H_{\tau\sigma}(M)=H_{\tau\sigma}(\bar{M})$$

and, for $f\in\mathrm{Hom}(\bar{M},\bar{N})$, $H_\sigma(\tau)=H_{\tau\sigma}(\tau f)$.

Let $g\in G(R)$; for any $f\colon M\to N$ in M_k, there is a commutative diagram

$$\begin{array}{ccc}
H_\sigma(M,R) & \xrightarrow{g_M} & H_\sigma(M,R)\\
\downarrow{\scriptstyle H_\sigma(f)} & & \downarrow{\scriptstyle H_\sigma(f)}\\
H_\sigma(N,R) & \xrightarrow{g_N} & H_\sigma(N,R).
\end{array}$$

Let $\tau=\pi(g)$, so that g acts on $\mathrm{Hom}(\bar{M},\bar{N})\subset\mathrm{Hom}(M,N)$ as τ. Then, for any $f\colon\bar{M}\to\bar{N}$ in $\mathsf{M}_{\bar{k}}$

$$\begin{array}{ccccc}
H_\sigma(\bar{M},R)g_M & \longrightarrow & H_\sigma(\bar{M},R) & = & H_{\tau\sigma}(\bar{M},R)\\
\downarrow{\scriptstyle H_\sigma(f)} & & \downarrow{\scriptstyle H_\sigma(\tau^{-1}f)} & & \downarrow{\scriptstyle H_{\tau\sigma}(f)}\\
H_\sigma(\bar{N},R) & \xrightarrow{g_N} & H_\sigma(\bar{N},R) & = & H_{\tau\sigma}(\bar{N},R).
\end{array}$$

commutes. The diagram shows that $g_M\colon H_\sigma(\bar{M},R)\to H_{\tau\sigma}(\bar{M},R)$ depends only on M as an object of $\mathsf{M}_{\bar{k}}$. We observed in the proof of (a) above that $\mathsf{M}_{\bar{k}}$ is generated by motives of the form $\bar{M}$, $M\in\mathsf{M}_k$. Thus g defines an element of $\underline{\mathrm{Hom}}^{\otimes}(H_\sigma,H_{\tau\sigma})(R)$, where H_σ and $H_{\tau\sigma}$ are to be regarded as functors on $\mathsf{M}_{\bar{k}}$. We have defined a map $\pi^{-1}(\tau)\to\underline{\mathrm{Hom}}^{\otimes}(H_\sigma,H_{\tau\sigma})$, and it is easy to see that it is surjective.

(d) After (c), we have to find a canonical element of $\mathrm{Hom}^{\otimes}(H_\ell(\sigma M),H_\ell(\tau\sigma M))$ depending functorially on $M\in\mathsf{M}_{\bar{k}}$. Extend τ to an automorphism $\bar{\tau}$ of $\mathbb{C}$. For any variety X over $\bar{k}$, there is a $\bar{\tau}^{-1}$-linear isomorphism $\sigma X\leftarrow\tau\sigma X$ which induces an isomorphism $\tau\colon H_\ell(\sigma X)\overset{\approx}{\to}H_\ell(\tau\sigma X)$. □

The "espoir" (Deligne 1979, 0.10) that every Hodge cycle is absolutely Hodge has a particularly elegant formulation in terms of motives.

CONJECTURE 6.24. *For any algebraically closed field k and embedding $\sigma\colon k\hookrightarrow\mathbb{C}$, the functor $H_\sigma\colon\mathsf{M}_k\to\mathsf{Hod}_{\mathbb{Q}}$ is fully faithful.*

The functor is obviously faithful. There is no description, not even conjectural, for the essential image of H_σ.

Motives of abelian varieties

Let $\mathsf{M}_k^{\mathrm{av}}$ be the Tannakian subcategory of M_k generated by motives of abelian varieties and Artin motives. The main theorem, 2.11, of Deligne 1982 has the following restatement.

THEOREM 6.25. *For any algebraically closed field k and embedding $\sigma\colon k\hookrightarrow\mathbb{C}$, the functor $H_\sigma\colon\mathsf{M}_k^{\mathrm{av}}\to\mathsf{Hod}_{\mathbb{Q}}$ is fully faithful.*

Therefore, for an algebraically closed k, the group $G^{\mathrm{av}}(\sigma)$ attached to $\mathsf{M}_k^{\mathrm{av}}$ and $\sigma\colon k \hookrightarrow \mathbb{C}$ is a connected pro-reductive group (see 6.22), and, for an arbitrary k, the sequence

$$1 \to G^{\mathrm{av}}(\sigma)^{\circ} \to G^{\mathrm{av}}(\sigma) \to \Gamma \to 1$$

is exact (see 6.23) (here $G^{\mathrm{av}}(\sigma)^{\circ}$ is the identity component of $G^{\mathrm{av}}(\sigma)$).

PROPOSITION 6.26. *The motive $h(X) \in \mathrm{ob}(\mathsf{M}_k^{\mathrm{av}})$ if*

(a) *X is a curve;*
(b) *X is a unirational variety of dimension ≤ 3;*
(c) *X is a Fermat hypersurface;*
(d) *X is a $K3$-surface.*

Before proving this, we note the following consequence.

COROLLARY 6.27. *Every Hodge cycle on a variety that is a product of abelian varieties, zero-dimensional varieties, and varieties of type (a), (b), (c), and (d) is absolutely Hodge.*

PROOF (OF 6.26.). Cases (a) and (b) follow immediately from (6.14) and (6.15), and (c) follows by induction (on n) from (6.16). In fact, one does not need the full strength of (6.16). There is a rational map

$$\begin{array}{ccc} X_d^r \times X_d^s & \dashrightarrow & X_d^{r+s} \\ (x_0:\ldots:x_{r+1}),(y_0:\ldots:y_{s+1}) & \longmapsto & (x_0y_{s+1}:\ldots:x_ry_{s+1}:\varepsilon x_{r+1}y_0:\ldots:\varepsilon x_{r+1}y_s) \end{array}$$

where ε is a primitive $2m$th root of 1. The map is not defined on the subvariety

$$Y\colon x_{r+1} = y_{s+1} = 0.$$

On blowing up $X_d^r \times X_d^s$ along the nonsingular centre Y, one obtains maps

$$\begin{tikzcd} Z_d^{r,s} \arrow[d] \arrow[dr] & \\ X_d^r \times X_d^s \arrow[r, dashed] & X_d^{r+s}. \end{tikzcd}$$

By induction, we can assume that the motives of X_d^r, X_d^s, and Y $(= X_d^{r-1} \times X_d^{s-1})$ are in $\mathsf{M}_k^{\mathrm{av}}$. Corollary (6.12) now shows that $h(Z_d^{r,s}) \in \mathrm{ob}(\mathsf{M}_k^{\mathrm{av}})$ and (6.9) that $h(X_d^{r+s}) \in \mathrm{ob}(\mathsf{M}_k^{\mathrm{av}})$.

For (d), we first note that the proposition is obvious if X is a Kummer surface, for then $X = \tilde{A}/\langle\sigma\rangle$ where $\tilde{A}$ is an abelian variety A with its 16 points of order ≤ 2 blown up and σ induces $a \mapsto -a$ on A.

Next consider an arbitrary $K3$-surface X, and fix a projective embedding of X. Then

$$h(X) = h(\mathbb{P}^2) \oplus h^2(X)_{\mathrm{prim}}$$

and so it suffices to show that $h^2(X)_{\mathrm{prim}}$ is in $\mathsf{M}_k^{\mathrm{av}}$. We can assume $k = \mathbb{C}$. It is known (Kuga and Satake 1967; Deligne 1972, 6.5) that there is a smooth connected variety S over $\mathbb{C}$ and families

$$f\colon Y \to S$$
$$a\colon A \to S$$

of polarized $K3$-surfaces and abelian varieties respectively parametrized by S having the following properties:

(a) for some $0 \in S$, $Y_0 \overset{\text{def}}{=} f^{-1}(0)$ is X together with its given polarization;
(b) for some $1 \in S$, Y_1 is a polarized Kummer surface;
(c) there is an inclusion $u\colon R^2 f_*\mathbb{Q}(1)_{\text{prim}} \hookrightarrow \underline{\text{End}}(R^1 a_*\mathbb{Q})$ compatible with the Hodge filtrations.

The map $u_0\colon H^2_{\text{B}}(X)(1)_{\text{prim}} \hookrightarrow \text{End}(H^1(A_0, \mathbb{Q}))$ is therefore defined by a Hodge cycle, and it remains to show that it is defined by an absolute Hodge cycle. But the initial remark shows that u_1, being a Hodge cycle on a product of Kummer and abelian surfaces, is absolutely Hodge, and Principle B (2.12 of Deligne 1982) completes the proof. □

Motives of abelian varieties of potential CM-type

An abelian variety A over k is said to be of ***potential CM-type*** if it becomes of CM-type over an extension of k. Let A be such an abelian variety defined over $\mathbb{Q}$, and let $\text{MT}(A)$ be the Mumford-Tate group of $A_{\mathbb{C}}$ (Deligne 1982, §5). Since $A_{\mathbb{C}}$ is of CM-type, $\text{MT}(A)$ is a torus, and we let $L \subset \mathbb{C}$ be a finite Galois extension of $\mathbb{Q}$ splitting $\text{MT}(A)$ and such that all the torsion points on A have coordinates in L^{ab}.[26] Let $\mathsf{M}_{\mathbb{Q}}^{A,L}$ be the Tannakian subcategory of $\mathsf{M}_{\mathbb{Q}}$ generated by A, the Tate motive, and the Artin motives split by L^{ab}, and let G^A be the affine group scheme associated with this Tannakian category and the fibre functor H_{B}.

PROPOSITION 6.28. *There is an exact sequence of affine group schemes*

$$1 \to \text{MT}(A) \xrightarrow{i} G^A \xrightarrow{\pi} \text{Gal}(L^{ab}/\mathbb{Q}) \to 1.$$

PROOF. Let $\mathsf{M}_{\mathbb{C}}^{A}$ be the image of $\mathsf{M}_{\mathbb{Q}}^{A,L}$ in $\mathsf{M}_{\mathbb{C}}$; then $\text{MT}(A)$ is the affine group scheme associated with $\mathsf{M}_{\mathbb{C}}^{A}$, and so the above sequence is a subsequence of the sequence in (6.23a). □

REMARK 6.29. If we identify $\text{MT}(A)$ with a subgroup of $\text{Aut}(H^1_{\text{B}}(A))$, then (as in 6.23a) $\pi^{-1}(\tau)$ becomes identified with the $\text{MT}(A)$-torsor whose R-points, for any $\mathbb{Q}$-algebra R, are the R-linear homomorphisms $a\colon H^1(A_{\mathbb{C}}, R) \to H^1(\tau A_{\mathbb{C}}, R)$ such that $a(s) = \tau s$ for all (absolute) Hodge cycles on $A_{\bar{\mathbb{Q}}}$. We can also identify $\text{MT}(A)$ with a subgroup of $\text{Aut}(H_1^B(A))$ and then it becomes more natural to identify $\pi^{-1}(\tau)$ with the torsor of R-linear isomorphisms $a^\vee\colon H_1(A_{\mathbb{C}}, R) \to H_1(\tau A_{\mathbb{C}}, R)$ preserving Hodge cycles.

On passing to the inverse limit over all A and L, we obtain an exact sequence

$$1 \to S^\circ \to S \to \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1$$

with S° and S respectively the connected Serre group and the Serre group. This sequence plays an important role in Articles III, IV, and V of Deligne et al. 1982.

Appendix: Terminology from nonabelian cohomology

We review some definitions from Giraud 1971.

[26] This condition was omitted in the original.

Fibred categories

Let $\alpha\colon \mathsf{F} \to \mathsf{A}$ be a functor. For an object U of A, we write F_U for the category whose objects are those in F in F such that $\alpha(F) = U$ and whose morphisms are those f such that $\alpha(f) = \mathrm{id}_U$. For any morphism $a\colon \alpha(F_1) \to \alpha(F_2)$, we write $\mathrm{Hom}_a(F_1, F_2)$ for the set of $f\colon F_1 \to F_2$ such that $\alpha(f) = a$. A morphism $f\colon F_1 \to F_2$ is said to be ***cartesian,*** and F_1 is said to be an ***inverse image*** $\alpha(f)^* F_2$ of F_2 relative to $\alpha(f)$ if, for every $F' \in \mathrm{ob}\,\mathsf{F}_{\alpha(F_1)}$ and $h \in \mathrm{Hom}_{\alpha(f)}(F', F_2)$, there exists a unique $g \in \mathrm{Hom}_{\mathrm{id}}(F', F_1)$ such that $f \circ g = h$:

$$\begin{tikzcd} F' \arrow[d, dashed, "g"'] \arrow[dr, "h"] & \\ F_1 \arrow[r, "f"] & F_2 \end{tikzcd}$$

$$\alpha(F_1) \xrightarrow{\alpha(f)} \alpha(F_2).$$

In other words,

$$\mathrm{Hom}_{\mathrm{id}}(F', F_1) \simeq \mathrm{Hom}_{\alpha(f)}(F', F_2)$$

for all F' lying over $\alpha(F_1)$.

DEFINITION. The functor $\alpha\colon \mathsf{F} \to \mathsf{A}$ is a ***fibred category*** if

(a) (existence of inverse images) for every morphism $a\colon V \to U$ in A and $F \in \mathrm{ob}(\mathsf{F}_V)$, an inverse image $a^* F$ of F exists;

(b) (transitivity of inverse images) the composite of two cartesian morphisms is cartesian.

In a fibred category, a^* can be made into a functor $\mathsf{F}_U \to \mathsf{F}_V$, and for every pair a, b of composable morphisms in A, $(a \circ b)^* \simeq b^* \circ a^*$.

Let $\alpha\colon \mathsf{F} \to \mathsf{A}$ and $\alpha'\colon \mathsf{F}' \to \mathsf{A}$ be fibred categories over A. A functor $\beta\colon \mathsf{F} \to \mathsf{F}'$ such that $\alpha' \circ \beta = \alpha$ is said to be ***cartesian*** if it maps cartesian morphisms to cartesian morphisms (in other words, it preserves inverse images).

Stacks (Champs)

Let S be the spectrum of a ring R, and let Aff_S be the category of affine schemes over S endowed with the fpqc topology (that for which the coverings are finite surjective families of flat morphisms $U_i \to U$).

Let $a\colon V \to U$ be a faithfully flat morphism of affine S-schemes, and let $F \in \mathrm{ob}(\mathsf{F}_U)$. A ***descent datum*** on F relative to a is an isomorphism

$$\phi\colon p_1^*(F) \to p_2^*(F)$$

satisfying the "cocycle" condition

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi)$$

where p_1, p_2 are the projections $V \times_U V \to V$ and the p_{ij} are the projections $V \times_U V \times_U V \to V$. With the obvious notion of morphism, the pairs (F, ϕ) form a category $\mathsf{Desc}(V/U)$. There is a functor $\mathsf{F}_U \to \mathsf{Desc}(V/U)$ under which an object F in F_U maps to $(a^* F, \phi)$ with ϕ the canonical isomorphism

$$p_1^*(a^* F) \simeq (a \circ p_1)^* F = (a \circ p_2)^* F \simeq p_2^*(a^* F).$$

DEFINITION. A ***stack*** is a fibred category $\alpha\colon \mathsf{F} \to \mathsf{Aff}_S$ such that, for all faithfully flat morphisms $a\colon V \to U$ in Aff_S, $\mathsf{F}_V \to \mathsf{Desc}(V/U)$ is an equivalence of categories.

Explicitly, this means the following:

(a) for an affine S-scheme U and objects F, G in F_U, the functor sending $a\colon V \to U$ to the set $\mathrm{Hom}(a^*F, a^*G)$ is a sheaf on U (for the fpqc topology);

(b) for every faithfully flat morphism $V \to U$ of affine S-schemes, descent is effective (that is, every descent datum for V/U is isomorphic to the descent datum defined by an object of F_U).

In other words, both morphisms and objects, given locally for the fpqc topology, patch to global objects.

EXAMPLE. (a) Let α:MOD$\to \mathsf{Aff}_S$ be the fibred category such that MOD_U is the category of finitely presented $\Gamma(U, \mathcal{O}_U)$-modules. Descent theory shows that this is a stack (Waterhouse 1979, 17.2; Bourbaki, Algèbre Commutative, I, 3.6).

(b) Let α:PROJ$\to \mathsf{Aff}_S$ be the fibred category such that PROJ_U is the category of finitely generated projective $\Gamma(U, \mathcal{O}_U)$. Descent theory again shows this to be a stack (ibid.).

(c) There is a stack AFF$\to \mathsf{Aff}_S$ for which $\mathrm{AFF}_T = \mathsf{Aff}_T$.

Gerbes

DEFINITION. A ***gerbe*** over S is a stack $\mathsf{G} \to \mathsf{Aff}_S$ such that

(a) in every category G_U, all morphisms are isomorphisms;

(b) there exists a faithfully flat morphism $U \to S$ such that G_U is nonempty;

(c) any two objects of a fibre G_U are locally isomorphic (i.e., their inverse images under some faithfully flat morphism $V \to U$ of affine S-schemes are isomorphic).

A ***morphism of gerbes*** over S is a cartesian functor, and an ***isomorphism of gerbes*** over S is a cartesian functor that is an equivalence of categories. A gerbe G$\to \mathsf{Aff}_S$ is ***neutral*** if G_S is nonempty.

EXAMPLE. Let F be a sheaf of groups on S (for the fpqc topology). The fibred category $\mathrm{TORS}(F) \to \mathsf{Aff}_S$ for which $\mathrm{TORS}(F)_U$ is the category of right F-torsors on U is a neutral gerbe. Conversely, let G be a neutral gerbe, and let $Q \in \mathrm{ob}(\mathsf{G}_S)$. If $F = \underline{\mathrm{Aut}}(Q)$ is a sheaf of commutative groups on S, then, for any $a\colon U \to S$ and $P \in \mathrm{ob}(\mathsf{G}_U)$, $\underline{\mathrm{Isom}}(a^*Q, a^*P)$ is an F-torsor, and the functor

$$P \rightsquigarrow \underline{\mathrm{Isom}}_U(a^*Q, a^*P)\colon \mathsf{G} \to \mathrm{TORS}(F)$$

is an isomorphism of gerbes.

Bands (Liens)

Let G be a gerbe over S. For $U \to S$ in Aff_S and x in G_U, the presheaf

$$V \rightsquigarrow \underline{\mathrm{Aut}}(x)(V) \stackrel{\text{def}}{=} \mathrm{Aut}(x|V)$$

is a sheaf of groups over U for the fpqc topology (because G is a stack). If x and y are isomorphic objects of G_U, then we have an isomorphism $\underline{\mathrm{Aut}}(x) \to \underline{\mathrm{Aut}}(y)$ unique up to composition with an inner automorphism. We are led to consider the category LI_U

whose objects are sheaves of groups on U, the morphisms $F \to G$ being the sections of the quotient sheaf

$$G\backslash\underline{\mathrm{Hom}}(F,G)/F$$

where F and G act by inner automorphisms. In this way, we get a fibred category LI $\to$ Aff_S in which the morphisms patch. A standard procedure allows us to add objects to the categories LI_U to obtain a fibred category LIEN $\to \mathsf{Aff}_S$ in which the objects also patch, i.e., which is a stack. An object of LIEN_U is called a ***band*** (lien) over U. By the above discussion, a gerbe G over S defines, up to a *unique* isomorphism, a band B over S. We say that G is ***banded*** by B, or that it is a B-gerbe.

We make this more explicit. Let F and G be sheaves of groups for the fpqc topology on S, and let G^{ad} be the quotient sheaf G/Z where Z is the centre of G. The action of G^{ad} on G induces an action of G^{ad} on the sheaf $\underline{\mathrm{Isom}}(F,G)$, and we set

$$\mathrm{Isex}(F,G) = \Gamma\left(S, G^{\mathrm{ad}}\backslash\underline{\mathrm{Isom}}(F,G)\right).$$

As G^{ad} acts faithfully on $\underline{\mathrm{Isom}}(F,G)$,

$$\mathrm{Isex}(F,G) = \varinjlim \mathrm{Ker}(G^{\mathrm{ad}}(T)\backslash\mathrm{Isom}(F|T,G|T) \rightrightarrows G^{\mathrm{ad}}(T\times T)\backslash\mathrm{Isom}(F|(T\times T),G|(T\times T))$$

where the limit is over all $T \to S$ faithfully flat and affine.

Every band B over S is defined by a triple (S',G,ϕ) where S' is an affine S-scheme, faithfully flat over S, G is a sheaf of groups on S', and $\phi \in \mathrm{Isex}(p_1^*G, p_2^*G)$ is such that

$$p_{31}^*(\phi) = p_{32}^*(\phi)\circ p_{21}^*(\phi).$$

(As before, the p_i and p_{ij} are the various projection maps $S'' \to S$ and $S''' \to S''$). If T is also a faithfully flat affine S-scheme, and $a\colon T \to S'$ is an S-morphism, then (S',G,ϕ) and $(T,a^*(G),(a\times a)^*(\phi))$ define the same band. If B_1 and B_2 are the bands defined by (S',G_1,ϕ_1) and (S',G_2,ϕ_2), then an element $\psi \in \mathrm{Isex}(G_1,G_2)$ such that $p_2^*(\psi)\circ\phi_1 = \phi_2\circ p_1^*(\psi)$ defines an isomorphism $B_1 \to B_2$.

When G is a sheaf of groups on S, we write $B(G)$ for the band defined by (S,G,id). One shows that

$$\mathrm{Isom}(B(G_1),B(G_2)) = \mathrm{Isex}(G_1,G_2).$$

Thus, $B(G_1)$ and $B(G_2)$ are isomorphic if and only if G_2 is an inner form of G_1, i.e., G_2 becomes isomorphic to G_1 on some faithfully flat S-scheme T, and the class of G_2 in $H^1(S,\underline{\mathrm{Aut}}(G_1))$ comes from $H^1(S,G_1^{\mathrm{ad}})$. When G_2 is commutative, then

$$\mathrm{Isom}(B(G_1),B(G_2)) = \mathrm{Isex}(G_1,G_2) = \mathrm{Isom}(G_1,G_2),$$

and we usually do not distinguish $B(G_2)$ from G_2.

The ***centre*** $Z(B)$ of the band B defined by (S',G,ϕ) is defined by $(S',Z,\phi|p_1^*Z)$ where Z is the centre of G. The above remark shows that $\phi|p_1^*Z$ lifts to an element $\phi_1 \in \mathrm{Isom}(p_1^*Z,p_2^*Z)$, and one checks immediately that $p_{31}^*(\phi_1) = p_{32}^*(\phi_1)\circ p_{21}^*(\phi_1)$. Thus $(S',Z,\phi|p_1^*Z)$ arises from a sheaf of groups on S, which we identify with $Z(B)$.

Let G be a gerbe on Aff_S. By (b) of the definition, there exists an object $Q \in \mathsf{G}_S$ for some $S' \to S$ faithfully flat and affine. Let $G = \underline{\mathrm{Aut}}(Q)$; it is a sheaf of groups on S'. Again, by definition, p_1^*Q and p_2^*Q are locally isomorphic on S'', and the locally-defined isomorphisms determine an element $\phi \in \mathrm{Isex}(p_1^*G,p_2^*G)$. The triple (S',G,ϕ) defines a band B which is uniquely determined up to a unique isomorphism. This is the band attached to the gerbe G.

A band B is said to be ***affine*** (resp. ***algebraic***) if it can be defined by a triple (S',G,ϕ) with G an affine (resp. algebraic) group scheme over S'. A gerbe is said to be ***affine*** (resp. ***algebraic***) if it is banded by an affine (resp. algebraic) band.

Cohomology

Let B be a band on Aff_S. Two gerbes G_1 and G_2 banded by B are said to be ***B-equivalent*** if there exists an isomorphism $m\colon \mathsf{G}_1 \to \mathsf{G}_2$ with the following property: for some triple (S',G,ϕ) defining B, there is an object $Q \in \mathsf{G}_{1S}$ such that the automorphism

$$G \simeq \underline{\mathrm{Aut}}(Q) \simeq \underline{\mathrm{Aut}}(m(Q)) \simeq G$$

defined by m is equal to id in $\mathrm{Isex}(G,G)$. The cohomology set $H^2(S,B)$ is defined to be the set of B-equivalence classes of gerbes banded by B. If Z is the centre of B, then $H^2(S,Z)$ is equal to the cohomology group of Z in the usual sense of the fpqc topology on S, and either $H^2(S,B)$ is empty or $H^2(S,Z)$ acts simply transitively on it (Giraud 1971, IV, 3.3.3).

PROPOSITION. *Let* G *be an affine algebraic gerbe over the spectrum of a field,* $S = \mathrm{Spec}\, k$*. There exists a finite field extension* k' *of* k *such that* $\mathsf{G}_{S'}$*,* $S' = \mathrm{Spec}\, k'$*, is nonempty.*

PROOF. By assumption, the band B of G is defined by a triple (S',G,ϕ) with G of finite type over S'. Let $S' = \mathrm{Spec}\, R'$; R' can be replaced by a finitely generated subalgebra, and then by a quotient modulo a maximal ideal, and so we may suppose that $S' = \mathrm{Spec}\, k'$ where k' is a finite field extension of k. We shall show that the gerbes G and $\mathrm{TORS}(G)$ become B-equivalent over some finite field extension of k'. The statement preceding the proposition shows that we have to prove that an element of $H^2(S',Z)$, Z the centre of B, is killed by a finite field extension of k'. But this assertion is obvious for elements of $H^1(S',Z)$ and is easy to prove for elements of the Čech groups $\check{H}^r(S',Z)$, and so the exact sequence

$$0 \to \check{H}^2(S',Z) \to H^2(S',Z) \to \check{H}^1(S',\mathcal{H}^1(Z))$$

completes the proof. See Saavedra Rivano 1972, III, 3.1, for more details. □

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Index of definitions