II TANNAKIAN CATEGORIES

by

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Introduction:

In the first section it is shown how to introduce on an abstract category operations of tensor products and duals having properties similar to the familiar operations on the category $\underline{\mathrm{Vec}}_k$ of finite-dimensional vector spaces over a field k. What complicates this is the necessity of including enough constraints so that, whenever an obvious isomorphism (e.g., $U \otimes (V \otimes W) \xrightarrow{\approx} (V \otimes U) \otimes W$) exists in $\underline{\mathrm{Vec}}_k$, a unique isomorphism is constrained to exist also in the abstract setting.

The next section studies the category $\underline{\mathrm{Rep}}_k(G)$ of finite-dimensional representations of an affine group scheme G over k and demonstrates necessary and sufficient conditions for a category $\underline{C}$ with a tensor product to be isomorphic to $\underline{\mathrm{Rep}}_k(G)$ for G; such a category $\underline{C}$ is then called a neutral Tannakian category.

A fibre functor on a Tannakian category $\underline{C}$ with values in a field $k' \supset k$ is an exact k-linear functor $\underline{C} \to \underline{Vec}_{k'}$, that commutes with tensor products. For example, the forgetful functor is a fibre functor on $\underline{Rep}_k(G)$. In the third section it is shown that fibre functors on $\underline{Rep}_k(G)$ are in one-to-one correspondence with the torsors of G. Also, the notion of a (non-neutral) Tannakian category as introduced.

The fourth section studies the notion of a polarization (compatible families of sesquilinear forms having certain positivity properties) on a Tannakian category, and the fifth studies the notion of a graded Tannakian category.

In the sixth section, motives are defined using absolute Hodge cycles, and the related motivic Galois groups discussed. In an appendix, some terminology from non-abelian cohomology is reviewed.

We note that the introduction of Saavedra [1] is an excellent summary of Tannakian categories, except that two changes are necessary: Théorème 3 is, unfortunately, only a conjecture; in Théorème 4 the requirement that G be abelian or connected can be dropped.

Notations: Functors between additive categories are assumed to be additive. In general, rings are commutative with 1 except in §2. A morphism of functors is also called a functorial or natural morphism. A strictly full subcategory is a full subcategory containing with any X, all objects isomorphic to X. The empty set is denoted by Φ.

Our notations agree with those of Saavedra [1] except that we have made some simplifications: What would be called a $\otimes$-widget ACU by Saavedra, here becomes a tensor widget, and $\underline{\mathrm{Hom}}^{\otimes,1}$ becomes $\underline{\mathrm{Hom}}^{\otimes}$.

$\underline{\mathrm{Vec}}_k$: Category of finite-dimensional vector spaces over k ;

$\underline{\mathrm{Rep}}_k(G)$: Category of finite-dimensional representation of G over k ;

$\underline{\mathrm{Mod}}_R$: Category of finitely generated R-modules;

$\underline{\mathrm{Proj}}_R$: Category of finitely generated projective R-modules;

$\underline{\mathrm{Set}}$: Category of sets

§1. Tensor categories

Let $\underline{C}$ be a category and

$$\otimes: \underline{C} \times \underline{C} \to \underline{C}, \quad (X,Y) \to X \otimes Y$$

a functor. An <u>associativity constraint</u> for $(\underline{C},\otimes)$ is a functorial isomorphism

$$\phi_{X,Y,Z}: X \otimes (Y \otimes Z) \xrightarrow{\approx} (X \otimes Y) \otimes Z$$

such that, for all objects X,Y,Z,T, the diagram

$$\begin{tikzcd}
X \otimes (Y\otimes(Z\otimes T)) \arrow[r, "\phi"] \arrow[d, "1 \otimes \phi"] & (X\otimes Y) \otimes (Z\otimes T) \arrow[r, "\phi"] & ((X\otimes Y)\otimes Z) \otimes T \\
X \otimes ((Y\otimes Z)\otimes T) \arrow[rr, "\phi"] & & (X\otimes(Y\otimes Z)) \otimes T \arrow[u, "\phi \otimes 1"']
\end{tikzcd} \tag{1.0.1}$$

is commutative (this is the pentagon axiom). Here, as in subsequent diagrams, we have omitted the obvious subscripts on the maps; for example, the ϕ at top-left is $\phi_{X,Y,Z\otimes T}$. A <u>commutativity constraint</u> for $(\underline{C},\otimes)$ is a functorial isomorphism

$$\psi_{X,Y}: X \otimes Y \xrightarrow{\approx} Y \otimes X$$

such that, for all objects X,Y, $\psi_{Y,X} \circ \psi_{X,Y} = id_{X\otimes Y}: X \otimes Y \to X \otimes Y$. An associativity constraint ϕ and a commutativity constraint ψ are compatible if, for all objects X,Y,Z, the diagram

$$\begin{tikzcd}
X \otimes (Y\otimes Z) \arrow[r, "\phi"] \arrow[d, "1 \otimes \psi"] & (X\otimes Y) \otimes Z \arrow[r, "\psi"] & Z \otimes (X\otimes Y) \arrow[d, "\phi"] \\
X \otimes (Z\otimes Y) \arrow[r, "\phi"] & (X\otimes Z) \otimes Y \arrow[r, "\psi\otimes 1"] & (Z\otimes X) \otimes Y
\end{tikzcd} \tag{1.0.2}$$

is commutative (hexagon axiom). A pair (U,u) comprising an object U of $\underline{C}$ and an isomorphism $u: U \to U \otimes U$ is an <u>identity object</u> of $(\underline{C},\otimes)$ if $X \longmapsto U \otimes X: \underline{C} \to \underline{C}$ is an equivalence of categories.

<u>Definition</u> 1.1. A system $(\underline{C},\otimes,\phi,\psi)$, in which ϕ and ψ are compatible associativity and commutativity constraints, is a <u>tensor category</u> if it has an identity object.

<u>Example</u> 1.2. The category $\underline{\text{Mod}}_R$ of finitely generated modules over a commutative ring R becomes a tensor category with the usual tensor product and the obvious constraints. (If one perversely takes ϕ to be the negative of the obvious isomorphism, then the pentagon (1.0.1) fails to commute by a sign.) A pair (U,u_0) comprising a free R-module U of rank 1 and a basis element u_0 determines an identity object (U,u) of $\underline{\text{Mod}}_R$ — take u to be the unique isomorphism $U \to U \otimes U$ mapping u_0 to $u_0 \otimes u_0$. Every identity element is of this form.

(For other examples, see the end of this section.)

<u>Proposition</u> 1.3. Let (U,u) be an identity object of the tensor category $(\underline{C},\otimes)$.

(a) There exists a unique functorial isomorphism

$$\ell_X: X \xrightarrow{\approx} U \otimes X$$

such that ℓ_U is u and the diagrams

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\ell"] \arrow[d, equal] & U \otimes (X\otimes Y) \arrow[d, "\phi"] \\
X \otimes Y \arrow[r, "\ell\otimes 1"] & (U\otimes X) \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "\ell\otimes 1"] \arrow[d, "1 \otimes \ell"] & (U\otimes X) \otimes Y \arrow[d, "\psi \otimes 1"] \\
X \otimes (U\otimes Y) \arrow[r, "\phi"] & (X\otimes U) \otimes Y
\end{tikzcd}$$

are commutative.

(b) If (U',u') is a second identity object of $(\underline{C},\otimes)$ then there is a unique isomorphism $a: U \xrightarrow{\approx} U'$ making

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "a"] & U \otimes U \arrow[d, "a \otimes a"] \\ U' \arrow[r, "u'"] & U' \otimes U' \end{tikzcd}$$

commute

<u>Proof</u> (a) We confine ourselves to defining ℓ_X . (See Saavedra [1,I2.2.5.1,2.4.1] for details.) As $X \longmapsto U \otimes X$ is an equivalence of categories, it suffices to define $1 \otimes \ell_X$: $U \otimes X \to U \otimes (U \otimes X)$; this we take to be

$$U \otimes X \xrightarrow{u \otimes 1} (U \otimes U) \otimes X \xrightarrow{\phi^{-1}} U \otimes (U \otimes X) .$$

(b) The map $U \xrightarrow{\ell_U} U' \otimes U \xrightarrow{\psi} U \otimes U' \xrightarrow{\ell_{U'}^{-1}} U'$ has the required properties.

The functorial isomorphism $r_X \overset{\text{df}}{=} \psi_{U,X} \circ \ell_X : X \to X \otimes U$ has analogous properties to ℓ_X . We shall often use $(\underline{1},e)$ to denote a (the) identity object of $(\underline{C},\otimes)$.

<u>Remark</u> 1.4. Our notion of a tensor category is the same as that of a "$\otimes$-catégorie AC unifère" in (Saavedra [1]) and, because of (1.3), is essentially the same as the notion of a "$\otimes$-catégorie ACU" defined in (Saavedra [1, I.2.4.1]) (cf. Saavedra [1,I.2.4.3]).

<u>Extending</u> $\otimes$

Let ϕ be an associativity constraint for $(\underline{C},\otimes)$. Any functor $\underline{C}^n \to \underline{C}$ defined by repeated application of $\otimes$ is called an <u>iterate</u> of $\otimes$. If $F, F' : \underline{C}^n \to \underline{C}$ are iterates of $\otimes$, then it is possible to construct an isomorphism of functors $\tau: F \xrightarrow{\approx} F'$ out of ϕ and ϕ^{-1} . The significance of the pentagon axiom is that it implies that τ is unique: any two iterates of $\otimes$ to C^n are isomorphic by a unique isomorphism of functors constructed out of ϕ and ϕ^{-1} (MacLane [1], [2,VII.2]). In other words, there is an essentially unique way of extending $\otimes$ to a functor $\bigotimes_{i=1}^{n} : \underline{C}^n \to \underline{C}$ when $n \geq 1$. Similarly, if $(\underline{C}, \otimes)$ is a tensor category, then it is possible to extend $\otimes$ in essentially one way to a functor $\bigotimes_{i \in I} : \underline{C}^I \to \underline{C}$ where I is any finite set: the tensor product of any finite family of objects of $\underline{C}$ is well-defined up to a unique isomorphism (MacLane [1]). We can make this more precise.

<u>Proposition</u> 1.5. The tensor structure on a tensor category $(\underline{C}, \otimes)$ can be extended as follows. For each finite set I there is to be a functor

$$\bigotimes_{i \in I} : \underline{C}^I \to \underline{C} ,$$

and for each map $\alpha: I \to J$ of finite sets there is to be a functorial isomorphism

$$\chi(\alpha): \bigotimes_{i \in I} X_i \xrightarrow{\approx} \bigotimes_{j \in J} \left(\bigotimes_{i \mapsto j} X_i \right)$$

satisfying the following conditions:

(a) if I consists of a single element, then $\bigotimes_{i \in I}$ is the identity functor $X \longmapsto X$; if α is a map between single-element sets, then $\chi(\alpha)$ is the identity automorphism of the identity functor;

(b) the isomorphisms defined by maps $I \xrightarrow{\alpha} J \xrightarrow{\beta} K$ give rise to a commutative diagram

$$\begin{array}{ccc}
\bigotimes_{i\in I} X_i & \xrightarrow{\quad\chi(\alpha)\quad} & \bigotimes_{j\in J}\left(\bigotimes_{i\mapsto j} X_i\right) \\
\Big\downarrow{\scriptstyle \chi(\beta\alpha)} & & \Big\downarrow{\scriptstyle \chi(\beta)} \\
\bigotimes_{k\in K}\left(\bigotimes_{i\mapsto k} X_i\right) & \xrightarrow{\quad\otimes(\chi(\alpha|I_k))\quad} & \bigotimes_{k\in K}\left(\bigotimes_{j\mapsto k}\left(\bigotimes_{i\mapsto j} X_i\right)\right) .
\end{array}$$

where $I_k = (\beta\alpha)^{-1}(k)$.

Proof: Omitted.

By $(\bigotimes_{i\in I}, \chi)$ being an extension of the tensor structure on $\underline{C}$ we mean that $\bigotimes_{i\in I} X_i = X_1 \otimes X_2$ when $I = \{1,2\}$ and that the isomorphisms $X \otimes (Y\otimes Z) \longrightarrow (X\otimes Y) \otimes Z$ and $X \otimes Y \longrightarrow Y \otimes X$ induced by χ are equal to ϕ and ψ respectively. It is automatic that $(\bigotimes_{\Phi} X_i, \chi(\Phi \to \{1,2\}))$ is an identity object and that $\chi(\{2\} \hookrightarrow \{1,2\})$ is $\ell_X\colon\ X \to \underline{1} \otimes X$. If $(\bigotimes'_{i\in I}, \chi')$ is a second such extension, then

there is a unique system of isomorphisms $\bigotimes_{i\in I} X_i \to \bigotimes'_{i\in I} X_i$ compatible with χ and χ' and such that, when $I = \{i\}$, the isomorphism is id_{X_i} .

When a tensor category $(\underline{C},\otimes)$ is given, we shall always assume that an extension as in (1.5) has been made. (We could, in fact, have defined a tensor category to be a system as in (1.5).)

Invertible objects

Let $(\underline{C},\otimes)$ be a tensor category. An object L of $\underline{C}$ is invertible if $X \longmapsto L \otimes X\colon \underline{C} \to \underline{C}$ is an equivalence of categories. Thus, if L is invertible, there exists an L' such that $L \otimes L' \approx \underline{1}$; the converse assertion is also true. An inverse of L is a pair (L^{-1},δ) where $\delta\colon \bigotimes_{i\in\{\pm\}} X_i \xrightarrow{\approx} \underline{1}$, $X_+ = L$, $X_- = L^{-1}$. Note that this definition is symmetric: (L,δ) is an inverse of L^{-1}. If (L_1,δ_1) and (L_2,δ_2) are both inverses of L, then there is a unique isomorphism $\alpha\colon L_1 \xrightarrow{\approx} L_2$ such that $\delta_1 = \delta_2 \circ (1\otimes\alpha)\colon L \otimes L_1 \to L \otimes L_2 \to \underline{1}$.

An object L of $\underline{\mathrm{Mod}}_R$ is invertible if and only if it is projective of rank 1. (Saavedra [1,0.2.2.2]).

Internal Hom

Let $(\underline{C},\otimes)$ be a tensor category.

Definition 1.6. If the functor $T \longmapsto \mathrm{Hom}(T\otimes X,Y)\colon \underline{C}^\circ \to \underline{\mathrm{Set}}$ is representable, then we denote by $\underline{\mathrm{Hom}}(X,Y)$ the representing object and by $ev_{X,Y}\colon \underline{\mathrm{Hom}}(X,Y) \otimes X \to Y$ the morphism corresponding to $id_{\underline{\mathrm{Hom}}(X,Y)}$.

Thus, to a g there corresponds a unique f such that $ev \circ (f\otimes id) = g$:

$$\begin{tikzcd} T \arrow[d, dashed, "f"] \\ \underline{\mathrm{Hom}}(X,Y) \end{tikzcd} \qquad \begin{tikzcd} T \otimes X \arrow[d, dashed, "f\otimes id"] \arrow[dr, "g"] & \\ \underline{\mathrm{Hom}}(X,Y) \otimes X \arrow[r, "ev"] & Y \end{tikzcd} \tag{1.6.1}$$

For example, in $\underline{\mathrm{Mod}}_R$, $\underline{\mathrm{Hom}}(X,Y) = \mathrm{Hom}_R(X,Y)$ regarded as an R-module, and ev is $f \otimes x \longmapsto f(x)$, whence its name.

Assume that in $(\underline{C},\otimes)$, $\underline{\mathrm{Hom}}(X,Y)$ exists for every pair (X,Y) . Then there is a composition map

$$\underline{\mathrm{Hom}}(X,Y) \otimes \underline{\mathrm{Hom}}(Y,Z) \to \underline{\mathrm{Hom}}(X,Z) \tag{1.6.2}$$

(corresponding to $\underline{\mathrm{Hom}}(X,Y) \otimes \underline{\mathrm{Hom}}(Y,Z) \otimes X \xrightarrow{ev} \underline{\mathrm{Hom}}(Y,Z) \otimes Y \xrightarrow{ev} Z$) and an isomorphism

$$\underline{\mathrm{Hom}}(Z,\underline{\mathrm{Hom}}(X,Y)) \xrightarrow{\approx} \underline{\mathrm{Hom}}(Z\otimes X,Y) \tag{1.6.3}$$

(inducing, for any object T,

$$\mathrm{Hom}(T,\underline{\mathrm{Hom}}(Z,\underline{\mathrm{Hom}}(X,Y))) \xrightarrow{\approx} \mathrm{Hom}(T\otimes Z,\underline{\mathrm{Hom}}(X,Y)) \xrightarrow{\approx} \mathrm{Hom}(T\otimes Z\otimes X,Y)$$
$$\xrightarrow{\approx} \mathrm{Hom}(T,\underline{\mathrm{Hom}}(Z\otimes X,Y)))$$

Note that

$$\mathrm{Hom}(\underline{1},\underline{\mathrm{Hom}}(X,Y)) = \mathrm{Hom}(\underline{1}\otimes X,Y) = \mathrm{Hom}(X,Y) \tag{1.6.4}$$

The <u>dual</u> $X^\vee$ of an object X is defined to be $\underline{\mathrm{Hom}}(X,\underline{1})$. There is therefore a map $ev_X\colon\ X^\vee \otimes X \to \underline{1}$ inducing a functorial isomorphism

$$\mathrm{Hom}(T,X^\vee) \xrightarrow{\approx} \mathrm{Hom}(T\otimes X,\underline{1}) \tag{1.6.5}$$

The map $X \longmapsto X^\vee$ can be made into a contravariant functor: to $f\colon\ X \to Y$ we associate the unique map ${}^tf\colon\ Y^\vee \to X^\vee$ rendering commutative

$$\begin{array}{ccc} Y^\vee \otimes X & \xrightarrow{{}^t f \otimes \mathrm{id}} & X^\vee \otimes X \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes f} & & \Big\downarrow{\scriptstyle ev_X} \\ Y^\vee \otimes Y & \xrightarrow{ev_Y} & \underline{1} \end{array} \tag{1.6.6}$$

For example, in $\underline{\mathrm{Mod}}_R$, $X^\vee = \mathrm{Hom}_R(X,R)$ and ${}^t f$ is determined by the equation $< {}^t f(y), x >_X = < y, f(x) >_Y$, $y \in Y^\vee$, $x \in X$, where we have written $< \, , \, >_X$ and $< \, , >_Y$ for ev_X and ev_Y .

If f is an isomorphism, we let $f^\vee = ({}^t f)^{-1}$: $X^\vee \to Y^\vee$, so that

$$ev_Y \circ (f^\vee \otimes f) = ev_X: \; X^\vee \otimes X \to \underline{1} \, . \tag{1.6.7}$$

(E.g. in $\underline{\mathrm{Mod}}_R$, $< f^\vee(x'), f(x) >_Y = < x', x >_X$, $x' \in X^\vee, x \in X$.)

Let i_X: $X \to X^{\vee\vee}$ be the map corresponding in (1.6.5) to $ev_X \circ \psi$: $X \otimes X^\vee \to \underline{1}$. If i_X is an isomorphism then X is said to be <u>reflexive</u>. If X has an inverse $(X^{-1}, \delta$: $X^{-1} \otimes X \xrightarrow{\approx} 1)$ then X is reflexive and δ determines an isomorphism $X^{-1} \xrightarrow{\approx} X^\vee$ as in (1.6.1).

For any finite families of objects $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ there is a morphism

$$\bigotimes_{i \in I} \underline{\mathrm{Hom}}(X_i, Y_i) \to \underline{\mathrm{Hom}}(\bigotimes_{i \in I} X_i, \bigotimes_{i \in I} Y_i) \tag{1.6.8}$$

corresponding in (1.6.1) to

$$(\bigotimes_{i \in I} \underline{\mathrm{Hom}}(X_i, Y_i)) \otimes (\bigotimes_{i \in I} X_i) \xrightarrow{\approx} \bigotimes_{i \in I} (\underline{\mathrm{Hom}}(X_i, Y_i) \otimes X_i) \xrightarrow{\otimes ev} \bigotimes_{i \in I} Y_i \, .$$

In particular, there are morphisms

$$\bigotimes_{i\in I} X_i^{\vee} \longrightarrow (\bigotimes_{i\in I} X_i)^{\vee} \tag{1.6.9}$$

and

$$X^{\vee} \otimes Y \longrightarrow \underline{\mathrm{Hom}}(X,Y) \tag{1.6.10}$$

obtained respectively by taking $Y_i = \underline{1}$ all i, and $X_1 = X$, $X_2 = \underline{1} = Y_1$, $Y_2 = Y$.

<u>Rigid tensor categories</u>

<u>Definition</u> 1.7. A tensor category $(\underline{C},\otimes)$ is <u>rigid</u> if $\underline{\mathrm{Hom}}(X,Y)$ exists for all objects X and Y, the maps (1.6.8) are isomorphisms for all finite families of objects, and all objects of $\underline{C}$ are reflexive.

In fact, it suffices to require that the maps (1.6.8) be isomorphisms in the special case that $I = \{1,2\}$.

Let $(\underline{C},\otimes)$ be a rigid tensor category. The functor

$$\{X,f\} \longmapsto \{X^{\vee},{}^{t}f\}\colon \underline{C}^{0} \longrightarrow \underline{C}$$

is an equivalence of categories because its composite with itself is isomorphic to the identity functor. (It is even an equivalence of tensor categories in the sense defined below—note that $\underline{C}^{\circ}$ has an obvious tensor structure for which $\otimes X_i^{\circ} = (\otimes X_i)^{\circ}$.) In particular

$$f \longmapsto {}^{t}f\colon \quad \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(Y^{\vee},X^{\vee}) \tag{1.7.1}$$

is an isomorphism. There is also a canonical isomorphism

$$\underline{\mathrm{Hom}}(X,Y) \xrightarrow{\approx} \underline{\mathrm{Hom}}(Y^{\vee},X^{\vee}) \ , \tag{1.7.2}$$

namely $\underline{\mathrm{Hom}}(X,Y) \xleftarrow[\approx]{1.6.10} X^{\vee} \otimes Y \xrightarrow{\approx} X^{\vee} \otimes Y^{\vee\vee} \xrightarrow{\approx}$ $Y^{\vee\vee} \otimes X^{\vee} \xrightarrow[\approx]{1.6.10} \underline{\mathrm{Hom}}(Y^{\vee},X^{\vee})$.

For any object X of $\underline{C}$, there is a morphism

$$\underline{\mathrm{Hom}}(X,X) \xrightarrow{1.6.10} X^{\vee} \otimes X \xrightarrow{ev} \underline{1} \ .$$

On applying the functor $\mathrm{Hom}(\underline{1},-)$ to this we obtain (see 1.6.4)) a morphism

$$\mathrm{Tr}_X\colon \ \mathrm{End}(X) \to \mathrm{End}(\underline{1}) \tag{1.7.3}$$

called the _trace_ morphism. The _rank_, $\mathrm{rk}(X)$, of X is defined to be $\mathrm{Tr}_X(\mathrm{id}_X)$. There are the formulas (Saavedra [1,I 5.1.4]):

$$\begin{gathered} \mathrm{Tr}_{X \otimes X'}(f \otimes f') = \mathrm{Tr}_X(f)\mathrm{Tr}_{X'}(f') \\ \mathrm{Tr}_{\underline{1}}(f) = f \end{gathered} \tag{1.7.4}$$

In particular,

$$\begin{gathered} \mathrm{rk}(X \otimes X') = \mathrm{rk}(X)\mathrm{rk}(X') \\ \mathrm{rk}(\underline{1}) = \mathrm{id}_{\underline{1}} \ . \end{gathered} \tag{1.7.5}$$

Tensor functors

Let $(\underline{C},\otimes)$ and $(\underline{C}',\otimes')$ be tensor categories.

Definition 1.8. A _tensor functor_ $(\underline{C},\otimes) \to (\underline{C}',\otimes')$ is a pair (F,c) comprising a functor $F\colon \ \underline{C} \to \underline{C}'$ and a functorial isomorphism $c_{X,Y}\colon \ F(X) \otimes F(Y) \xrightarrow{\approx} F(X \otimes Y)$ with the properties:

(a) for all $X,Y,Z \in ob(\underline{C})$, the diagram

$$\begin{tikzcd} FX \otimes (FY\otimes FZ) \arrow[r, "id\otimes c"] \arrow[d, "\phi'"] & FX \otimes F(Y\otimes Z) \arrow[r, "c"] & F(X\otimes(Y\otimes Z)) \arrow[d, "F(\phi)"] \\ (FX\otimes FY) \otimes FZ \arrow[r, "c\otimes id"] & F(X\otimes Y) \otimes FZ \arrow[r, "c"] & F((X\otimes Y)\otimes Z) \end{tikzcd}$$

is commutative;

(b) for all $X,Y \in ob(\underline{C})$, the diagram

$$\begin{tikzcd} FX \otimes FY \arrow[r, "c"] \arrow[d, "\psi'"] & F(X\otimes Y) \arrow[d, "F(\psi)"] \\ FY \otimes FX \arrow[r, "c"] & F(Y\otimes X) \end{tikzcd}$$

is commutative;

(c) if (U,u) is an identity object of $\underline{C}$ then $(F(U),F(u))$ is an identity object of $\underline{C}'$.

In (Saavedra [1,I4.2.4]) a tensor functor is called a "$\otimes$-foncteur ACU".

Let (F,c) be a tensor functor $\underline{C} \to \underline{C}'$. The conditions (a), (b), (c) imply that, for any finite family $(X_i)_{i\in I}$ of objects of $\underline{C}$, c gives rise to a well-defined isomorphism

$$c: \bigotimes_{i\in I} F(X_i) \xrightarrow{\approx} F(\bigotimes_{i\in I} X_i) ;$$

moreover, for any map $\alpha: I \to J$, the diagram

$$\begin{array}{ccccc}
\bigotimes_{i\in I} F(X_i) & & \xrightarrow[\approx]{c} & & F(\bigotimes_{i\in I} X_i) \\
\approx \Big\downarrow \chi'(\alpha) & & & & \approx \Big\downarrow F(\chi(\alpha)) \\
\bigotimes_{j\in J} (\bigotimes_{i\mapsto j} F(X_i)) & \xrightarrow[\approx]{c} & \bigotimes_{j\in J} (F(\bigotimes_{i\mapsto j} X_i)) & \xrightarrow[\approx]{c} & F(\bigotimes_{j\in J} (\bigotimes_{i\mapsto j} X_i))
\end{array}$$

is commutative. In particular, (F,c) maps inverse objects to inverse objects. Also, the morphism $F(ev): F(\underline{Hom}(X,Y)) \otimes F(X) \to F(Y)$ gives rise to morphisms $F_{X,Y}: F(\underline{Hom}(X,Y)) \to \underline{Hom}(FX,FY)$ and $F_X: F(X^\vee) \to F(X)^\vee$.

<u>Proposition 1.9</u>. Let $(F,c): \underline{C} \to \underline{C}'$ be a tensor functor. If $\underline{C}$ and $\underline{C}'$ are rigid, then $F_{X,Y}: F(\underline{Hom}(X,Y)) \to \underline{Hom}(FX,FY)$ is an isomorphism for all $X,Y \in ob(\underline{C})$.

<u>Proof</u>: It suffices to show that F preserves duality, but this is obvious from the following characterization of the dual of X : it is a pair $(Y, Y\otimes X \xrightarrow{ev} \underline{1})$, for which there exists $\varepsilon : \underline{1} \to X \otimes Y$ such that $X = \underline{1} \otimes X \xrightarrow{\varepsilon\otimes id} (X\otimes Y)\otimes X = X\otimes(Y\otimes X) \xrightarrow{id\otimes ev} X$, and the same map with X and Y interchanged, are identity maps.

<u>Definition</u> 1.10. A tensor functor $(F,c) : \underline{C} \to \underline{C}'$ is a <u>tensor equivalence</u> (or an <u>equivalence of tensor categories</u>) if $F : \underline{C} \to \underline{C}'$ is an equivalence of categories.

The definition is justified by the following proposition.

Proposition 1.11. Let $(F,c)\colon \underline{C} \to \underline{C}'$ be a tensor equivalence; then there is a tensor functor $(F',c')\colon \underline{C}' \to \underline{C}$ and isomorphisms of functors $F' \circ F \xrightarrow{\approx} \mathrm{id}_{\underline{C}}$ and $F \circ F' \xrightarrow{\approx} \mathrm{id}_{\underline{C}'}$ commuting with tensor products (i.e. isomorphisms of tensor functors; see below).

Proof: Saavedra [1, I4.4].

A tensor functor $F\colon \underline{C} \to \underline{C}'$ of rigid tensor categories induces a morphism $F\colon \mathrm{End}(\underline{1}) \to \mathrm{End}(\underline{1}')$. The following formulas hold:

$$\mathrm{Tr}_{F(X)} F(f) = F(\mathrm{Tr}_X(f))$$

$$\mathrm{rk}(F(X)) = F(\mathrm{rk}(X)) \; .$$

Morphisms of tensor functors

Definition 1.12. Let (F,c) and (G,d) be tensor functors $\underline{C} \to \underline{C}'$; a morphism of tensor functors $(F,c) \longrightarrow (G,d)$ is a morphism of functors $\lambda\colon F \to G$ such that, for all finite families $(X_i)_{i\in I}$ of objects in $\underline{C}$, the diagram

$$\begin{tikzcd} \bigotimes_{i\in I} F(X_i) \arrow[r, "c"] \arrow[d, "\otimes\lambda_{X_i}"] & F(\bigotimes_{i\in I} X_i) \arrow[d, "\lambda_{\otimes X_i}"] \\ \bigotimes_{i\in I} G(X_i) \arrow[r, "c"] & G(\bigotimes_{i\in I} X_i) \end{tikzcd} \qquad (1.12.1)$$

is commutative.

In fact, it suffices to require that the diagram (1.12.1) be commutative when I is $\{1,2\}$ or the empty set. For the empty set, (1.12.1) becomes

$$\begin{array}{ccc} \underline{1}' & \xrightarrow{\approx} & F(\underline{1}) \\ \| & & \downarrow \lambda_{\underline{1}} \\ \underline{1}' & \xrightarrow{\approx} & G(\underline{1}) \end{array} \tag{1.12.2}$$

in which the horizontal maps are the unique isomorphisms compatible with the structures of $\underline{1}'$, $F(\underline{1})$, and $G(\underline{1})$ as identity objects of $\underline{C}'$. In particular, when (1.12.2) commutes, $\lambda_{\underline{1}}$ is an isomorphism.

We write $\mathrm{Hom}^{\otimes}(F,F')$ for the set of morphisms of tensor functors $(F,c) \to (G,d)$.

<u>Proposition</u> 1.13. Let (F,c) and (G,d) be tensor functors $\underline{C} \to \underline{C}'$. If $\underline{C}$ and $\underline{C}'$ are rigid, then any morphism of tensor functors λ: $F \to G$ is an isomorphism.

<u>Proof</u>: The morphism μ: $G \to F$, making the diagrams

$$\begin{array}{ccc} F(\check{X}) & \xrightarrow{\lambda_{\check{X}}} & G(\check{X}) \\ \downarrow\approx & & \downarrow\approx \\ F(X)^{\vee} & \xrightarrow{{}^{t}(\mu_X)} & G(X)^{\vee} \end{array}$$

commutative for all $X \in \mathrm{ob}(\underline{C})$, is an inverse for λ.

For any field k and k-algebra R, there is a canonical tensor functor ϕ_R: $\underline{\mathrm{Vec}}_k \to \underline{\mathrm{Mod}}_R$, $V \longmapsto V \otimes_k R$. If (F,c) and (G,d) are tensor functors $\underline{C} \to \underline{\mathrm{Vec}}_k$, then we define $\underline{\mathrm{Hom}}^{\otimes}(F,G)$ to be the functor of k-algebras such that

$$\underline{\mathrm{Hom}}^{\otimes}(F,G)(R) = \mathrm{Hom}^{\otimes}(\phi_R \circ F, \phi_R \circ G) \tag{1.13.1}$$

Tensor subcategories

Definition 1.14. Let $\underline{C}'$ be a strictly full subcategory of a tensor category $\underline{C}$. We say $\underline{C}'$ is a tensor subcategory of $\underline{C}$ if it is closed under the formation of finite tensor products (equivalently, if it contains an identity object of $\underline{C}$ and if $X_1 \otimes X_2 \in \mathrm{ob}(\underline{C}')$ whenever $X_1, X_2 \in \mathrm{ob}(\underline{C}')$) . A tensor subcategory of rigid tensor category is said to be a rigid tensor subcategory if it contains $X^\vee$ whenever it contains X .

A tensor subcategory becomes a tensor category under the induced tensor structure, and similarly for rigid tensor subcategories.

When $(\underline{C},\otimes)$ is abelian (see below), then we say that a family $(X_i)_{i\in I}$ of objects of $\underline{C}$ is a tensor generating family for $\underline{C}$ if every object of $\underline{C}$ is isomorphic to a subquotient of $P(X_i)$, $P(t_i) \in \mathbb{N}[t_i]_{i\in I}$, where in $P(X_i)$ multiplication is interpreted as $\otimes$ and addition as $\oplus$.

Abelian tensor categories; End ($\underline{1}$)

Our convention, that functors between additive categories are to be additive, forces the following definition.

Defintion 1.15. An additive (resp. abelian) tensor category is a tensor category $(\underline{C},\otimes)$ such that $\underline{C}$ is an additive (resp. abelian) category and $\otimes$ is a bi-additive functor.

If $(C,\otimes)$ is such a category, then $R = \mathrm{End}(\underline{1})$ is a ring which acts, via $\ell_X : X \to \underline{1} \otimes X$, on each object X. The action of R on X commutes with endomorphisms of X and so, in particular, R is commutative. The category $\underline{C}$ is R-linear and $\otimes$ is R-bilinear. When $\underline{C}$ is rigid, the trace morphism is an R-linear map $\mathrm{Tr}\colon \mathrm{End}(X) \to R$.

<u>Proposition 1.16</u>. Let $(\underline{C},\otimes)$ be a rigid tensor category. If $\underline{C}$ is abelian then $\otimes$ is bi-additive and commutes with direct and inverse limits in each factor; in particular it is exact.

<u>Proof</u>: The functor $X \mapsto X \otimes Y$ has a right adjoint, namely $Z \mapsto \underline{\mathrm{Hom}}(Y,Z)$, and therefore commutes with direct limits and is additive. By considering the opposite category $\underline{C}^\circ$, one deduces that it also commutes with inverse limits. (In fact, $Z \mapsto \underline{\mathrm{Hom}}(Y,Z)$ is also a left adjoint for $X \mapsto X \otimes Y$).

<u>Proposition 1.17</u>. Let $(\underline{C},\otimes)$ be a rigid abelian tensor category. If U is a subobject of $\underline{1}$, then $\underline{1} = U \oplus U^{\perp}$ where $U^{\perp} = \ker(\underline{1} \to U^{\vee})$. Consequently $\underline{1}$ is a simple object if $\mathrm{End}(\underline{1})$ is a field.

<u>Proof</u>: Let $V = \mathrm{coker}\,(U \to \underline{1})$. On tensoring $0 \to U \to \underline{1} \to V \to 0$ with itself, we obtain an exact commulative diagram

$$\begin{tikzcd}
U \otimes V \arrow[r] & V \arrow[r] & V \otimes V \\
U \arrow[u] \arrow[r] \arrow[ur, dashed, "0"] & \underline{1} \arrow[u] \arrow[r] & V \arrow[u] \\
U \otimes U \arrow[u] \arrow[r, hook] & U \arrow[u, hook] \arrow[r, two heads] \arrow[ur, dashed, "0"] & V \otimes U \arrow[u, hook]
\end{tikzcd},$$

from which it follows that $U \otimes V = 0$ and that $U \otimes U = U$ as a subobject of $\underline{1} \otimes \underline{1} = \underline{1}$.

For any X , the largest subobject Y of X such that $U \otimes Y = 0$ is also the largest subobject for which the map $U \otimes Y \hookrightarrow Y$ $(=(U \hookrightarrow \underline{1}) \otimes Y)$ is zero or, equivalently, such that $Y \hookrightarrow Y \otimes U^{\vee}$ is zero; hence $Y = \ker(X \twoheadrightarrow X \otimes U^{\vee}) = X \otimes U^{\perp}$. On applying this remark with $X = V$, and using that $U \otimes V = 0$, we find that $V \otimes U^{\perp} = V$; on applying it with $X = U$, we find $U \otimes U^{\perp} = 0$. From

$$0 \to U \otimes U^{\perp} \to \underline{1} \otimes U^{\perp} \to V \otimes U^{\perp} \to 0$$

we deduce that $U^{\perp} \xrightarrow{\approx} V$, and that $\underline{1} = U \oplus U^{\perp}$.

<u>Remark</u> 1.18. The proposition shows that there is a one-to-one correspondence between subobjects of $\underline{1}$ and idempotents in $\mathrm{End}(\underline{1})$. Such an idempotent e determines a decomposition of tensor categories $\underline{C} = \underline{C}' \times \underline{C}''$ in which $\mathrm{ob}(\underline{C}')$ is the set of objects of $\underline{C}$ on which e acts as the identity map.

Proposition 1.19. Let $\underline{C}$ and $\underline{C}'$ be rigid abelian tensor categories and assume that $\mathrm{End}(\underline{1})$ is a field and that $\underline{1}' \neq 0$, where $\underline{1}$ and $\underline{1}'$ are identity objects in $\underline{C}$ and $\underline{C}'$. Then any exact tensor functor $F : \underline{C} \to \underline{C}'$ is faithful.

Proof: The criterion in $\underline{C}$,

$$X \neq 0 \Leftrightarrow X \otimes \check{X} \to \underline{1} \text{ surjective}$$

is respected by F.

A criterion to be a rigid tensor category

Proposition 1.20. Let $\underline{C}$ be a k-linear abelian category, where k is a field, and let $\otimes\colon \underline{C} \times \underline{C} \to \underline{C}$ be a k-bilinear functor. Suppose there are given a faithful exact k-linear functor $F\colon \underline{C} \to \underline{\mathrm{Vec}}_k$, a functorial isomorphism $\phi_{X,Y,Z}\colon X \otimes (Y\otimes Z) \to (X\otimes Y) \otimes Z$, and a functorial isomorphism $\psi_{X,Y}\colon X \otimes Y \to Y \otimes X$ with the following properties:

(a) $F \circ \otimes = \otimes \circ F \times F$;

(b) $F(\phi_{X,Y,Z})$ is the usual associativity isomorphism in $\underline{\mathrm{Vec}}_k$;

(c) $F(\psi_{X,Y})$ is the usual commutativity isomorphism in $\underline{\mathrm{Vec}}_k$;

(d) there exists a $U \in \mathrm{ob}(\underline{C})$ such that $k \to \mathrm{End}(U)$ is an isomorphism and $F(U)$ has dimension 1;

(e) if $F(L)$ has dimension 1, then there exists an object L^{-1} in $\underline{C}$ such that $L \otimes L^{-1} \approx U$.

Then $(C,\otimes,\phi,\psi)$ is a rigid abelian tensor category.

Proof: It is not difficult to prove this directly — essentially one only has to show that the object U of (d) is an identity object and that (e) is sufficient to show that $\underline{C}$ is rigid — but we shall indicate a more elegant approach in (2.18) below.

Examples

(1.21) $\underline{\mathrm{Vec}}_k$, for k a field, is rigid abelian tensor category and $\mathrm{End}(\underline{1}) = k$. All of the above definitions take on a familiar meaning when applied to $\underline{\mathrm{Vec}}_k$. For example, $\mathrm{Tr}: \mathrm{End}(X) \to k$ is the usual trace map.

(1.22) $\underline{\mathrm{Mod}}_R$ is an abelian tensor category and $\mathrm{End}(\underline{1}) = R$. In general it will not be rigid because not all R-modules will be reflexive.

(1.23) The category $\underline{\mathrm{Proj}}_R$ of projective modules of finite type over a commutative ring R is a rigid additive tensor category and $\mathrm{End}(\underline{1}) = R$. The rigidity follows easily from considering the objects of $\underline{\mathrm{Proj}}_R$ as locally-free modules of finite rank on spec(R).

(1.24) Let G be an affine group scheme over a field k and let $\underline{\mathrm{Rep}}_k(G)$ be the category of finite-dimensional representations of G over k. Thus an object of $\underline{\mathrm{Rep}}_k(G)$ consists of a finite-dimensional vector space V over k and a homomorphism $g \longmapsto g_V : G \to GL(V)$ of affine group schemes over k. Then $\underline{\mathrm{Rep}}_k(G)$ is a rigid abelian tensor category and $\mathrm{End}(\underline{1}) = k$. These categories, and more generally the categories of representations of affine gerbs (see §3), are the main topic of study of this article.

(1.25) (Vector spaces graded by $\mathbb{Z}/2\mathbb{Z}$). Let $\underline{C}$ be the category whose objects are pairs (V^0, V^1) of finite dimensional vector spaces over a field k. We give $\underline{C}$ the tensor structure whose commutativity constraint is determined by the Koszul rule of signs, i.e., that defined by the isomorphisms

$$v \otimes w \mapsto (-1)^{ij}\, w \otimes v : V^i \otimes W^j \to W^j \otimes V^i .$$

Then $\underline{C}$ is a rigid abelian tensor category and $\mathrm{End}(\underline{1}) = k$, but it is not of the form $\underline{\mathrm{Rep}}_k(G)$ for any G because $\mathrm{rk}(V^\circ, V^1) = \dim(V^\circ) - \dim(V^1)$ may not be positive.

(1.26) The rigid additive tensor category freely generated by an object T is a pair $(\underline{C}, T)$ comprising a rigid additive tensor category $\underline{C}$ and that $\mathrm{End}(\underline{1}) = \mathbb{Z}[t]$ and an object T having the property that

$$F \longmapsto F(T) : \mathrm{Hom}^{\otimes}(\underline{C}, \underline{C}') \to \underline{C}'$$

is an equivalence of categories for all rigid additive tensor categories $\underline{C}'$ (t will turn out to be the rank of T). We show how to construct such a pair $(\underline{C}, T)$ — clearly it is unique up to a unique equivalence of tensor categories preserving T.

Let V be a free module of finite rank over a commutative ring k and let $T^{a,b}(V)$ be the space $V^{\otimes a} \otimes \check{V}^{\otimes b}$ of tensors with covariant degree a and contravariant degree b. A map $f : T^{a,b}(V) \to T^{c,d}(V)$ can be identified with a tensor "f" in $T^{b+c,a+d}(V)$. When $a+d = b+c$, $T^{b+c,a+d}(V)$ contains a special element, namely the $(a+d)^{th}$ tensor power of "id" $\in T^{1,1}(V)$, and other elements can be obtained by allowing an element of the symmetric group S_{a+d} to permute the contravariant components of this special element. We have therefore a map

$$\varepsilon : S_{a+d} \to \mathrm{Hom}(T^{a,b},T^{c,d}) \quad \text{(when } a+d = b+c\text{)} .$$

The induced map $k[S_{a+d}] \to \mathrm{Hom}(T^{a,b},T^{c,d})$ is injective provided $\mathrm{rk}(V) \geq a+d$. One checks that the composite of two such maps $\varepsilon(\sigma) : T^{a,b}(V) \to T^{c,d}(V)$ and $\varepsilon(\tau) : T^{c,d}(V) \to T^{e,f}(V)$ is given by a universal formula

$$\varepsilon(\tau)\cdot\varepsilon(\sigma) = (\mathrm{rk}\ V)^N\ \varepsilon(\rho) \tag{1.26.1}$$

with ρ and N depending only on a,b,c,d,e,f,σ, and τ .

We define $\underline{C}'$ to be the category having as objects symbols $T^{a,b}$ $(a,b, \in \mathbb{N})$, and for which $\mathrm{Hom}(T^{a,b},T^{c,d})$ is the free $\mathbb{Z}[t]$-module with basis S_{a+d} if $a+d = b+c$ and is zero otherwise. Composition of morphisms is defined to be $\mathbb{Z}[t]$-bilinear and to agree on basis elements with the universal formula (1.26.1) with rk V replaced by the inderterminate t . The associativity law holds for this composition because it does whenever t is replaced by a large enough positive integer (it becomes the associativity law in a category of modules). Tensor products are defined by $T^{a,b} \otimes T^{c,d} = T^{a+d,b+d}$ and by an obvious rule for morphisms. We define T to be $T^{1,0}$.

The category $\underline{C}$ is deduced from $\underline{C}'$ by formally adjoining direct sums of objects. Its universality follows from the fact that the formula (1.26.1) holds in any rigid additive category.

(1.27) ($\underline{GL}_t$) . Let n be an integer, and use $t \longmapsto n : \mathbb{Z}[t] \to \mathbb{C}$ to extend the scalars in the above example from $\mathbb{Z}[t]$ to $\mathbb{C}$. If V is an n-dimensional complex vector space, and if $a+d \leq n$, then

$$\mathrm{Hom}(T^{a,b}, T^{c,d}) \otimes_{\mathbb{Z}[t]} \mathbb{C} \to \mathrm{Hom}_{GL(V)}(T^{a,b}(V), T^{c,d}(V))$$

is an isomorphism. For any sum T' of $T^{a,b}$s and large enough integer n , $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{C}$ is therefore a product of matrix algebras. This implies that $\mathrm{End}(T') \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t)$ is a semisimple algebra.

After extending the scalars in $\underline{C}$ to $\mathbb{Q}(t)$ (i.e., replacing $\mathrm{Hom}(T',T'')$ with $\mathrm{Hom}(T',T'') \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t)$) and passing to the pseudo-abelian (Karoubian) envelope (i.e., formally adjoining images of idempotents), we obtain a semisimple rigid abelian tensor category $\underline{GL}_t$. The rank of T in $\underline{GL}_t$ is $t \notin \mathbb{N}$ and so, although $\mathrm{End}(\underline{1}) = \mathbb{Q}(t)$ is a field, $\underline{GL}_t$ is not of the form $\underline{\mathrm{Rep}}_k(G)$ for any group scheme (or gerb) G .

§2. Neutral Tannakian categories

Throughout this section, k will be a field.

Affine group schemes

Let $G = \mathrm{spec}\, A$ be an affine group scheme over k . The maps mult: $G \times G \to G$, identity $\{1\} \to G$, inverse: $G \to G$ induce maps of k-algebras

$$\Delta: \quad A \to A \otimes_k A, \qquad \varepsilon: \quad A \to k, \qquad S: \quad A \to A$$

(the comultiplication, coidentity, and coinverse maps) such that

$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta: \quad A \to A \otimes A \rightrightarrows A \otimes A \otimes A$

(coassociativity axiom),

$\mathrm{id} = (\varepsilon \otimes \mathrm{id})\Delta: A \to A \otimes A \to k \otimes A = A$ (coidentity axiom), and

$(A \xrightarrow{\Delta} A \otimes A \xrightarrow{(S,\mathrm{id})} A) = (A \xrightarrow{\varepsilon} k \hookrightarrow A)$ (coinverse axiom).

We define a <u>bialgebra</u> over k to be a k-algebra A together with maps Δ, ε, and S satisfying the three axioms. (This terminology is not standard).

<u>Proposition</u> 2.1. The functor $A \mapsto \operatorname{spec} A$ defines an equivalence between the category of k-bialgebras and the category of affine group schemes over k.

<u>Proof</u>: Obvious.

If A is finitely generated (as a k-algebra) we say that G is <u>algebraic</u> or that it is an <u>algebraic group</u>.

We define a <u>coalgebra</u> over k to be a vector space C over k together with k-linear maps $\Delta: C \to C \otimes_k C$ and $\varepsilon: C \to k$ satisfying the coassociativity and coidentity axioms.

A <u>comodule</u> over C is a vector space V over k together with a k-linear map $\rho: V \to V \otimes C$ such that $(\mathrm{id} \otimes \varepsilon)\rho: V \to V \otimes C \to V \otimes k = V$ is the identity map and $(\mathrm{id} \otimes \Delta)\rho = (\rho \otimes \mathrm{id})\rho : V \to V \otimes C \otimes C$.

For example, Δ defines an C-comodule structure on C.

<u>Proposition</u> 2.2. Let $G = \operatorname{spec} A$ be an affine group scheme over k and let V be a vector space over k. There is a canonical one-to-one correspondence between the A-comodule structures on V and the linear representations of G on V.

Proof. Let $G \to GL(V)$ be a representation. The element $id \in Mor(G,G) = G(A)$ maps to an element of $GL(V \otimes A)$ whose restriction to $V = V \otimes k \subset V \otimes A$ is a comodule structure on V. Conversely, a comodule structure ρ on V determines a representation of G on V such that, for R a k-algebra and $g \in G(R)$, the restriction of $g_V\colon V \otimes R \to V \otimes R$ to $V = V \otimes k \subset V \otimes R$ is

$$(id\otimes g)\rho\colon\ V \to V \otimes A \to V \otimes R\ .$$

Proposition 2.3. Let C be a k-coalgebra and (V,ρ) a comodule over C. Any finite subset of V is contained in a sub-comodule of V having finite dimension over k.

Proof: Let $\{a_i\}$ be a basis for C over k. If v is in the finite subset, write $\rho(v) = \sum v_i \otimes a_i$ (finite sum). The k-space generated by the v and the v_i is a sub-comodule.

Corollary 2.4. Any k-rational representation of an affine group scheme is a directed union of finite-dimensional subrepresentations V_i.

Proof: Combine (2.2) and (2.3).

Corollary 2.5. An affine group scheme G is algebraic if and only if it has a faithful finite-dimensional representation over k.

Proof: The sufficiency is obvious. For the necessity, let V be the regular representation of G, and write $V = \bigcup V_i$ with the V_i as in (2.4). Then $\bigcap_i \operatorname{Ker}(G \to GL(V_i)) = \{1\}$ because V is a faithful representation, and it follows that

$\mathrm{Ker}(G \to GL(V_{i_0})) = \{1\}$ for some i_0 because G is Noetherian as a topological space.

<u>Proposition</u> 2.6. Let A be a k-bialgebra. Any finite subset of A is contained in a sub-bialgebra that is finitely generated as an algebra over k.

<u>Proof:</u> According to (2.3), the finite subset is contained in a finite-dimensional subspace V of A such that $\Delta(V) \subset V \otimes A$. Let $\{v_i\}$ be a basis for V and let $\Delta(v_j) = \sum v_i \otimes a_{ij}$. The subalgebra $k[v_j, a_{ij}, Sv_j, Sa_{ij}]$ of A is a sub-bialgebra. (See Waterhouse [1,3.3]).

<u>Corollary</u> 2.7. Any affine group scheme G over k is a directed inverse limit $G = \varprojlim G_i$ of affine algebraic groups over k in which the transition maps $G_i \leftarrow G_j$, $i \leq j$, are surjective.

<u>Proof</u>: The functor spec transforms a direct limit $A = \cup A_i = \varinjlim A_i$ into an inverse limit $G = \varprojlim G_i$. The transition map $G_i \leftarrow G_j$ is surjective because A_j is faithfully flat over its subalgebra A_i (Waterhouse [1, 14.1]).

The converse to (2.7) is also true; in fact the inverse limit of any family of affine group schemes is again an affine group scheme.

<u>Determining a group scheme from its representations</u>.

Let G be an affine group scheme over k and let ω, or ω^G, be the forgetful functor $\underline{\mathrm{Rep}}_k(G) \to \underline{\mathrm{Vec}}_k$. For R a k-algebra, $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ consists of families (λ_X), $X \in \mathrm{ob}(\underline{\mathrm{Rep}}_k(G))$, where λ_X is an R-linear automorphism of $X \otimes R$ such that $\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}$, $\lambda_{\underline{1}}$ is the identity map (on R), and

$$\lambda_Y \circ (\alpha\otimes 1) = (\alpha\otimes 1) \circ \lambda_X\colon \ X \otimes R \to Y \otimes R$$

for all G-equivariant k-linear maps $\alpha\colon X \to Y$ (see 1.12). Clearly any $g \in G(R)$ defines an element of $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$.

Proposition 2.8. The natural map $G \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an isomorphism of functors of k-algebras.

Proof: Let $X \in \underline{\mathrm{Rep}}_k(G)$ and let $\underline{C}_X$ be the strictly full subcategory of $\underline{\mathrm{Rep}}_k(G)$ of objects isomorphic to a subquotient of $P(X,X^{\vee})$, $P \in \mathbb{N}[t,s]$ (cf. the discussion following (1.14)). The map $\lambda \longmapsto \lambda_X$ identifies $\underline{\mathrm{Aut}}^{\otimes}(\omega|\underline{C}_X)(R)$ with a subgroup of $GL(X \otimes R)$. Let G_X be the image of G in $GL(X)$; it is a closed algebraic subgroup of G, and clearly

$$G_X(R) \subset \underline{\mathrm{Aut}}^{\otimes}(\omega|\underline{C}_X)(R) \subset GL(X \otimes R) .$$

If $V \in \mathrm{ob}(\underline{C}_X)$ and $t \in V$ is fixed by G, then $a \longmapsto at\colon\ k \xrightarrow{\alpha} V$ is G-equivariant, and so $\lambda_V(t \otimes 1) = (\alpha \otimes 1)\lambda_{\underline{1}}(1) = t \otimes 1$. Now (I.3.2) shows that $G_X = \underline{\mathrm{Aut}}^{\otimes}(\omega|\underline{C}_X)$.

If $X' = X \oplus Y$ for some representation Y of G, then $\underline{C}_X \subset \underline{C}_{X'}$, and there is a commutative diagram

$$\begin{array}{ccc} G_{X'} & \xrightarrow{\approx} & \underline{\mathrm{Aut}}^{\otimes}(\omega|\underline{C}_{X'}) \\ \downarrow & & \downarrow \\ G_X & \xrightarrow{\approx} & \underline{\mathrm{Aut}}^{\otimes}(\omega|\underline{C}_X) \end{array} .$$

It is clear from (2.5) and (2.7) that $G = \varprojlim G_{X'}$, and so, on passing to the inverse limit over these diagrams, we obtain an isomorphism $G \xrightarrow{\approx} \underline{Aut}^{\otimes}(\omega)$.

A homomorphism $f\colon G \to G'$ defines a functor $\omega^f\colon \underline{Rep}_k(G') \to \underline{Rep}_k(G)$, namely $(G' \to GL(V)) \mapsto (G \xrightarrow{f} G' \to GL(V))$, such that $\omega^G \circ \omega^f = \omega^{G'}$.

Corollary 2.9. Let G and G' be affine group schemes over k and let $F\colon \underline{Rep}_k(G') \to \underline{Rep}_k(G)$ be a tensor functor such that $\omega^G \circ F = \omega^{G'}$. Then there is a unique homomorphism $f\colon G \to G'$ such that $F = \omega^f$.

Proof: For $\lambda \in \underline{Aut}^{\otimes}(\omega^G)(R)$, R a k-algebra, define $F^*(\lambda) \in \underline{Aut}^{\otimes}(\omega^{G'})(R)$ by the rule $F^*(\lambda)_{X'} = \lambda_{F(X')}$. The proposition allows us to regard F^* as a homomorphism $G \to G'$, and clearly $F \mapsto F^*$ and $f \mapsto \omega^f$ are inverse maps.

Remark 2.10. Proposition 2.8 shows that G is determined by the triple $(\underline{Rep}_k(G), \otimes, \omega^G)$; it can be shown that the coalgebra of G is already determined by $(\underline{Rep}_k(G), \omega^G)$ (cf. the proof of Theorem 2.11).

The main theorem

Theorem 2.11. Let $\underline{C}$ be a rigid abelian tensor category such that $k = \underline{End}(\underline{1})$, and let $\omega\colon \underline{C} \to \underline{Vec}_k$ be an exact faithful k-linear tensor functor. Then,

(a) the functor $\underline{Aut}^{\otimes}(\omega)$ of k-algebras is representable by an affine group scheme G;

(b) ω defines an equivalence of tensor categories $\underline{C} \to \underline{\mathrm{Rep}}_k(G)$.

Proof: We first construct the coalgebra A of G without using the tensor structure on $\underline{C}$. The tensor structure then enables us to define an algebra structure on A , and the rigidity of $\underline{C}$ implies that spec A is a group scheme (rather than a monoid scheme). The following easy observation will allow us to work initially with algebras rather than coalgebras: for a finite-dimensional (not necessarily commutative) k-algebra A and its dual coalgebra $A^{\vee} \overset{\mathrm{df}}{=} \mathrm{Hom}(A,k)$, the bijection

$$\mathrm{Hom}(A \otimes_k V,\ V) \xleftrightarrow{\approx} \mathrm{Hom}(V, A^{\vee} \otimes_k V)$$

determines a one-to-one correspondence between the A-module structures on a vector space V and the $A^{\vee}$-comodule structure on V .

We begin with some constructions that are valid in any k-linear abelian category $\underline{C}$. For V a finite-dimensional vector space over k and X and object of $\underline{C}$, we define $V \otimes X$ to be the system $((X^n)_\alpha, \phi_{\beta,\alpha})$ where α runs through the isomorphisms $k^n \xrightarrow{\approx} V$, $(X^n)_\alpha = X^n \overset{\mathrm{df}}{=} X \oplus \ldots \oplus X$ (n copies), and $\phi_{\beta,\alpha}\colon\ (X^n)_\alpha \to (X^n)_\beta$ is defined by $\beta^{-1} \circ \alpha \in GL_n(k)$. Note that $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$. A morphism $V \otimes X \to T$ or $T \to V \otimes X$, where $T \in \mathrm{ob}(\underline{C})$, is a family of morphisms compatible with the $\phi_{\beta,\alpha}$. There is a canonical k-linear map $V \to \mathrm{Hom}(X, V \otimes X)$ under which $v \in V$ maps to $(X \xrightarrow{\psi_\alpha} (X^n)_\alpha)$ where ψ_α is defined by $\alpha^{-1}(v) \in k^n$. This map induces a functorial isomorphism

$\mathrm{Hom}(V\otimes X,T) \xrightarrow{\approx} \mathrm{Hom}(V,\mathrm{Hom}(X,T))$, $T \in \mathrm{ob}(\underline{C})$. Any k-linear functor $F\colon \underline{C} \to \underline{C}'$ has the property that $F(V\otimes X) = V \otimes F(X)$. When $\underline{C}$ is $\underline{\mathrm{Vec}}_k$, $V \otimes X$ can be identified with the usual object.

For $V \in \mathrm{ob}(\underline{\mathrm{Vec}}_k)$ and $X \in \mathrm{ob}(\underline{C})$, we define $\underline{\mathrm{Hom}}(V,X)$ to be $V^V \otimes X$. If $W \subset V$ and $Y \subset X$, then the subobject of $\underline{\mathrm{Hom}}(V,X)$ mapping W into Y is defined to be

$$(Y:W) = \mathrm{Ker}(\underline{\mathrm{Hom}}(V,X) \to \underline{\mathrm{Hom}}(W,X/Y)) .$$

Lemma 2.12. Let $\underline{C}$ be a k-linear abelian category and $\omega\colon \underline{C} \to \underline{\mathrm{Vec}}_k$ a k-linear exact faithful functor. Then, for any $X \in \mathrm{ob}(\underline{C})$, the following two objects are equal:

(a) the largest subobject P of $\underline{\mathrm{Hom}}(\omega(X),X)$ whose image in $\underline{\mathrm{Hom}}(\omega(X)^n,X^n)$ (embedded diagonally) is contained in $(Y\colon \omega(Y))$ for all $Y \subset X^n$;

(b) the smallest subobject P' of $\underline{\mathrm{Hom}}(\omega(X),X)$ such that the subspace $\omega(P')$ of $\mathrm{Hom}(\omega(X),\omega(X))$ contains $\mathrm{id}\colon \omega(X) \to \omega(X)$.

Proof: Clearly $\omega(X) = 0$ implies $\mathrm{End}(X) = 0$, which implies $X = 0$. Thus if $X \subset Y$ and $\omega(X) = \omega(Y)$ then $X = Y$, and it follows that all objects of $\underline{C}$ are both Artinian and Noetherian. The objects P and P' therefore obviously exist.

The functor ω maps $\underline{\mathrm{Hom}}(V,X)$ to $\mathrm{Hom}(V,\omega(X))$ and $(Y:W)$ to $(\omega(Y):W)$ for all $W \subset V \in \mathrm{ob}(\underline{\mathrm{Vec}}_k)$ and $Y \subset X \in \mathrm{ob}(\underline{C})$. It therefore maps $P \overset{\mathrm{df}}{=} \cap\, (\underline{\mathrm{Hom}}(\omega(X),X) \cap (Y:\omega(Y)))$ to $\cap\, (\mathrm{End}(\omega(X)) \cap (\omega Y:\omega Y))$. This means ωP is the largest subring of $\mathrm{End}(\omega(X))$ stabilizing $\omega(Y)$ for all $Y \subset X^n$. Hence $\mathrm{id} \in \omega(P)$ and $P \supset P'$.

Let V be a finite-dimensional vector space over k; there is an obvious map $\underline{\mathrm{Hom}}(\omega(X),X) \to \underline{\mathrm{Hom}}(\omega(V\otimes X),V\otimes X)$ (inducing $f \longmapsto 1 \otimes f\colon \mathrm{End}(\omega(X)) \to \mathrm{End}(V\otimes\omega(X))$ and $\omega(P) \subset \mathrm{End}(\omega(X))$ stabilizes $\omega(Y)$ for all $Y \subset V \otimes X$. On applying this remark to a $Q \subset \underline{\mathrm{Hom}}(\omega(X),X) = \omega(X)^{\vee} \otimes X$, we find that $\omega(P)$, when acting by left multiplication on $\mathrm{End}(\omega(X))$, stablizes $\omega(Q)$. Therefore, if $\omega(Q)$ contains 1, then $\omega(P) \subset \omega(Q)$, and $P \subset Q$; this shows that $P \subset P'$.

Let $P_X \subset \underline{\mathrm{Hom}}(\omega(X),X)$ be the subobject defined in (a) (or (b)) of the lemma, and let $A_X = \omega(P_X)$; it is the largest subalgebra of $\mathrm{End}(\omega(X))$ stabilizing $\omega(Y)$ for all $Y \subset X^n$. Let $\langle X \rangle$ be the strictly full subcategory of $\underline{C}$ such that $\mathrm{ob}(\langle X \rangle)$ consists of the objects of $\underline{C}$ that are isomorphic to subquotients of X^n, $n \in \mathbb{N}$. Then $\omega|\langle X \rangle\colon \langle X \rangle \to \underline{\mathrm{Vec}}_k$ factors through $\underline{\mathrm{Mod}}_{A_X}$.

Lemma 2.13. Let $\omega\colon \underline{C} \to \underline{\mathrm{Vec}}_k$ be as in (2.12). Then ω defines an equivalence of categories $\langle X \rangle \to \underline{\mathrm{Mod}}_{A_X}$ carrying $\omega|\langle X \rangle$ into the forgetful functor. Moreover $A_X = \mathrm{End}(\omega|\langle X \rangle)$.

Proof: The right action $f \longmapsto f \circ a$ of A_X on $\underline{\mathrm{Hom}}(\omega(X),X)$ stabilizes P_X because obviously $(Y:\omega(Y))(\omega(Y):\omega(Y)) \subset (Y:\omega(Y))$. If M is an A_X-module we define

$$P_X \otimes_{A_X} M = \mathrm{Coker}(P_X \otimes A_X \otimes M \rightrightarrows P_X \otimes M) .$$

Then $\omega(P_X \otimes_{A_X} M) = \omega(P_X) \otimes_{A_X} M = A_X \otimes_{A_X} M = M$. Recall that

$P_X \otimes_{A_X} M$ is a family (Y_α) of objects of $\underline{C}$ with given compatible isomorphisms $Y_\alpha \to Y_\beta$. If we choose one α, then $\omega(Y_\alpha) \approx M$, which shows that ω is essentially surjective. A similar argument shows that $<X> \to \underline{\mathrm{Mod}}_{A_X}$ is full.

Clearly any element of A_X defines an endomorphism of $\omega| <X>$. On the other hand an element λ of $\mathrm{End}(\omega| <X>)$ is determined by $\lambda_X \in \mathrm{End}(\omega(X))$; thus $\mathrm{End}(\omega(X)) \supset \mathrm{End}(\omega| <X>) \supset A_X$. But λ_X stabilizes $\omega(Y)$ for all $Y \subset X^n$, and so $\mathrm{End}(\omega| <X>) \subset A_X$. This completes the proof of the lemma.

Let $B_X = A_X^\vee$. The remark at the start of the proof allows us to restate (2.13) as follows: ω defines an equivalence

$$(<X>, \omega|<X>) \to (\underline{\mathrm{Comod}}_{B_X}, \text{forget})$$

where $\underline{\mathrm{Comod}}_{B_X}$ is the category of B_X-comodules of finite dimension over k.

On passing to the inverse limit over X (cf. the proof of (2.8)), we obtain the following result.

Proposition 2.14. Let $(\underline{C}, \omega)$ be as in (2.12) and let $B = \varinjlim \mathrm{End}(\omega| <X>)^\vee$. Then ω defines an equivalence of categories $\underline{C} \to \underline{\mathrm{Comod}}_B$ carrying ω into the forgetful functor.

Example 2.15. Let A be a finite-dimensional k-algebra and let ω be the forgetful functor $\underline{\mathrm{Mod}}_A \to \underline{\mathrm{Vec}}_k$. For R

a commutative k-algebra, let ϕ_R be the functor $R \otimes -: \underline{\mathrm{Vec}}_k \to \underline{\mathrm{Mod}}_R$. There is a canonical map $\alpha: R \otimes_k A \to \mathrm{End}(\phi_R \circ \omega)$, which we shall show to be an isomorphism by defining an inverse β. For $\lambda \in \mathrm{End}(\phi_R \circ \omega)$, set $\beta(\lambda) = \lambda_A(1)$. Clearly $\beta\alpha = \mathrm{id}$, and so we have to show $\alpha\beta = \mathrm{id}$. For $M \in \mathrm{ob}(\underline{\mathrm{Mod}}_A)$, let $M_0 = \omega(M)$. The A-module $A \otimes_k M_0$ is a direct sum of copies of A, and the additivity of λ shows that $\lambda_{A \otimes M_0} = \lambda_A \otimes \mathrm{id}_{M_0}$. The map $a \otimes m \longmapsto am: A \otimes_k M_0 \to M$ is A-linear, and hence

$$\begin{array}{ccc} R \otimes A \otimes M_0 & \longrightarrow & R \otimes M \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ R \otimes A \otimes M_0 & \longrightarrow & R \otimes M \end{array}$$

is commutative. Therefore $\lambda_M(m) = \lambda_A(1)m = (\alpha\beta(\lambda))_M(m)$ for $m \in R \otimes M$.

In particular, $A \xrightarrow{\approx} \mathrm{End}(\omega)$, and it follows that, if in (2.13) we take $\underline{C} = \underline{\mathrm{Mod}}_A$ so that $\underline{C} = \langle A \rangle$, then the equivalence of categories obtained is the identity functor.

Let B be a coalgebra over k and let ω be the forgetful functor $\underline{\mathrm{Comod}}_B \to \underline{\mathrm{Vec}}_k$. The above discussion shows that $B = \varinjlim \mathrm{End}(\omega | \langle X \rangle)^\vee$. We deduce, as in (2.9), that every functor $\underline{\mathrm{Comod}}_B \to \underline{\mathrm{Comod}}_{B'}$, carrying the forgetful functor into the forgetful functor arises from a unique homomorphism $B \to B'$.

Again let B be a coalgebra over k. A homomorphism $u\colon B \otimes_k B \to B$ defines a functor

$$\phi^u\colon \underline{\mathrm{Comod}}_B \times \underline{\mathrm{Comod}}_B \to \underline{\mathrm{Comod}}_B$$

sending (X,Y) to $X \otimes_k Y$ with the B-comodule structure

$$X \otimes Y \xrightarrow{\rho_X \otimes \rho_Y} X \otimes B \otimes Y \otimes B \xrightarrow{1 \otimes u} X \otimes Y \otimes B .$$

Proposition 2.16. The map $u \longmapsto \phi^u$ defines a one-to-one correspondence between the set of homomorphisms $B \otimes_k B \to B$ and the set of functors $\phi\colon \underline{\mathrm{Comod}}_B \times \underline{\mathrm{Comod}}_B \to \underline{\mathrm{Comod}}_B$ such that $\phi(X,Y) = X \otimes_k Y$ as k vector spaces. The natural associativity and commutativity constraints on $\underline{\mathrm{Vec}}_k$ induce similar contraints on $(\underline{\mathrm{Comod}}_B, \phi^u)$ if and only if the multiplication defined by u on B is associative and commutative; there is an identity object in $(\underline{\mathrm{Comod}}_B, \phi^u)$ with underlying vector space k if and only if B has an identity element.

Proof: The pair $(\underline{\mathrm{Comod}}_B \times \underline{\mathrm{Comod}}_B, \omega\otimes\omega)$, with $(\omega \otimes \omega)(X \otimes Y)$ $\omega(X) \otimes \omega(Y)$ (as a k vector space), satisfies the conditions of (2.14), and $\varinjlim \mathrm{End}(\omega\otimes\omega | < (X,Y) >)^{\vee} = B \otimes B$. Thus the first statement of the proposition follows from (2.15). The remaining statements are easy.

Let $(\underline{C},\omega)$ and B be as in (2.14) except now assume that $\underline{C}$ is a tensor category and ω is a tensor functor. The tensor structure on $\underline{C}$ induces a similar structure on

$\underline{\text{Comod}}_B$ and hence, because of (2.16), the structure of an associative commutative k-algebra with identity element on B. Thus B lacks only a coinverse map S to be a bialgebra, and $G = \operatorname{spec} B$ is an affine monoid scheme. Using (2.15) we find that, for any k-algebra R, $\underline{\text{End}}(\omega)(R) \overset{\text{df}}{=} \text{End}(\phi_R \circ \omega) = \varprojlim \text{Hom}_{k\text{-lin}}(B_X, R) = \text{Hom}_{k\text{-lin}}(B,R)$. An element $\lambda \in \text{Hom}_{k\text{-lin}}(B,R)$ corresponds to an element of $\underline{\text{End}}(\omega)(R)$ commuting with the tensor structure if and only if λ is a k-algebra homomorphism; thus $\underline{\text{End}}^{\otimes}(\omega)(R) = \text{Hom}_{k\text{-alg}}(B,R) = G(R)$. We have shown that if in the statement of (2.11) the rigidity condition is omitted, then one can conclude that $\underline{\text{End}}^{\otimes}(\omega)$ is representable by an affine monoid scheme $G = \operatorname{spec} B$ and ω defines an equivalence of tensor categories $\underline{C} \to \underline{\text{Comod}}_B = \underline{\text{Rep}}_k(G)$. If we now assume that $(\underline{C}, \otimes)$ is rigid, then (1.13) shows that $\underline{\text{End}}^{\otimes}(\omega) = \underline{\text{Aut}}^{\otimes}(\omega)$, and the theorem follows.

Remark 2.17. Let $(\underline{C}, \omega)$ be $(\text{Rep}_k(G), \omega^G)$. On following through the proof of (2.11) in this case one recovers (2.8): $\underline{\text{Aut}}^{\otimes}(\omega^G)$ is represented by G.

Remark 2.18. Let $(\underline{C}, \otimes, \phi, \psi, F)$ satisfy the conditions of (1.20). Then $(\underline{C}, \otimes, \phi, \psi)$ is obviously a tensor category, and the proof of (2.11) shows that F defines an equivalence of tensor categories $\underline{C} \to \underline{\text{Rep}}_k(G)$ where G is an affine group monoid representing $\underline{\text{End}}^{\otimes}(\omega)$. We can assume that $\underline{C} = \underline{\text{Rep}}_k(G)$.

Let $\lambda \in G(R)$. If $L \subset \underline{\mathrm{Rep}}_k(G)$ has dimension 1 , then $\lambda_L\colon R \otimes L \to R \otimes L$ is invertible, as follows easily from the existence of a G-isomorphism $L \otimes L^{-1} \xrightarrow{\approx} k$. It follows that λ_X is invertible for any $X \in \mathrm{ob}(\underline{\mathrm{Rep}}_k(G))$ because $\det(\lambda_X) \stackrel{\mathrm{df}}{=} \Lambda^d \lambda_X = \lambda_{\Lambda^d X}$, where $d = \dim X$, is invertible.

Definition 2.19. A neutral Tannakian category over k is a rigid abelian k-linear tensor category $\underline{C}$ for which there exists an exact faithful k-linear tensor functor $\omega\colon \underline{C} \to \underline{\mathrm{Vec}}_k$. Any such functor ω is said to be a fibre functor for $\underline{C}$.

Thus (2.11) shows that any neutral Tannakian category is equivalent (in possibly many different ways) to the category of finite-dimensional representations of an affine group scheme.

Properties of G and of $\underline{\mathrm{Rep}}_k(G)$.

In view of the last remark, it is natural to ask how properties of G are reflected in $\underline{\mathrm{Rep}}_k(G)$.

Proposition 2.20. Let G be an affine group scheme over k .

(a) G is finite if and only if there exists an object X of $\underline{\mathrm{Rep}}_k(G)$ such that every object of $\underline{\mathrm{Rep}}_k(G)$ is isomorphic to a subquotient of X^n , some $n \geq 0$.

(b) G is algebraic if and only if there exists an object X of $\underline{\mathrm{Rep}}_k(G)$ that is a tensor generator for $\underline{\mathrm{Rep}}_k(G)$.

Proof (a). If G is finite then the regular representation of G has the required properties. Conversely if, with the notations of the proof of (2.11), $\underline{\mathrm{Rep}}_k(G) = \langle X \rangle$, then $G = \mathrm{spec}\, B$ where B is the linear dual of the finite k-algebra A_X .

(b) If G is algebraic, then it has a finite-dimensional faithful representation X (2.5), and one shows as in (I.3.1a) that $X \oplus X^{\vee}$ is a tensor generator for $\underline{\mathrm{Rep}}_k(G)$. Conversely, if X is a tensor generator for $\underline{\mathrm{Rep}}_k(G)$ then it is a faithful representation of G.

Proposition 2.21. Let $f\colon G \to G'$ be a homomorphism of affine group schemes over k, and let ω^f be the corresponding functor $\underline{\mathrm{Rep}}_k(G') \to \underline{\mathrm{Rep}}_k(G)$.

(a) f is faithfully flat if and only if ω^f is fully faithful and every subobject of $\omega^f(X')$, for $X' \in \mathrm{ob}(\underline{\mathrm{Rep}}_k(G'))$, is isomorphic to the image of a subobject of X'.

(b) f is a closed immersion if and only if every object of $\underline{\mathrm{Rep}}_k(G)$ is isomorphic to a subquotient of an object of the form $\omega^f(X')$, $X' \in \mathrm{ob}(\underline{\mathrm{Rep}}_k(G'))$.

Proof (a). If $G \xrightarrow{f} G'$ is faithfully flat, and therefore is an epimorphism, then $\underline{\mathrm{Rep}}_k(G')$ can be identified with the subcategory of $\underline{\mathrm{Rep}}_k(G)$ of representations $G \to GL(V)$ factoring through G'. It is therefore obvious that ω^f is fully faithful etc. Conversely, if ω^f is fully faithful, it defines an equivalence of $\mathrm{Rep}_k(G')$ with a full subcategory of $\underline{\mathrm{Rep}}_k(G)$, and the second condition shows that, for $X' \in \mathrm{ob}(\mathrm{Rep}_k(G'))$, $\langle X' \rangle$ is equivalent to $\langle \omega^f(X') \rangle$. Let $G = \mathrm{spec}\, B$ and $G' = \mathrm{spec}\, B'$; then (2.15) shows that

$$B' = \varinjlim \mathrm{End}(\omega' | \langle X' \rangle)^{\vee} = \varinjlim \mathrm{End}(\omega | \langle \omega^f(X') \rangle)^{\vee} \subset \varinjlim \mathrm{End}(\omega | \langle X \rangle)^{\vee} = B,$$

and $B' \to B$ being injective implies that $G \to G'$ is faithfully flat (Waterhouse [1,14]).

(b) Let $\underline{C}$ be the strictly full subcategory of $\underline{\mathrm{Rep}}_k(G)$ whose objects are isomorphic to subquotients of objects of the form $\omega^f(X')$. The functors

$$\underline{\mathrm{Rep}}_k(G') \to \underline{C} \to \underline{\mathrm{Rep}}_k(G)$$

correspond (see (2.14,2.15)) to homomorphisms of k-coalgebras

$$B' \to B'' \to B$$

where $G = \mathrm{spec}\, B$ and $G' = \mathrm{spec}\, B'$. An argument as in the above proof shows that $B'' \to B$ is injective. Moreover, for $X' \in \mathrm{ob}(\underline{\mathrm{Rep}}_k(G'))$, $\mathrm{End}(\omega | \langle \omega^f(X) \rangle) \to \mathrm{End}(\omega' | \langle X' \rangle)$ is injective, and so $B' \to B''$ is surjective. If f is a closed immersion, then $B' \to B$ is surjective and it follows that $B'' \xrightarrow{\approx} B$, and $\underline{C} = \underline{\mathrm{Rep}}_k(G)$. Conversely, if $\underline{C} = \underline{\mathrm{Rep}}_k(G)$, then $B'' = B$ and $B' \to B$ is surjective.

<u>Corollary</u> 2.22 Assume k has characteristic zero; then G is connected if and only if, for any representation X of G on which G acts non-trivially, the strictly full subcategory of $\underline{\mathrm{Rep}}_k(G)$ whose objects are isomorphic to subquotients of X^n , $n \geq 0$, is not stable under $\otimes$.

<u>Proof</u>: G is connected if and only if there is no non-trivial epimorphism $G \to G'$ with G' finite. According to (2.21a), this is equivalent to $\underline{\mathrm{Rep}}_k(G)$ having no non-trivial subcategory of the type described in (2.20a).

Proposition 2.23. Assume k has characteristic zero and that G is connected; then G is pro-reductive if and only if $\mathrm{Rep}_k(G)$ is semisimple.

Proof: As every finite-dimensional representation $G \to GL(V)$ of G factors through an algebraic quotient of G, we can assume that G itself is an algebraic group.

Lemma 2.24. Let X be a representation of G; a subspace $Y \subset X$ is stable under G if and only if it is stable under $\mathrm{Lie}(G)$.

Proof: Standard.

Lemma 2.25. Let $\bar{k}$ be the algebraic closure of k; then $\mathrm{Rep}_k(G)$ is semisimple if and only if $\mathrm{Rep}_{\bar{k}}(G_{\bar{k}})$ is semisimple.

Proof: Let $U(G)$ be the universal enveloping algebra of $\mathrm{Lie}(G)$, and let X be a finite-dimensional representation of G. The last lemma shows that X is semisimple as a representation of G if and only if it is semisimple as a representation of $\mathrm{Lie}(G)$, or of $U(G)$. But X is a semisimple $U(G)$-module if and only if $\bar{k} \otimes_k X$ is a semisimple $\bar{k} \otimes U(G)$-module (Bourbaki [1,13.4]). Since $\bar{k} \otimes U(G) = U(G_{\bar{k}})$, this shows that if $\mathrm{Rep}_{\bar{k}}(G_{\bar{k}})$ is semisimple then so is $\mathrm{Rep}_k(G)$. For the converse, let $\bar{X}$ be an object of $\mathrm{Rep}_{\bar{k}}(G_{\bar{k}})$. There is a finite extension k' of k and a representation X' of $G_{k'}$ over k' giving $\bar{X}$ by extension of scalars. If we regard X' as a vector space over k then we obtain a k-representation X of G. By assumption, X is semisimple and, as was observed above, this implies that $\bar{k} \otimes_k X$ is semisimple. Since $\bar{X}$ is a quotient of $\bar{k} \otimes_k X$, $\bar{X}$ is semisimple.

Lemma 2.26 (Weyl). Let L be a semisimple Lie algebra over an algebraically closed field k (of characteristic zero). Any finite-dimensional representation of L is semisimple.

Proof: For an algebraic proof, see for example (Humphries [1, 6.3]). Weyl's original proof was as follows: we can assume $k = \mathbb{C}$; let L_0 be a compact real form of L and G_0 a connected simply-connected real Lie group with Lie algebra L_0 ; as G_0 is compact, any finite-dimensional representation of it carries a positive-definite form (see (I3.6)) and therefore is semisimple; thus any finite-dimensional (real or complex) representation of L_0 is semisimple, and it is then obvious that any (complex) representation of L is semisimple.

For the remainder of the proof, we assume that k is algebraically closed.

Lemma 2.27. If N is a normal closed subgroup of G and $\rho : G \to GL(X)$ is semisimple, then $\rho|N$ is semisimple.

Proof: We can assume X is a simple G-module. Let Y be a nonzero simple N-submodule of X. For any $g \in G(k)$, gY is an N-module and is simple because $S \mapsto g^{-1}S$ maps N-submodules of gY to N-submodules of Y. The sum $\Sigma\, gY$, $g \in G(k)$, is G-stable and nonzero, and therefore equals X. Thus X, being a sum of simple N-submodules, is semisimple.

We now prove the proposition. If G is reductive, then $G = Z.G'$ where Z is the centre of G and G' is the derived subgroup of G. Let $\rho\colon G \to GL(X)$ be a finite-dimensional representation of G. As Z is a torus, $\rho|Z$ is diagonalizable: $X = \oplus X_i$ as a Z-module, where any $z \in Z$ acts on X_i as a scalar $\chi_i(z)$. Each X_i is G'-stable and, as G' is semisimple, is a direct sum of simple G'-modules. It is now clear that X is semisimple as a G-module.

Conversely, assume that $\underline{\mathrm{Rep}}_k(G)$ is semisimple and choose a faithful representation X of G. Let N be the unipotent radical of G. Lemma 2.27 shows that X is semisimple as an N-module: $X = \oplus X_i$ where each X_i is a simple N-module. As N is solvable, the Lie-Kolchin theorem shows that each X_i has dimension one, and as N is unipotent, it has a fixed vector in each X_i. Therefore N acts trivially on each X_i, and on X, and, as X is faithful, this shows that $N = \{1\}$.

Remark 2.28. The proposition can be strengthened as follows: assume that k has characteristic zero; then the identity component G^0 of G is pro-reductive if and only if $\underline{\mathrm{Rep}}_k(G)$ is semisimple.

To prove this one has to show that $\underline{\mathrm{Rep}}_k(G)$ is semisimple if and only if $\underline{\mathrm{Rep}}_k(G^\circ)$ is semisimple. The necessity follows from (2.27). For the sufficiency, let X be a representation of G (where G is assumed to be algebraic) and let Y be a G-stable subspace of X. By assumption, there is a G°-equivariant map $p : X \to Y$ such that $p|Y = \mathrm{id}$. Define

$$q : \bar{k} \otimes X \to \bar{k} \otimes Y, \; q = \frac{1}{n} \sum_{g} g_Y p g_X^{-1}$$

where $n = (G(\bar{k}) : G^\circ(\bar{k}))$ and g runs over a set of coset representatives for $G^\circ(\bar{k})$ in $G(\bar{k})$. One checks easily that q has the following properties:

(i) it is independent of the choice of the coset representatives;

(ii) for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, $\sigma(q) = q$;

(iii) for all $y \in \bar{k} \otimes Y$, $q(y) = q$;

(iv) for all $g \in G(\bar{k})$, $g_Y \cdot q = q \cdot g_X$.

Thus q is defined over k , restricts to the identity map on Y , and is G-equivariant.

Remark 2.29. When, as in the above remark, $\underline{\mathrm{Rep}}_k(G)$ is semisimple, the second condition in (2.21a) is superfluous; thus $f : G \to G'$ is faithfully flat if and only if ω^f is fully faithful.

Examples.

(2.30) (Graded vector spaces) Let $\underline{C}$ be the category whose objects are families $(V^n)_{n \in \mathbb{Z}}$ of vector spaces over k with finite-dimensional sum $V = \oplus V^n$. There is an obvious rigid tensor structure on $\underline{C}$ for which $\mathrm{End}(\underline{1}) = k$ and $\omega : (V^n) \longmapsto \oplus V^n$ is a fibre functor. Thus, according to (2.11), there is an equivalence of tensor categories $\underline{C} \to \underline{\mathrm{Rep}}_k(G)$ for some G. This equivalence is easy to describe: Take $G = \mathbb{G}_m$ and make (V^n) correspond to the representation of $\mathbb{G}_m$ on $\oplus V^n$ for which $\mathbb{G}_m$ acts on V^n through the character $\lambda \longmapsto \lambda^n$.

(2.31) A real Hodge structure is a finite dimensional vector space V over $\mathbb{R}$ together with a decomposition $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$ such that $V^{p,q}$ and $V^{q,p}$ are conjugate complex subspaces of $V \otimes \mathbb{C}$. There is an obvious rigid tensor structure on the category $\underline{\mathrm{Hod}}_{\mathbb{R}}$ of real Hodge structures and $\omega : (V,(V^{p,q})) \longmapsto V$ is a fibre functor. The group corresponding to $\underline{\mathrm{Hod}}_{\mathbb{R}}$ and ω is the real algebraic group $\mathbb{S}$ obtained from $\mathbb{G}_m$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$: $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. The real Hodge structure $(V,(V^{p,q}))$ corresponds to the representation of $\mathbb{S}$ on V such that an element $\lambda \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $V^{p,q}$ as $\lambda^{-p}\bar{\lambda}^{-q}$. We can write $V = \oplus V^n$ where $V^n \oplus \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$. The functor $(V,(V^{p,q})) \longmapsto (V^n)$ from $\underline{\mathrm{Hod}}_{\mathbb{R}}$ to the category of real graded vector spaces corresponds to the homomorphism $\mathbb{G}_m \to \mathbb{S}$ which, on real points, is $t \longmapsto t^{-1}$: $\mathbb{R}^\times \to \mathbb{C}^\times$.

(2.32) The preceding examples have a common generalization. Recall that an algebraic group G is of <u>multiplicative type</u> if $G_{\bar{k}}$, where $\bar{k}$ is the separable algebraic closure of k, is diagonalizable in some faithful representation, and that the character group $X(G) \overset{\mathrm{df}}{=\!=} \mathrm{Hom}(G_{\bar{k}}, \mathbb{G}_m)$ of such a G is a finitely generated abelian group on which $\Gamma = \mathrm{Gal}(\bar{k}/k)$ acts continuously. Write $M = X(G)$, and let $k' \subset \bar{k}$ be a Galois extension of k over which all elements of M are defined. For any finite-dimensional representation V of G, $V \otimes_k k' = \bigoplus_{m \in M} V^m$ where $V^m = \{v \in V \otimes_k k' \mid gv = m(g)v \text{ , all } g \in G(k')\}$. A finite-dimensional vector space V over k together with a decomposition $k' \otimes V = \oplus V^m$ arises from a representation of G if and only if $V^{\sigma(m)} = \sigma V^m (\overset{\text{df}}{=} V^m_{\otimes_{k',\sigma}} k')$ for all $m \in M$ and $\sigma \in \Gamma$. Thus an object of $\underline{\mathrm{Rep}}_k(G)$ can be identified with a finite-dimensional vector space V over k together with an M-grading on $V \otimes_k k'$ that is compatible with the action of the Galois group.

(2.33) (Tannakian duality) Let K be a topological group. The category $\underline{\mathrm{Rep}}_{\mathbb{R}}(K)$ of continuous representations of K on finite-dimensional real vector spaces is, in a natural way, a neutral Tannakian category with the forgetful functor as fibre functor. There is therefore a real affine algebraic group $\tilde{K}$, called the <u>real algebraic envelope</u> of K, for which there exists an equivalence $\underline{\mathrm{Rep}}_{\mathbb{R}}(K) \tilde{\to} \underline{\mathrm{Rep}}_{\mathbb{R}}(\tilde{K})$. There is also a map $K \to \tilde{K}(\mathbb{R})$, which is an isomorphism when K is compact.

In general, a real algebraic group G is said to be <u>compact</u> if $G(\mathbb{R})$ is compact and the natural functor $\underline{\mathrm{Rep}}_{\mathbb{R}}(G(\mathbb{R})) \to \underline{\mathrm{Rep}}_{\mathbb{R}}(G)$ is an equivalence. The second condition is equivalent to each connected component of $G(\mathbb{C})$ containing a real point (or to $G(\mathbb{R})$ being Zariski dense in G). We note for reference that Deligne [1, 2.5] shows that a subgroup of a compact real algebraic group is compact.

(2.34) (The true fundamental group). Recall that a vector bundle E on a curve C is <u>semi-stable</u> if for every sub-bundle $E' \subset E$, $(\deg E')/(\operatorname{rank} E') \leq (\deg E)/(\operatorname{rank} E)$. Let X be a complete connected reduced k-scheme, where k is assumed to be perfect. A vector bundle E on X will be said to be semi-stable if for every nonconstant morphism $f : C \to X$ with C a projective smooth connected curve, f^*E is semi-stable of degree zero. Such a bundle E is <u>finite</u> if there exist polynomials $g,h \in \mathbb{N}[t]$, $g \neq h$, such that $g(E) \approx h(E)$. Let $\underline{C}$ be the category of semi-stable vector bundles on X that are isomorphic to a subquotient of a finite vector bundle. If X has a k-rational point x then $\underline{C}$ is a neutral Tannakian category over k with fibre functor $\omega : E \longmapsto E_x$. The group associated with $(\underline{C},\omega)$ is a pro-finite group scheme over k , called the <u>true fundamental group</u> $\pi_1(X,x)$ of X , which classifies all G-coverings of X with G a finite group scheme over k . The maximal pro-étale quotient of $\pi_1(X,x)$ is the usual étale fundamental group of X . See Nori [1].

(2.35) Let K be a field of characteristic zero, complete with respect to a discrete valuation, whose residue field is algebraically closed of characteristic $p \neq 0$. The Hodge-Tate modules for K from a neutral Tannakian category over $\mathbb{Q}_p$ (see Serre [2]).

§3. Fibre Functors; the general notion of a Tannakian category

Throughout this section, k denotes a field

Fibre functors

Let G be an affine group scheme over k and let $U = \operatorname{spec} R$ be an affine k-scheme. A G-<u>torsor</u> over U (for the f.p.q.c. topology) is an affine scheme T, faithfully flat over S, together with a morphism $T \times_U G \to T$ such that $(t,g) \longmapsto (t,tg) : T \times_U G \to T \times T$ is an isomorphism. Such a scheme T is determined by its points functor, $h_T = (R' \longmapsto T(R'))$. A non-vacuous set-valued functor h of R-algebras with functorial pairing $h(R') \times G(R') \to h(R')$ arises from a G-torsor if

(3.1a) For each R-algebra R' such that $h(R')$ is non-empty, $G(R')$ acts simply transitively on $h(R')$, and

(3.1b) h is respectable by an affine scheme faithfully flat over U. Descent theory shows that (3.1b) can be replaced by the condition that h be a sheaf for the f.p.q.c. topology on U (see Waterhouse [1,V]). There is an obvious notion of a morphism of G-torsors.

Let $\underline{C}$ be a k-linear abelian tensor category; a <u>fibre functor</u> on $\underline{C}$ with values in a k-algebra R is a k-linear exact faithful tensor functor $\eta : \underline{C} \to \underline{\mathrm{Mod}}_R$ that takes values in the subcategory $\underline{\mathrm{Proj}}_R$ of $\underline{\mathrm{Mod}}_R$. Assume now that $\underline{C}$ is a neutral Tannakian category over k. There then exists a fibre functor ω with values in k and we proved in the last section that if we let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, ω defines an equivalence $\underline{C} \xrightarrow{\sim} \underline{\mathrm{Rep}}_k(G)$. For any fibre functor η with values in R, composition defines a pairing

$$\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta) \times \underline{\mathrm{Aut}}^{\otimes}(\omega) \to \underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$$

of functors of R-algebras. Proposition 1.13 shows that

$\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta) = \underline{\mathrm{Isom}}^{\otimes}(\omega,\eta)$, and therefore that $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ satisfies (3.1a).

Theorem 3.2. Let $\underline{C}$ be a neutral Tannakian category over k .
(a) For any fibre functor η on $\underline{C}$ with values in R , $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ is representable by an affine scheme faithfully flat over $\operatorname{spec} R$; it is therefore a G-torsor.
(b) The functor $\eta \longmapsto \underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ determines an equivalence between the category of fibre functors on $\underline{C}$ with values in R and the category of G-torsors over R .

Proof: Let $X \in \mathrm{ob}(\underline{C})$, and, with the notations of the proof of (2.11), define

$$\begin{cases} A_X \subset \mathrm{End}(\omega(X)) \ , \ A_X = \bigcap\limits_{Y} (\omega(Y) : \omega(Y)) \ , \ Y \subset X^n \ , \\ P_X \subset \underline{\mathrm{Hom}}(\omega(X), X) \ , \ P_X = \bigcap\limits_{Y} (Y : \omega(Y)) \ , \ Y \subset X^n \ . \end{cases}$$

Then $\omega(P_X) = A_X$ and $P_X \in \mathrm{ob}(\langle X\rangle)$. For any R-algebra R' , $\underline{\mathrm{Hom}}(\omega|\langle X\rangle, \eta|\langle X\rangle)(R')$ is the subspace of $\mathrm{Hom}(\omega(P_X)\otimes_k R', \eta(P_X)\otimes_R R')$ of maps respecting all $Y \subset X^n$; it therefore equals $\eta(P_X) \otimes R'$. Thus

$$\underline{\mathrm{Hom}}(\omega|\langle X\rangle, \eta|\langle X\rangle)(R') \xrightarrow{\approx} \mathrm{Hom}_{R\text{-lin}}(\eta(P_X^\vee), R') \ .$$

Let Q be the ind-object $(P_X^\vee)_X$, and let $B = \varinjlim A_X^\vee$. As we saw in the last section, the tensor structure on $\underline{C}$ defines an algebra structure on B ; it also defines a ring structure on Q (i.e., a map $Q \otimes Q \to Q$ in $\mathrm{Ind}(\underline{C})$) making $\omega(Q) \stackrel{\approx}{\to} B$ into an isomorphism of k-algebras. We have

$$\begin{aligned}\underline{\mathrm{Hom}}(\omega,\eta)(R') &= \varprojlim \underline{\mathrm{Hom}}(\omega|\langle X\rangle,\ \eta|\langle X\rangle)(R') \\ &= \varprojlim \mathrm{Hom}_{R\text{-lin}}(\eta(P_X^\vee),\ R') \\ &= \mathrm{Hom}_{R\text{-lin}}(\eta(Q),\ R)\end{aligned}$$

where $\eta(Q) \stackrel{\mathrm{df}}{=} \varinjlim \eta(P_X^\vee)$. Under this correspondence,

$$\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)(R') = \mathrm{Hom}_{R\text{-alg}}(\eta(Q),R') ,$$

and so $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ is represented by $\eta(Q)$. By definition $\eta(P_X^\vee)$ is a projective R-module, and so $\eta(Q) = \varinjlim \eta(P_X^\vee)$ is flat over R . For each X there is a surjection $P_X \twoheadrightarrow \underline{1}$, and the exact sequence

$$0 \to \underline{1} \to P_X^\vee \to P_X^\vee/\underline{1} \to 0$$

gives rise to an exact sequence

$$0 \to \eta(\underline{1}) \to \eta(\underline{P}_X^\vee) \to \eta(\underline{P}_X^\vee/\underline{1}) \to 0$$

As $\eta(\underline{1}) = R$ and $\eta(\underline{P}_X^\vee/\underline{1})$ is flat, this shows that $\eta(\underline{P}_X^\vee)$ is a faithfully flat R-module. Hence $\eta(Q)$ is faithfully flat over R , which completes the proof that $\underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ is a G-torsor.

To show that $\eta \mapsto \underline{\mathrm{Hom}}^{\otimes}(\omega,\eta)$ is an equivalence, we construct a quasi-inverse. Let T be a G-torsor over R. For a fixed X, define $R' \mapsto \eta_T(X)(R')$ to be the sheaf associated with

$$R' \longmapsto (\omega(X) \otimes R') \times T(R')/G(R') .$$

Then $X \mapsto \eta_T(X)$ is a fibre functor on $\underline{C}$ with values in R.

Remark 3.3

(a) Define

$$A_X \subset \underline{\mathrm{Hom}}(X,X), \ \underline{A}_X = \bigcap (Y:Y) \ , \ Y \subset X^n .$$

Then $\underline{A}_X$ is a ring in $\underline{C}$ such that $\omega(\underline{A}_X) = A_X$ (as k-algebras). Let $\underline{B}$ be the ind-object $(\underline{A}_X^{\vee})$. Then

$$\underline{\mathrm{End}}^{\otimes}(\omega) = \mathrm{spec}\ \omega(\underline{B}) = G$$
$$\underline{\mathrm{End}}^{\otimes}(\eta) = \mathrm{spec}\ \eta(B) .$$

(b) The proof of (3.2) can be made more concrete by using (2.11) to replace $(\underline{C},\omega)$ with $(\underline{\mathrm{Rep}}_k(G),\omega^G)$.

Remark 3.4. The situation described in the theorem is analogous to the following. Let X be a connected topological space and let $\underline{C}$ be the category of locally constant sheaves of $\mathbb{Q}$ vector spaces on X. For any $x \in X$, there is a fibre functor $\omega_x : \underline{C} \to \underline{\mathrm{Vec}}_{\mathbb{Q}}$, and ω_x defines an equivalence of categories

$\underline{C} \tilde{\to} \underline{\mathrm{Rep}}_{\mathbb{Q}}(\pi_1(X,x))$. Let $\Pi_{x,y}$ be the set of homotopy classes of paths from x to y ; then $\Pi_{x,y} \approx \underline{\mathrm{Isom}}(\omega_x,\omega_y)$, and $\Pi_{x,y}$ is a $\pi_1(X,x)$-torsor.

Question 3.5. Let $\underline{C}$ be a rigid abelian tensor category whose objects are of finite length and which is such that $\mathrm{End}(\underline{1})=k$ and $\otimes$ is exact. (Thus $\underline{C}$ lacks only a fibre functor with values in k to be a neutral Tannakian category). As in (3.3) one can define

$$\underline{A}_X \subset \underline{\mathrm{Hom}}\ (X,X)\,,\ \underline{A}_X = \bigcap (Y:Y)\,,\ Y \subset X^n$$

and hence obtain a bialgebra $\underline{B} = \text{"}\varinjlim\text{"}\ A_X^\vee$ in $\mathrm{Ind}(\underline{C})$ which can be thought of as defining an affine group scheme G in $\mathrm{Ind}(\underline{C})$.

Is it true that for $X \subset X'$, $A_{X'} \to A_X$ is an epimorphism?

For any X in $\underline{C}$, there is a morphism $X \xrightarrow{\rho} X \otimes \underline{B}$, which can be regarded as a representation of G . Define X^G , the subobject fixed by G , to be the largest subobject of X such that $X^G \to X \otimes B_X$ factors through $X^G \otimes 1 \hookrightarrow X \otimes B_X$. Is it true that $\underline{\mathrm{Hom}}(\underline{1},X) \otimes_k \underline{1} \to X^G$ is an isomorphism?

If for all X there exists an N such that $\wedge^N X = 0$, is $\underline{C}$ Tannakian in the sense of Definition 3.7 below? (See note at the end of the article.)

The general notion of a Tannakian category

In this subsection, we need to use some terminology from non-abelian 2-cohomology, for which we refer the reader to the

Appendix. In particular $\underline{\mathrm{Aff}}_S$ or $\underline{\mathrm{Aff}}_k$ denotes the category of affine schemes over $S = \operatorname{spec} k$ and PROJ is the stack over $\underline{\mathrm{Aff}}_S$ such that $\mathrm{PROJ}_U = \underline{\mathrm{Proj}}_R$ for $R = \Gamma(U, \mathcal{O}_U)$. For any gerb $\underline{G}$ over $\underline{\mathrm{Aff}}_k$ (for the f.p.q.c. topology) we let $\underline{\mathrm{Rep}}_k(\underline{G})$ denote the category of cartesian functors $\underline{G} \to \mathrm{PROJ}$. Thus an object Φ of $\underline{\mathrm{Rep}}_k(G)$ determines (and is determined by) functors $\Phi_R : \underline{G}_R \to \underline{\mathrm{Proj}}_R$, one for each k-algebra R, and functorial isomorphisms $\Phi_{R'}(g^*Q) \xleftrightarrow{\approx} \Phi_R(Q) \otimes_R R'$ defined whenever $g : R \to R'$ is a homomorphism of k-algebras and $Q \in \mathrm{ob}(\underline{G}_R)$. There is an obvious rigid tensor structure on $\underline{\mathrm{Rep}}_k(G)$, and $\mathrm{End}(\underline{1}) = k$.

Example 3.6. Let G be an affine group scheme over k, and let $\mathrm{TORS}(G)$ be the gerb over $\underline{\mathrm{Aff}}_S$ such that $\mathrm{TORS}(G)_U$ is the category of G-torsors over U. Let G_r be G regarded as a right G-torsor, and let Φ be an object of $\underline{\mathrm{Rep}}_k(\mathrm{TORS}(G))$. The isomorphism $G \xrightarrow{\approx} \underline{\mathrm{Aut}}(G_r)$ defines a representation of G on the vector space $\Phi_k(G_r)$, and it is not difficult to show that $\Phi \longmapsto \Phi_k(G_r)$ extends to an equivalence of categories $\underline{\mathrm{Rep}}_k(\mathrm{TORS}(G)) \xrightarrow{\approx} \underline{\mathrm{Rep}}_k(G)$.

Let $\underline{C}$ be a rigid abelian tensor category with $\mathrm{End}(\underline{1}) = k$. For any k-algebra R, the fibre functors on $\underline{C}$ with values in R form a category $\mathrm{FIB}(\underline{C})_R$, and the collection of these categories forms in a natural way a fibred category $\mathrm{FIB}(\underline{C})$ over $\underline{\mathrm{Aff}}_k$. Descent theory for projective modules shows that $\mathrm{FIB}(\underline{C})$ is a stack, and (1.13) shows that its fibres are groupoids. There is a canonical k-linear tensor functor $\underline{C} \to \underline{\mathrm{Rep}}_k(\mathrm{FIB}(C))$ associating to $X \in \mathrm{ob}(\underline{C})$ the family of functors $\omega \longmapsto \omega(X) : \mathrm{FIB}(\underline{C})_R \to \underline{\mathrm{Proj}}_R$.

<u>Definition</u> 3.7. A <u>Tannakian category</u> over k is a rigid abelian tensor category $\underline{C}$ with $\mathrm{End}(\underline{1}) = k$ such that $\mathrm{FIB}(\underline{C})$ is an affine gerb and $\underline{C} \to \underline{\mathrm{Rep}}_k(\mathrm{FIB}(\underline{C}))$ is an equivalence of categories.

<u>Example</u> 3.8. Let $\underline{C}$ be a neutral Tannakian category over k . Theorem 3.2 shows that the choice of a fibre functor ω with values in k determines an equivalence of fibred categories $\mathrm{FIB}(\underline{C}) \xrightarrow{\sim} \mathrm{TORS}(G)$ where G represents $\underline{\mathrm{Aut}}^{\otimes}(\omega)$. Thus $\mathrm{FIB}(\underline{C})$ is an affine gerb and the commutative diagram of functors

$$\begin{array}{ccc} \underline{C} & \to & \underline{\mathrm{Rep}}_k(\mathrm{FIB}(\underline{C})) \\ \sim\downarrow\omega & & \downarrow\sim \\ \underline{\mathrm{Rep}}_k(G) & \xrightarrow{\sim} & \underline{\mathrm{Rep}}_k(\mathrm{TORS}(G)) \end{array}$$

shows that $\underline{C}$ is a Tannakian category. Thus a Tannakian category in the sense of (3.7) is a neutral Tannakian category in the sense of (2.19) if an only if it has a fibre functor with values in k .

<u>Remark</u> 3.9. The condition in (3.7) that $\mathrm{FIB}(\underline{C})$ is a gerb means that $\underline{C}$ has a fibre functor ω with values in some field $k' \supset k$ and that any two fibre functors are locally isomorphic for the f.p.q.c. topology. The condition that the gerb $\mathrm{FIB}(\underline{C})$ be affine means that $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is representable by an affine group scheme over k' .

Remark 3.10. A Tannakian category $\underline{C}$ over k is said to be *algebraic* if $\mathrm{FIB}(\underline{C})$ is an algebraic gerb. There then exists a finite field extension k' of k and a fibre functor ω with values in k' (App., Proposition), and the algebraicity of $\underline{C}$ means that $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is an algebraic group over k'. As in the neutral case (2.20), a Tannakian category is algebraic if and only if it has a tensor generator. Consequently, any Tannakian category is a filtered union of algebraic Tannakian categories.

Tannakian categories neutralized by a finite extension

Let $\underline{C}$ be a k-linear category, and let A be a commutative k-algebra. An *A-module* in $\underline{C}$ is a pair (X, α_X) with X an object of $\underline{C}$ and α_X a homomorphism $A \to \mathrm{End}(X)$. For example, an A-module in $\underline{\mathrm{Vec}}_{k'}$, where $k' \supset k$, is simply an $A \otimes_k k'$-module that is of finite dimension over k'. With an obvious notion of morphism, the A-modules in $\underline{C}$ form an A-linear category $\underline{C}_{(A)}$. If $\underline{C}$ is abelian so also is $\underline{C}_{(A)}$, and if $\underline{C}$ has a tensor structure and its objects have finite length then we define $(X, \alpha_X) \otimes (Y, \alpha_Y)$ to be the A-module in $\underline{C}$ with object the largest quotient of $X \otimes Y$ to which $\alpha_X(a) \otimes \mathrm{id}$ and $\mathrm{id} \otimes \alpha_Y(a)$ agree for all $a \in A$.

Now let $\underline{C}$ be a Tannakian category over k, and let k' be a finite field extension of k. As the tensor operation on $\underline{C}$ commutes with direct limits (1.16), it extends to $\mathrm{Ind}(\underline{C})$, which is therefore an abelian tensor category. The functor $\underline{C} \to \mathrm{Ind}(\underline{C})$ defines an equivalence between $\underline{C}$ and the strictly full subcategory $\underline{C}^e$ of $\mathrm{Ind}(\underline{C})$ of essentially constant ind-objects. In $\underline{C}^e$ it is possible to define external tensor products with objects of $\underline{\mathrm{Vec}}_k$ (cf. the proof of (2.11)) and hence a functor

$$X \mapsto i(X) = (k' \otimes_k X,\ a' \mapsto a' \otimes \mathrm{id}) : \underline{C}^e \to \underline{C}^e_{(k')} .$$

This functor is left adjoint to

$$(X,\alpha) \mapsto j(X,\alpha) = X : \underline{C}^e_{(k')} \to \underline{C}^e$$

and has the property that $k' \otimes_k \mathrm{Hom}(X,Y) \xrightarrow{\approx} \mathrm{Hom}(i(X),\ i(Y))$. Let ω be a fibre functor on $\underline{C}^e$ (or $\underline{C}$) with values in k' . For any $(X,\alpha) \in \mathrm{ob}(\underline{C}^e_{(k')})$, $(\omega(X),\ \omega(\alpha))$ is a k'-module in $\underline{\mathrm{Vec}}_{k'}$, i.e., it is a $k' \otimes_k k'$-module. If we define

$$\omega'(X,\alpha) = k' \otimes_{k'\otimes k'} \omega(X) \qquad (3.10.1)$$

Then

$$\begin{tikzcd} \underline{C}^e \arrow[r] \arrow[dr, "\omega"'] & \underline{C}^e_{(k')} \arrow[d, "\omega'"] \\ & \underline{\mathrm{Vec}}_{k'} \end{tikzcd}$$

commutes up to a canonical isomorphism.

Proposition 3.11. Let $\underline{C}$ be a Tannakian category over k and let ω be a fibre functor on $\underline{C}$ with values in a finite field extension k' of k ; extend ω' to $\underline{C}_{(k')}$ using the formula (3.10.1) ; then ω' defines an equivalence of tensor categories $\underline{C}_{(k')} \tilde{\to} \underline{\mathrm{Rep}}_{k'}(G)$ where $G = \underline{\mathrm{Aud}}^{\otimes}(\omega)$. In particular, ω' is exact.

Proof: One has simply to compose the following functors:

$$\underline{C}_{(k')} \tilde{\to} \underline{\mathrm{Rep}}_k(G)_{(k')}$$

arising from the equivalence $\underline{C} \tilde{\to} \underline{\mathrm{Rep}}_k(\underline{G})$ $(\underline{G} = \mathrm{FIB}(\underline{C}))$ in the definition (3.7);

$$\mathrm{Rep}_k(\underline{G})_{(k')} \tilde{\to} \underline{\mathrm{Rep}}_{k'}(\underline{G},/k')$$

where $\underline{G}/k'$ denotes the restriction of $\underline{G}$ to $\underline{\mathrm{Aff}}_{k'}$ (the functor sends $(\Phi,\alpha) \in \mathrm{ob}(\underline{\mathrm{Rep}}_k(\underline{G})_{(k')})$ to Φ' where, for any k'-algebra R and $Q \in \underline{G}_R$, $\Phi'_R(Q) = R \otimes_{k'\otimes R}\Phi_R(Q))$;

$$\underline{\mathrm{Rep}}_{k'}(\underline{G}/k') \tilde{\to} \underline{\mathrm{Rep}}_{k'}(\mathrm{TORS}(G))$$

arising from $\mathrm{TORS}(G) \tilde{\to} \underline{G}/k'$;

$$\underline{\mathrm{Rep}}_{k'}(\mathrm{TORS}(G)) \tilde{\to} \underline{\mathrm{Rep}}_{k'}(G) \qquad \text{(see 3.6).}$$

<u>Remark</u> 3.12. Let $\underline{C} = \underline{\mathrm{Rep}}_k(G)$ and let k' be a finite extension of k . Then $\underline{C}_{(k')} = \underline{\mathrm{Rep}}_{k'}(G)$ and $i : \underline{C} \to \underline{C}_{(k')}$ is $X \mapsto k' \otimes_k X$. Let ω be the fibre functor $X \mapsto k' \otimes_k X : \underline{\mathrm{Rep}}_k(G) \to \underline{\mathrm{Vec}}_{k'}$. Then $G_{k'} = \mathrm{Aut}^{\otimes}(\omega)$ and the equivalence $\underline{C}_{(k')} \xrightarrow{\sim} \underline{\mathrm{Rep}}_{k'}(G_{k'})$ defined in the proposition is

$$X \mapsto k' \otimes_{k'\otimes k'} X : \underline{\mathrm{Rep}}_{k'}(G) \to \underline{\mathrm{Rep}}_{k'}(G_{k'}) .$$

<u>Descent of Tannakian categories</u>

Let k'/k be a finite Galois extension with Galois group Γ , and let $\underline{C}'$ be a Tannakian category over k' . A <u>descent datum</u> on $\underline{C}'$ relative to k'/k is

(3.13a) a family $(\beta_\gamma)_{\gamma\in\Gamma}$ of equivalences of tensor categories $\beta_\gamma : \underline{C}' \to \underline{C}'$, β_γ being semi-linear relative to γ , together with

(3.13b) a family $(\mu_{\gamma',\gamma})$ of isomorphisms of tensor functors $\mu_{\gamma',\gamma} : \beta_{\gamma'\gamma} \xrightarrow{\approx} \beta_{\gamma'} \circ \beta_\gamma$ such that

$$\begin{array}{ccc}
\beta_{\gamma''\gamma'\gamma}(X) & \xrightarrow{\mu_{\gamma'',\gamma'\gamma}(X)} & \beta_{\gamma''}(\beta_{\gamma'\gamma}(X)) \\
\Big\downarrow{\scriptstyle \mu_{\gamma''\gamma',\gamma}(X)} & & \Big\downarrow{\scriptstyle \beta_{\gamma''}(\mu_{\gamma',\gamma}(X))} \\
\beta_{\gamma''\gamma'}(\beta_\gamma(X)) & \xrightarrow{\mu_{\gamma''\gamma'}(\beta_\gamma(X))} & \beta_{\gamma''}(\beta_{\gamma'}(\beta_\gamma(X)))
\end{array}$$

commutes for all $X \in \mathrm{ob}(\underline{C})$.

A Tannakian category $\underline{C}$ over k gives rise to a Tannakian category $\underline{C}' = \underline{C}_{(k')}$ over k' together with a descent datum for which $\beta_\gamma(X,\alpha_X) = (X,\alpha_X\circ\gamma^{-1})$. Conversely, a Tannakian category $\underline{C}'$ over k' together with a descent datum relative to k'/k gives rise to a Tannakian category $\underline{C}$ over k whose objects are pairs $(X,(a_\gamma))$, where $X \in ob(\underline{C}')$ and $(a_\gamma : X \to \beta_\gamma(X))_{\gamma\in\Gamma}$ is such that $(\mu_{\gamma',\gamma})_X\circ a_{\gamma'\gamma} = \beta_{\gamma'}(a_\gamma)\circ a_{\gamma'}$, and whose morphisms are morphisms in $\underline{C}'$ commuting with the a_γ . These two operations are quasi-inverse, so that to give a Tannakian category over k (up to a tensor equivalence, unique up to a unique isomorphism) is the same as to give a Tannakian category over k' together with a descent datum relative to k'/k (Saavedra [1, III 1.2]). On combining this statement with (3.11) we see that to give a Tannakian category over k together with a fibre functor with values in k' is the same as to give an affine group scheme G over k' together with a descent datum on the Tannakian category $\underline{Rep}_{k'}(G)$.

Questions

(3.14) Let $\underline{G}$ be an affine gerb over k . There is a morphism of gerbs

$$\underline{G} \to FIB(\underline{Rep}_k(\underline{G})) \tag{3.14.1}$$

which, to an object Q of $\underline{G}$ over $S = spec\ R$, associates the fibre functor $F \longmapsto F(Q)$ with values in R . Is (3.14.1) an equivalence of gerbs? If $\underline{G}$ is algebraic, or if the band of $\underline{G}$ is defined by an affine group scheme over k , then it is (Saavedra

[1, III 3.2.5]) but the general question is open. A positive answer would provide the following classification of Tannakian categories: The maps $\underline{C} \mapsto FIB(\underline{C})$ and $\underline{G} \mapsto \underline{Rep}_k(\underline{G})$ determine a one-to-one correspondence between the set of tensor equivalence classes of Tannakian categories over k and the set of equivalence classes of affine gerbs over k ; the affine gerbs bound by a given band B are classified by $H^2(S,B)$, and $H^2(S,B)$ is a pseudo-torsor over $H^2(S,Z)$ where Z is the centre of B .

(3.15) In [1, III 3.2.1] Saavedra defines a Tannakian category over k to be a k-linear rigid abelian tensor category $\underline{C}$ for which there exists a fibre functor with values in a field $k' \supset k$. He then claims to prove (ibid. 3.2.3.1) that $\underline{C}$ satisfies the conditions we have used to define a Tannakian category. This is false. For example, $\underline{Vec}_{k'}$ for k' a field containing k is a Tannakian category over k according to his definition but the fibre functors $V \mapsto \sigma V \overset{df}{=} V \otimes_{k',\sigma} k'$ for $\sigma \in Aut(k'/k)$ are not locally isomorphic for the f.p.q.c. topology on spec k' . There is an error in the proof (ibid. p. 197, ℓ.7) where it is asserted that "par définition" the objects of G_S are locally isomorphic.

The question remains of whether Saavedra's conditions plus the condition that $End(\underline{1}) = k$ imply our conditions. As we noted in (3.8), when there is a fibre functor with values in k they do, but the general question is open. The essential point is the following: Let $\underline{C}$ be a rigid abelian tensor category with $End(\underline{1}) = k$ and let ω be a fibre functor with values in a finite field extension k' of k ; is the functor ω' ,

$$X \longmapsto k' \otimes_{k'\otimes k'} \omega(X) \; : \; \underline{C}_{(k')} \to \underline{\mathrm{Vec}}_{k'} \; ,$$

exact? (See Saavedra [1, p. 195]; the proof there that ω' is faithful is valid.) The answer is yes if $\underline{C} = \underline{\mathrm{Rep}}_k(G)$, G an affine group scheme over k , but we know of no proof simpler than to say that ω' is defined by a G-torsor on k' , and $\underline{C}_{(k')} = \underline{\mathrm{Rep}}_{k'}(G)$. (See note at end.)

§4. Polarizations

Throughout this section $\underline{C}$ will be an algebraic Tannakian category over $\mathbb{R}$ and $\underline{C}'$ will be its extension to $\mathbb{C}$: $\underline{C}' = \underline{C}_{(\mathbb{C})}$.

Tannakian categories over $\mathbb{R}$

According to (3.13) and the paragraph following it, to give $\underline{C}$ is the same as to give the following data:

(4.1a) A Tannakian category $\underline{C}'$ over $\mathbb{C}$;

(4.1b) A semi-linear tensor functor $X \longmapsto \bar{X} : \underline{C}' \to \underline{C}'$;

(4.1c) A functorial tensor isomorphism $\mu_X : X \xrightarrow{\approx} \bar{\bar{X}}$ such that $\mu_{\bar{X}} = \bar{\mu}_X$.

An object of $\underline{C}$ can be identified with an object X of $\underline{C}'$ together with a descent datum (an isomorphism $a : X \xrightarrow{\approx} \bar{X}$ such that $\bar{a} \circ a = \mu_X$) . Note that $\underline{C}'$ is automatically neutral (3.10).

Example 4.2. Let G be an affine group scheme over $\mathbb{C}$ and let $\sigma : G \to G$ be a semi-linear isomorphism (meaning $f \longmapsto \sigma \circ f$: $\Gamma(G, O_G) \to \Gamma(G, O_G)$ is a semi-linear isomorphism). Assume there is given $c \in G(\mathbb{C})$ such that

$$\sigma^2 = \underline{\mathrm{ad}}(c) \ , \ \sigma(c) = c \qquad (4.2.1).$$

From (G,σ,c) we can construct data as in (4.1):

(a) define $\underline{C}'$ to be $\underline{\mathrm{Rep}}_{\mathbb{C}}(G)$;

(b) for any vector space V over $\mathbb{C}$ there is an (essentially) unique vector space $\bar{V}$ and semi-linear isomorphism $v \longmapsto \bar{v} : V \xrightarrow{\approx} \bar{V}$; if V is a G-representation, we define a representation of G on $\bar{V}$ by the rule $\overline{gv} = \sigma(g)\bar{v}$;

(c) define μ_V to be the map $cv \longmapsto \bar{\bar{v}} : V \xrightarrow{\approx} \bar{\bar{V}}$.

Let $m \in G(\mathbb{C})$. Then $\sigma' = \sigma \circ \underline{\mathrm{ad}}(m)$ and $c' = \sigma(m)cm$ again satisfy (4.2.1). The element m defines an isomorphism of the functor $V \longmapsto \bar{V}$ (rel. to (σ,c)) with the functor $V \longmapsto \bar{V}$ (rel. to (σ',c')) by $\overline{mv} \longmapsto \bar{v} : \bar{V}$ (rel. to (σ,c)) $\to \bar{V}$ (rel. to (σ',c')) . This isomorphism carries μ_V (rel. to (σ,c)) to μ_V (rel. to (σ',c')) , and hence defines an equivalence $\underline{C}$ (rel. to (σ,c)) with $\underline{C}$ (rel. to (σ',c')).

Proposition 4.3. Let $\underline{C}$ be an algebraic Tannakian category over $\mathbb{R}$, and let $\underline{C}' = \underline{C}_{(\mathbb{C})}$. Choose a fibre functor ω on $\underline{C}'$ with values in $\mathbb{C}$ and let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$.

(a) There exists a pair (σ,c) satisfying (4.2.1) and such that under the equivalence $\underline{C}' \xrightarrow{\approx} \underline{\mathrm{Rep}}_{\mathbb{C}}(G)$ defined by ω , $X \longmapsto \bar{X}$ corresponds to $V \longmapsto \bar{V}$ and $\omega(\mu_X) = \mu_{\omega(X)}$.

(b) The pair (σ,c) in (a) is uniquely determined up to replacement by a pair (σ',c') with $\sigma' = \sigma \circ \underline{\mathrm{ad}}(m)$ and $c' = \sigma(m)cm$, some $m \in G(\mathbb{C})$.

Proof: (a) Let $\bar{\omega}$ be the fibre functor $X \longmapsto \overline{\omega(\bar{X})}$ and let $T = \underline{\mathrm{Hom}}^{\otimes}(\omega, \bar{\omega})$. According to (3.2), T is a G-torsor, and the Nullstellensatz shows that it is trivial. The choice of a trivialization provides us with a functorial isomorphism $\omega(X) \xrightarrow{\approx} \bar{\omega}(X)$ and therefore with a semi-linear functorial isomorphism $\lambda_X : \omega(X) \xrightarrow{\approx} \omega(\bar{X})$. Define σ by the condition that $\sigma(g)_{\bar{X}} = \lambda_X \circ g_X \circ \lambda_X^{-1}$ for all $g \in G(\mathbb{C})$, and let c be such that $c_X = \omega(\mu_X)^{-1} \circ \lambda_{\bar{X}} \circ \lambda_X$.

(b) The choice of a different trivialization of T replaces λ_X with $\lambda_X \circ m_X$ for some $m \in G(\mathbb{C})$, and σ with $\sigma \circ \underline{\mathrm{ad}}(m)$ and c with $\sigma(m)\, cm$.

Sesquilinear forms

Let $\underline{1}$ (with $e : \underline{1} \otimes \underline{1} \xrightarrow{\approx} \underline{1}$) be an identity object for $\underline{C}'$. Then $\bar{\underline{1}}$ (with $\bar{e}$) is again an identity object, and the unique isomorphism of identity objects $a : \underline{1} \to \bar{\underline{1}}$ is a descent datum. It will be used to identify $\underline{1}$ and $\bar{\underline{1}}$.

A sesquilinear form on an object X of C' is a morphism

$$\phi : X \otimes \bar{X} \to \underline{1}.$$

On applying $-$, we obtain a morphism $\bar{X} \otimes \bar{\bar{X}} \to \bar{\underline{1}}$, which can be identified with a morphism

$$\bar{\phi} : \bar{X} \otimes X \to \underline{1}.$$

There are associated with ϕ two morphisms $\phi^{\sim}, {}^{\sim}\phi : X \to \bar{X}^{\vee}$ determined by

$$\phi^{\sim}(x)(y) = \phi(x \otimes y)$$

$$^{\sim}\phi(x)(y) = \bar{\phi}(y \otimes x) \tag{4.3.1}$$

The form ϕ is said to be <u>non-degenerate</u> if $\phi^{\sim}$ (equivalently ${}^{\sim}\phi$) is an isomorphism. The <u>parity</u> of a non-degenerate sesquilinear form ϕ is the unique morphism $\varepsilon_{\phi} : X \to X$ such that

$$\phi^{\sim} = {}^{\sim}\phi \circ \varepsilon_{\phi} \quad , \quad \phi(x,y) = \bar{\phi}(y, \varepsilon_{\phi} x) \ . \tag{4.3.2}$$

Note that

$$\phi \circ (\varepsilon_{\phi} \otimes \bar{\varepsilon}_{\phi}) = \phi, \ \phi(\varepsilon_{\phi} x, \bar{\varepsilon}_{\phi} y) = \phi(x,y) . \tag{4.3.3}$$

The <u>transpose</u> u^{ϕ} of $u \in \mathrm{End}(X)$ relative to ϕ is determined by

$$\phi \circ (u \otimes \mathrm{id}_{\bar{X}}) = \phi \circ (\mathrm{id}_X \otimes \overline{u^{\phi}}) \ , \ \phi(ux,y) = \phi(x, \overline{u^{\phi}} y) \ . \tag{4.3.4}$$

There are the formulas

$$(uv)^{\phi} = v^{\phi} u^{\phi} \ , \ (\mathrm{id})^{\phi} = \mathrm{id}, \ (u^{\phi})^{\phi} = \varepsilon_{\phi} u \varepsilon_{\phi}^{-1}, (\varepsilon_{\phi})^{\phi} = \varepsilon_{\phi}^{-1} \tag{4.3.5}$$

and $u \mapsto u^{\phi}$ is a semi-linear bijection $\mathrm{End}(X) \to \mathrm{End}(X)$.

If ϕ is a non-degenerate sequilinear form on X , then any other non-degenerate sequilinear form can be written

$$\phi_{\alpha} = \phi\circ(\alpha\otimes \mathrm{id}) \ , \ \phi_{\alpha}(x,y) = \phi(\alpha x,y) = \phi(x,\overline{\alpha^{\phi}y}) \qquad (4.3.6)$$

for a uniquely determined automorphism α of X . There are the formulas

$$u^{\phi\alpha} = (\alpha u\alpha^{-1})^{\phi} \ , \ \varepsilon_{\phi_{\alpha}} = (\alpha^{\phi})^{-1}\,\varepsilon_{\phi}\alpha \ . \qquad (4.3.7)$$

When ε_{ϕ} is in the centre of $\mathrm{End}(X)$, ϕ_{α} has the same parity as ϕ if and only if $\alpha^{\phi} = \alpha$.

Remark 4.4. There is also the notion of a bilinear form on an object X of a tensor category: It is a morphism $X \otimes X \to \underline{1}$. Most of the notions associated with bilinear forms on vector spaces make sense in the context of Tannakian categories; see Saavedra [1, V 2.1].

Weil forms

A non-degenerate sesquilinear form ϕ on X is a Weil form if its parity ε_{ϕ} is in the centre of $\mathrm{End}(X)$ and if for all nonzero $u \in \mathrm{End}(X)$, $\mathrm{Tr}_X(uu^{\phi})>0$.

Proposition 4.5. Let ϕ be a Weil form on X .

(a) The map $u \mapsto u^{\phi}$ is an involution on $\mathrm{End}(X)$ inducing complex conjugation on $\mathbb{C} = \mathbb{C}.\mathrm{id}_X$, and $(u,v) \mapsto \mathrm{Tr}(uv^{\phi})$ is a positive definite Hermitian form on $\mathrm{End}(X)$.

(b) $\mathrm{End}(X)$ is a semisimple $\mathbb{C}$-algebra.

(c) Any commutative sub-$\mathbb{R}$-algebra A of $\mathrm{End}(X)$ composed of symmetric elements (i.e., such that $u^{\phi}=u$) is a product of copies of $\mathbb{R}$.

Proof. (a) is obvious.

(b) Let I be a nilpotent ideal in $\mathrm{End}(X)$; we have to show that $I = 0$. Suppose on the contrary that there is a $u \neq 0$ in I . Then $v \overset{\mathrm{df}}{=} u\,u^{\phi} \in I$ and is nonzero because $\mathrm{Tr}(v)>0$. As $v = v^{\phi}$, we have $\mathrm{Tr}(v^2)>0$, $\mathrm{Tr}(v^4)>0,\ldots,$ contradicting the nilpotence of I .

(c) The argument used in (b) shows that A is semisimple and is therefore a product of fields. If $\mathbb{C}$ occurs as a factor of A , then $\mathrm{Tr}_X|\mathbb{C}$ is a multiple of the identity map, and $\mathrm{Tr}(u^2) = \mathrm{Tr}(uu^{\phi})>0$ is impossible.

Two Weil forms, ϕ on X and ψ on Y , are said to be compatible if the sesquilinear form $\phi\oplus\psi$ on $X\oplus Y$ is a Weil form. Note that if $\mathrm{Hom}(X,Y)=0=\mathrm{Hom}(Y,X)$, then ϕ and ψ are automatically compatible.

Proposition 4.6. Let ϕ be a Weil form on X ; then $\alpha \mapsto \phi_{\alpha} \overset{\mathrm{df}}{=} \phi\circ\alpha\otimes 1$ induces a bijection between

$$\{\alpha \in \mathrm{Aut}(X) \mid \alpha^{\phi} = \alpha, \ \alpha \text{ is a square in } \mathbb{R}[\alpha] \subset \mathrm{End}(X)\}$$

and the set of Weil forms on X that have the same parity as ϕ and are compatible with ϕ .

<u>Proof</u>: We saw in (4.3.6) that any non-degenerate sesquilinear form on X is of the form ϕ_α for a unique automorphism α of X . Moreover, ϕ_α has the same parity as ϕ if and only if $\alpha = \alpha^{\phi}$. Assume $\alpha=\alpha^{\phi}$ then $u^{\phi_\alpha} = \alpha u^{\phi}\alpha^{-1}$ and so ϕ_α is a Weil form if and only if $\mathrm{Tr}(u\alpha u^{\phi}\alpha^{-1})>0$ for all $u \neq 0$. Let $v = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \in \mathrm{End}(X \oplus X)$; then $v^{\phi\oplus\phi_\alpha} = \begin{pmatrix} 0 & u^{\phi}\alpha \\ 0 & 0 \end{pmatrix}$ and $\mathrm{Tr}_{X\oplus X}(v^{\phi\oplus\phi_\alpha}v) = \mathrm{Tr}(u^{\phi}\alpha u)$. Therefore if ϕ_α is compatible with ϕ , then $\mathrm{Tr}_X(u^{\phi}\alpha u)>0$ for all $u \neq 0$. One checks easily that the converse statement also holds.

Now assume α to be symmetric and equal to β^2 with $\beta \in \mathbb{R}[\alpha]$. Then $\mathrm{Tr}(u\alpha u^{\phi}\alpha^{-1}) = \mathrm{Tr}((u\beta)\beta u^{\phi}\alpha^{-1}) = \mathrm{Tr}(\beta u^{\phi}\alpha^{-1}(u\beta)) = \mathrm{Tr}((\beta^{-1}u\beta)^{\phi}\beta^{-1}u\beta) > 0$ for $u \neq 0$, and $\mathrm{Tr}(u^{\phi}\alpha u) = \mathrm{Tr}((\beta u)^{\phi}\beta u)>0$ for $u \neq 0$. Hence ϕ_α is a Weil form and is compatible with ϕ . Conversely, if ϕ_α has the same parity as ϕ and is compatible with it, then α is symmetric and $\mathrm{Tr}_X(u^2\alpha)>0$ for all $u \neq 0$ in $\mathbb{R}[\alpha]$; this last statement implies that α is a square in $\mathbb{R}[\alpha]$.

<u>Corollary</u> 4.7. Let ϕ and ϕ' be compatible Weil forms on X with the same parity, and let ψ be a Weil form on Y . If ϕ is compatible with ψ , then so also is ϕ' . In particular,

compatibility is an equivalence relation for Weil forms on X having a given parity.

Proof: This follows easily from writing $\phi' = \phi_\alpha$.

Example 4.8. Let X be a simple object in $\underline{C}'$, so that $\mathrm{End}(X) = \mathbb{C}$, and let $\varepsilon \in \mathrm{End}(X)$. If $\bar{X}$ is isomorphic to $\check{X}$, then (4.3.6) shows that the sesquilinear forms on X form a complex line; (4.3.7) shows that if there is a nonzero such form with parity ε , then the set of sesquilinear forms on X with parity ε is a real line; (4.6) shows that if there is a Weil form with parity ε , then the set of such forms falls into two compatibility classes, each parametrized by $\mathbb{R}_{>0}$.

Remark 4.9. Let X_o be an object in $\underline{C}$ and let ϕ_o be a non-degenerate bilinear form $\phi_o : X_o \otimes X_o \to \underline{1}$ The parity of ϕ_o is defined by the equation $\phi_o(x,y) = \phi_o(y,\varepsilon x)$. The form ϕ_o is said to be a Weil form on X_o if ε is in the centre of $\mathrm{End}(X_o)$ and if for all nonzero $u \in \mathrm{End}(X_o)$, $\mathrm{Tr}(uu^{\phi_o})>0$. Two Weil forms ϕ_o and ψ_o are said to be compatible if $\phi_o \oplus \psi_o$ is also a Weil form.

Let X_o correspond to the pair (X,a) with $X \in \mathrm{ob}(\underline{C}')$. Then ϕ_o defines a bilinear form ϕ on X , and $\psi \stackrel{\mathrm{df}}{=} (X \otimes \bar{X} \xrightarrow{1\otimes a^{-1}} X \otimes X \xrightarrow{\phi} \underline{1})$ is a non-degenerate sesquilinear form on X . If ϕ_o is a Weil form, then ψ is a Weil form on X which is compatible with its conjugate $\bar{\psi}$, and every such ψ arises from a ϕ_o; moreover $\varepsilon(\psi) = \varepsilon(\phi_o)$.

Polarizations.

Let Z be the centre of the band associated with $\underline{C}$ (see the appendix). Thus Z is a commutative algebraic group over $\mathbb{R}$ such that $Z(\mathbb{C})$ is the centre of $\mathrm{Aut}^{\otimes}(\omega)$ for any fibre functor on $\underline{C}'$ with values in $\mathbb{C}$. Moreover, Z represents $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\underline{C}})$.

Let $\varepsilon \in Z(\mathbb{R})$ and, for each $X \in \mathrm{ob}(\underline{C}')$, let $\pi(X)$ be an equivalence class (for the relation of compatibility) of Weil forms on X with parity ε; we say that π is a (homogeneous) polarization on $\underline{C}$ if

(4.10a) for all X, $\bar{\phi} \in \pi(X)$ whenever $\phi \in \pi(\bar{X})$, and

(4.10b) for all X and Y, $\phi \oplus \psi \in \pi(X \oplus Y)$ and $\phi \otimes \psi \in \pi(X \otimes Y)$ whenever $\phi \in \pi(X)$ and $\psi \in \pi(Y)$.

We call ε the parity of π and say that ϕ is positive for π if $\phi \in \pi(X)$. Thus the conditions require that $\bar{\phi}$, $\phi \oplus \psi$, and $\phi \otimes \psi$ be positive for π whenever ϕ and ψ are.

Proposition 4.11. Let π be a polarization on $\underline{C}$.

(a) The categories $\underline{C}$ and $\underline{C}'$ are semisimple.

(b) If $\phi \in \pi(X)$ and $Y \subset X$ then $X = Y \oplus Y^{\perp}$ and the restriction ϕ_Y of ϕ to Y is in $\pi(Y)$.

Proof. (a) Let X be an object of $\underline{C}'$; let Y be a nonzero simple subobject of X and let $u : Y \hookrightarrow X$ denote the inclusion map. Choose $\phi \in \pi(Y)$ and $\psi \in \pi(X)$. Consider $v = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$: $X \oplus Y \to X \oplus Y$ and let $u' : X \to Y$ be such that $v^{\psi\oplus\phi} = \begin{pmatrix} 0 & 0 \\ u' & 0 \end{pmatrix}$.

Then $\mathrm{Tr}_Y(u'u) = \mathrm{Tr}_{Y\oplus X}(v^{\psi\oplus\phi}v) > 0$, and so $u'u$ is an automorphism w of Y . The map $p = w^{-1}\circ u'$ projects X onto Y , which shows that Y is a direct summand of X , and X is semisimple.

The same argument, using the bilinear forms (4.9) shows that $\underline{C}$ is semisimple.

(b) Let $Y' = Y \cap Y^{\perp}$, where $Y^{\perp}$ is the largest subobject of X such that ϕ is zero on $Y \otimes \bar{Y}^{\perp}$, and let $p : X \to X$ project X onto Y' (by which we mean that $p(X) \subset Y'$ and $p|Y' = \mathrm{id}$). As ϕ is zero on $Y' \otimes \bar{Y}'$, $0 = \phi\circ(p\otimes\bar{p}) = \phi\circ(\mathrm{id}\otimes\overline{p^{\phi}p})$, and so $p^{\phi}p = 0$. Therefore $\mathrm{Tr}(p^{\phi}p) = 0$ and so p , and Y', are zero. Thus $X = Y \oplus Y^{\perp}$ and $\phi = \phi_Y \oplus \phi_{Y^{\perp}}$. Let $\phi_1 \in \pi(Y)$ and $\phi_2 \in \pi(Y^{\perp})$. Then $\phi_1 \oplus \phi_2$ is compatible with ϕ , and this implies that ϕ_1 is compatible with ϕ_Y .

<u>Remark</u> 4.12. Suppose $\underline{C}$ is defined by a triple (G,σ,c) , as in (4.1), so that $\underline{C}' = \underline{\mathrm{Rep}}_{\mathbb{C}}(G)$. A sesquilinear form $\phi: X \otimes \bar{X} \to \underline{1}$ defines a sesquilinear form ϕ' on X in the usual, vector space, sense by the formula

$$\phi'(x,y) = \phi(x \otimes \bar{y}) \ , \ x,y \in X \qquad (4.12.1).$$

The conditions that ϕ be a G-morphism and have a parity $\varepsilon \in Z(\mathbb{R})$ become respectively

$$\phi'(x,y) = \phi'(gx,\sigma^{-1}(g)y) \ , \ g \in G(\mathbb{C}) \ , \qquad (4.12.2)$$

$$\overline{\phi'(y,x)} = \phi'(x, \varepsilon c^{-1}y) \qquad (4.12.3).$$

When G acts trivially on X, then (4.12.3) becomes

$$\overline{\phi'(y,x)} = \phi'(x,y) \ ,$$

and so ϕ' is a Hermitian form in the usual sense on X. If X is one-dimensional and $\phi \in \pi(X)$, then ϕ' is positive-definite (for otherwise $\phi \otimes \phi \notin \pi(X)$). Now (4.11b) shows that the same is true for any X, and (4.6) shows that $\{\phi' | \phi \in \pi(X)\}$ is the complete set of positive-definite Hermitian forms on X (when G acts trivially on X). In particular, $\underline{\mathrm{Vec}}_{\mathbb{R}}$ has a unique polarization.

Remark 4.13. A polarization π on $\underline{C}$ with parity ε defines, for each simple object X of $\underline{C}'$, an orientation of the real line of sesquilinear forms on X with parity ε (see 4.8), and π is obviously determined by this family of orientations. Choose a fibre functor ω for $\underline{C}'$, and choose for each simple object X_i a $\phi_i \in \pi(X_i)$. Then

$$\pi(X_i) = \{r\,\phi_i | r \in \mathbb{R}_{>0}\} \ .$$

If X is isotypic of type X_i, so that $\omega(X) = W \otimes \omega(X_i)$ where $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ acts trivially on W, then

$$\{\omega(\phi)' | \phi \in \pi(X)\} = \{\psi \otimes \omega(\phi_i)' | \psi \text{ Hermitian } \psi>0\} \ .$$

If $X = \oplus X^{(i)}$ where the $X^{(i)}$ are the isotypic components of X, then

$$\pi(X) = \oplus\ \pi(X^{(i)}) \ .$$

Remark 4.14. Let $\varepsilon \in Z(\mathbb{R})$ and, for each $X_o \in ob(\underline{C})$, let $\pi(X_o)$ be an equivalence class of bilinear Weil forms on X_o with parity ε (see (4.9)). One says that π is a homogeneous polarization on $\underline{C}$ if $\phi_o \oplus \psi_o \in \pi(X \oplus Y)$ and $\phi_o \otimes \psi_o \in \pi(X \otimes Y)$ whenever $\phi_o \in \pi(X)$ and $\psi_o \in \pi(Y)$. As $\{X | (X,a) \in ob(\underline{C})\}$ generates $\underline{C}'$, the relation between bilinear and sesquilinear forms noted in (4.9) establishes a one-to-one correspondence between polarizations in this bilinear sense and in the sesquilinear sense of (4.10).

In the situation of (4.12), a bilinear form ϕ_o on X_o defines a sesquilinear form ψ' on $X = \mathbb{C} \otimes X_o$ (in the usual vector space sense) by the formula

$$\psi'(z_1 v_1, z_2 v_2) = z_1 \bar{z}_2\ \phi_o(v_1, v_2),\ v_1, v_2 \in X_o,\ z_1, z_2 \in \mathbb{C} \ .$$

Description of polarizations

Let $\underline{C}$ be defined by a triple (G,σ,c) satisfying (4.2.1), and let K be a maximal compact subgroup of $G(\mathbb{C})$. As all maximal compact subgroups of $G(\mathbb{C})$ are conjugate (Hochschild [1, XV. 3.1]), there exists $m \in G(\mathbb{C})$ such that $\sigma^{-1}(K) = m\, K\, m^{-1}$. Therefore, after replacing σ by $\sigma \circ \underline{ad}(m)$, we can assume that

$\sigma(K) = K$. Subject to this constraint, (σ,c) is determined up to modification by an element m in the normalizer of K .

Assume that $\underline{C}$ is polarizable. Then (4.11a) and (2.28) show that G^{o} is reductive, and it follows that K is a compact real form G , i.e., K has the structure of a compact real algebraic group in the sense of (2.33) and $K_{\mathbb{C}} = G$ (see Springer [1, 5.6]). Let σ_K be the semi-linear automorphism of G such that, for $g \in G(\mathbb{C})$, $\sigma_K(g)$ is the conjugate of g relative to the real structure on G defined by K ; note that σ_K determines K . The normalizer of K is $K.Z(\mathbb{C})$, and so $c \in K.Z(\mathbb{C})$.

Fix a polarization π on $\underline{C}$ with parity ε . If X is an irreducible representation of G and ψ is a positive-definite K-invariant Hermitian form on X , then for any $\phi \in \pi(X)$,

$$(\phi(x \otimes \bar{y} \overset{\mathrm{df}}{=\!=}) \phi'(x,y) = \psi(x,\beta y)$$

for some $\beta \in \mathrm{Aut}(X)$. Equations (4.12.2) and (4.12.3) can be re-written as

$$\beta g_X = \sigma(g)_X\, \beta \ , \ g \in K(\mathbb{R}) \qquad (4.14.1)$$

$$\beta^* = \beta\, \varepsilon_X\, c_X^{-1} \qquad (4.14.2)$$

where β^* is the adjoint of β relative to ψ : $\psi(\beta x,y) = \psi(x,\beta^* y)$. As $K(\mathbb{R})$ is Zariski dense in $K(\mathbb{C})$, X is also irreducible as a representation of $K(\mathbb{R})$, and so the set $c(X,\pi)$ of such β's is parametrized by $\mathbb{R}_{>0}$. An arbitrary finite-dimensional representation X of G can be written

$$X = \oplus_i W_i \otimes X_i$$

where the sum is over the non-isomorphic irreducible representations X_i of G, and G acts trivially on each W_i; let ψ_i and ψ_i' respectively be K-invariant positive-definite Hermitian forms on X_i and W_i, and let $\psi = \oplus \psi_i' \otimes \psi_i$; then for any $\phi \in \pi(X)$, $\phi'(x,y) = \psi(x,\beta y)$ where $\beta = \oplus \beta_i' \otimes \beta_i$ with $\beta_i \in c(X_i,\pi)$ and β_i' is positive-definite and Hermitian relative to ψ_i'. We let $c(X,\pi)$ denote the set of β as ϕ runs through $\pi(X)$. The condition (4.10b) that $\pi(X_1) \otimes \pi(X_2) \subset \pi(X_1 \otimes X_2)$ becomes $c(X_1,\pi) \otimes c(X_2,\pi) \subset c(X_1 \otimes X_2,\pi)$.

<u>Lemma</u> 4.15. There exists a $b \in K$ with the following properties:

(a) $b_X \in c(X,\pi)$ for all irreducible X;

(b) $\sigma = \sigma_K \circ \underline{\mathrm{ad}}\, b$, where σ_K denotes complex conjugation on G relative to K;

(c) $\varepsilon^{-1}c = \sigma b.b = b^2$.

<u>Proof</u>: Let $a = \varepsilon c^{-1} \in G(\mathbb{C})$. When X is irreducible, (4.14.1) applied twice shows that $\beta^2 gx = \sigma^2(g)\beta^2 x = cgc^{-1}\beta^2 x$ for $\beta \in c(X,\pi)$, $g \in K$, and $x \in X$; therefore $(c^{-1}\beta^2)gx = g(c^{-1}\beta^2)x$, and so $c^{-1}\beta^2$ acts as a scalar on X. Hence $a\beta^2 = \varepsilon\, c^{-1}\beta^2$ also acts as a scalar. Moreover, $\beta^2 a = \beta\beta^*$ (by 4.14.2) and so $\mathrm{Tr}_X(a\beta^2) = \mathrm{Tr}_X(\beta^2 a) > 0$; we conclude that $a_X\beta^2 \in \mathbb{R}_{>0}$. It follows that there is a unique $\beta \in c(X,\pi)$ such that $a_X = \beta^{-2}$, $\beta g_X = \sigma(g)_X\beta$ $(g \in K)$, and $\beta^* = \beta^{-1}$ (so β is unitary).

For an arbitrary X we write $X = \oplus W_i \otimes X_i$ as before, and set $\beta = \oplus\, \mathrm{id} \otimes \beta_i$, where β_i is the canonical element of

$c(X_i,\pi)$ just defined. We still have $a_X = \beta^{-2}$, $\beta g_X = \sigma(g)_X\beta$ $(g \in K)$, and $\beta \in c(X,\pi)$. Moreover, these conditions characterize β : if $\beta' \in c(X,\pi)$ has the same properties, then $\beta' = \Sigma\, \gamma_i \otimes \beta_i$ (this expresses that $\beta' g_X = \sigma(g)_X\beta'$, $g \in K$) with $\gamma_i^2 = 1$ (as $\beta'^2 = a_X^{-1}$) and γ_i positive-definite and Hermitian; hence $\gamma_i = 1$.

The conditions are compatible with tensor products, and so the canonical β are compatible with tensor products: they therefore define an element $b \in G(\mathbb{C})$. As b is unitary on all irreducible representations, it lies in K . The equations $\beta^2 = a_X^{-1}$ show that $b^2 = a^{-1} = \varepsilon^{-1}c$. Finally, $\beta g_X = \sigma(g)_X\beta$ implies that $\sigma(g) = \underline{\mathrm{ad}}\, b(g)$ for all $g \in K$; therefore $\sigma\circ\underline{\mathrm{ad}}\, b^{-1}$ fixes K and, as it has order 2, it must equal σ_K .

<u>Theorem</u> 4.16. Let $\underline{C}$ be an algebraic Tannakian category over $\mathbb{R}$ and let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ where ω is a fibre functor on $\underline{C}$ with values in $\mathbb{C}$; let π be a polarization on $\underline{C}$ with parity ε . For any compact real-form K of G , the pair (σ_K,ε) satisfies (4.2.1), and the equivalence $\underline{C}' \xrightarrow{\sim} \underline{\mathrm{Rep}}_{\mathbb{C}}(G)$ defined by ω carries the descent datum on $\underline{C}'$ defined by $\underline{C}$ into that on $\underline{\mathrm{Rep}}_{\mathbb{C}}(G)$ defined by (σ_K,ε) : $\omega(\bar{X}) = \overline{\omega(X)}$, $\omega(\mu_X) = \mu_{\omega(X)}$. For any simple X in $\underline{C}'$, $\{\omega(\phi)' \mid \phi \in \pi(X)\}$ is the set of K-invariant positive-definite Hermitian forms on $\omega(X)$.

<u>Proof</u>: Let $(\underline{C},\omega)$ correspond to a triple (G,σ_1,c_1) (see (4.3a)), and let $b \in K$ be the element constructed in the lemma. Then $\sigma_1 = \sigma_K\circ\underline{\mathrm{ad}}\, b$ and $c = \varepsilon.\sigma b.b = \sigma b.\varepsilon.b$. Therefore (σ_K,ε) has the same property as (σ_1,c_1) (see (4.3b)), which proves the first assertion. The second assertion follows from the fact that $b \in c(\omega(X),\pi)$ for any simple X .

Classification of polarized Tannakian categories

Theorem 4.17 (a) The category $\underline{C}$ is polarizable if and only if its band is defined by a compact real algebraic group K .

(b) Let K be a compact real algebraic group, and let $\varepsilon \in Z(\mathbb{R})$ where Z is the centre of K ; there exists a Tannakian category $\underline{C}$ over $\mathbb{R}$ whose gerb is bound by the band $B(K)$ of K and a polarization π on $\underline{C}$ with parity ε .

(c) Let $(\underline{C}_1,\pi_1)$ and $(\underline{C}_2,\pi_2)$ be polarized algebraic Tannakian categories over $\mathbb{R}$, and let $B \xrightarrow{\approx} B_1$ and $B \xrightarrow{\approx} B_2$ be the identifications of the bands of $\underline{C}_1$ and $\underline{C}_2$ with a given band B . If $\varepsilon(\pi_1) = \varepsilon(\pi_2)$ in $Z(B)(\mathbb{R})$ then there is a tensor equivalence $\underline{C}_1 \xrightarrow{\sim} \underline{C}_2$ respecting the polarizations and the actions of B (i.e., such that $FIB(\underline{C}_2) \xrightarrow{\sim} FIB(\underline{C}_1)$ is a B-equivalence), and this equivalence is unique up to isomorphism.

Proof: We have already seen that if $\underline{C}$ is polarizable, then $\underline{C}'$ is semisimple, and so, for any fibre functor ω with values in $\mathbb{C}$, (the identity component of) $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is reductive, and hence has a compact real form K . This proves half of (a). Part (b) is proved in the first lemma below, and the sufficiency in (a) follows from (b) and the second lemma below. Part (c) is essentially proved by (4.16).

Lemma 4.18. Let K be a compact real algebraic group and let $G = K_{\mathbb{C}}$; let $\sigma(g) = \sigma'(\bar{g})$ where σ' is a Cartan involution for K , and let $\varepsilon \in Z(\mathbb{R})$ where Z is the centre of K . Then (σ,ε) satisfies (4.2.1) and the Tannakian category $\underline{C}$ defined by (G,σ,ε) has a polarization with parity ε .

Proof: Since $\sigma^2 = \mathrm{id}$ and σ fixes all elements of K, (4.2.1) is obvious. There exists a polarization π on $\underline{C}$ such that, for all simple X, $\{\phi' \mid \phi \in \pi(X)\}$ is the set of positive-definite K-invariant Hermitian forms on X. (In the notation of (4.15), b=1.) This polarization has parity ε.

Let $\underline{C}$ correspond to $(\underline{C}', X \mapsto \bar{X}, \mu)$; for any $z \in Z(\mathbb{R})$, where Z is the centre of the band B of $\underline{C}$, $(\underline{C}', X \mapsto \bar{X}, \mu \circ z)$ defines a new Tannakian category ${}^z\underline{C}$ over $\mathbb{R}$.

Lemma 4.19. Every Tannakian category over $\mathbb{R}$ whose gerb is bound by B is of the form ${}^z\underline{C}$ for some $z \in Z(\mathbb{R})$; there is a tensor equivalence ${}^z\underline{C} \xrightarrow{\approx} {}^{z'}\underline{C}$ respecting the action of B if and only if $z'z^{-1} \in Z(\mathbb{R})^2$.

Proof: Let ω be a fibre functor on $\underline{C}$, and let $(\underline{C}, \omega)$ correspond to (G, σ, c); we can assume that a second category $\underline{C}_1$ corresponds to (G, σ_1, c_1). Let γ and γ_1 be the functors $V \mapsto \bar{V}$ defined by (σ, c) and (σ_1, c_1) respectively. Then $\gamma_1^{-1} \circ \gamma$ defines a tensor automorphism of ω, and so corresponds to an element $m \in G(\mathbb{C})$. We have $\sigma = \sigma_1 \circ \underline{\mathrm{ad}}(m)$, and so we can modify (σ_1, c_1) in order to get $\sigma_1 = \sigma$. Let μ and μ_1 be the functorial isomorphisms $V \to \bar{\bar{V}}$ defined by (σ, c) and (σ, c_1) respectively. Then $\mu_1^{-1} \circ \mu$ defines a tensor automorphism of $\mathrm{id}_{\underline{C}}$, and so $\mu_1^{-1} \circ \mu = z^{-1}$, $z \in Z(\mathbb{R})$. We have $\mu_1 = \mu \circ z$.

The second part of the lemma is obvious.

Remark 4.20. Some of the above results can be given a more cohomological interpretation. Let B be the band defined by a compact real group K, and let Z be the centre of B; let $\underline{C}$ be a Tannakian category, whose gerb is B.

(a) As Z is a subgroup of a compact real algebraic group, it is also compact (see (2.33)). It is easy to compute its cohomology; one finds that

$$H^1(\mathbb{R},Z) = {}_2Z(\mathbb{R}) \stackrel{\text{df}}{=} \ker(2:Z(\mathbb{R}) \to Z(\mathbb{R}))$$

$$H^2(\mathbb{R},Z) = Z(\mathbb{R})/Z(\mathbb{R})^2$$

(b) The general theory shows that there is an isomorphism $H^1(\mathbb{R},Z) \to \mathrm{Aut}_B(\underline{C})$, which can be described explicitly as the map associating to $z \in Z(\mathbb{R})_2$ the automorphism w_z

$$\begin{cases} (X,\ a_X) \longmapsto (X,\ a_X\, z_X) \\ f \longmapsto f\ . \end{cases}$$

(c) The Tannakian categories bound by B, up to B-equivalence, are classified by $H^2(\mathbb{R},B)$, and $H^2(\mathbb{R},B)$ if nonempty is an $H^2(\mathbb{R},Z)$-torsor; the action of $H^2(\mathbb{R},Z) = Z(\mathbb{R})/Z(\mathbb{R})^2$ on the categories is made explicit in (4.19).

(d) Let $\mathrm{Pol}(\underline{C})$ denote the set of polarizations on $\underline{C}$. For $\pi \in \mathrm{Pol}(\underline{C})$ and $z \in Z(\mathbb{R})$ we define $z\pi$ to be the polarization such that

$$\phi(x,y) \in z\pi(X) \iff \phi(x,zy) \in \pi(X) \ ;$$

it has parity $\varepsilon(z\pi) = z^2 \varepsilon(\pi)$. The pairing

$$(z,\pi) \mapsto z\pi : Z(\mathbb{R}) \times \mathrm{Pol}(\underline{C}) \to \mathrm{Pol}(\underline{C})$$

makes $\mathrm{Pol}(\underline{C})$ into a $Z(\mathbb{R})$-torsor.

(e) Let $\pi \in \mathrm{Pol}(\underline{C})$ and let $\varepsilon = \varepsilon(\pi)$; then $\underline{C}$ has a polarization with parity $\varepsilon' \in Z(\mathbb{R})$ if and only if $\varepsilon' = \varepsilon z^2$ for some $z \in Z(\mathbb{R})$.

Remark 4.21. In Saavedra [1, V. 1] there is a table of Tannakian categories whose bands are simple, from which it is possible to read off those that are polarizable (loc. cit. V. 2.8.3).

Neutral polarized categories

The above results can be made more explicit when $\underline{C}$ has a fibre functor with values in $\mathbb{R}$.

Let G be an algebraic group over $\mathbb{R}$, and let $C \in G(\mathbb{R})$. A G-invariant sesquilinear form $\psi : V \times V \to \mathbb{C}$ on $V \in \mathrm{ob}(\underline{\mathrm{Rep}}_{\mathbb{C}}(G))$ is said to be a C-polarization if

$$\psi^C(x,y) \stackrel{\mathrm{df}}{=} \psi(x,Cy)$$

is a positive-definite Hermitian form on V. If every object of $\underline{\mathrm{Rep}_{\mathbb{C}}}(G)$ has a C-polarization then C is called a Hodge element.

Proposition 4.22. Assume that $G(\mathbb{R})$ contains a Hodge element C. There is then a polarization π_C on $\underline{\mathrm{Rep}_{\mathbb{R}}}(G)$ for which the positive forms are exactly the C-polarizations; the parity of π_C is C^2; for any $g \in G(\mathbb{R})$ and $z \in Z(\mathbb{R})$, where Z is the centre of G, $C' = zgCg^{-1}$ is also a Hodge element and $\pi_{C'} = z\pi_C$; every polarization on $\underline{\mathrm{Rep}_{\mathbb{R}}}(G)$ is of the form $\pi_{C'}$ for some Hodge element C'.

Proof: Let ψ be a C-polarization on $V \in \mathrm{ob}(\underline{\mathrm{Rep}_{\mathbb{C}}}(G))$; then $\psi(x,y) = \psi(Cx,Cy)$ because ψ is G-invariant, and $\psi(Cx,Cy) = \psi^C(Cx,y) = \overline{\psi^C(y,Cx)} = \overline{\psi(y,C^2x)}$. This shows that ψ has parity C^2. For any V, $\overline{\psi(y,C^2x)} = \psi(x,y) = \psi(gx,gy) = \overline{\psi(gy,C^2gx)} = \overline{\psi(y,g^{-1}C^2gx)}$, $g \in G(\mathbb{R})$, x, $y \in V$; this shows that $C^2 \in Z(\mathbb{R})$. For any $u \in \mathrm{End}(V)$, $u^\psi = u^{\psi^C}$, and so $\mathrm{Tr}(uu^\psi) > 0$ if $u \neq 0$. This shows that ψ is a Weil form with parity C^2. The first assertion of the proposition is now easy to check. The third assertion is straightforward to prove, and the fourth follows from it and (4.19).

Proposition 4.23. The following conditions on G are equivalent:

(a) there is a Hodge element in $G(\mathbb{R})$;

(b) the category $\underline{\mathrm{Rep}_{\mathbb{R}}}(G)$ is polarizable;

(c) G is an inner form of a compact real algebraic group K.

Proof: (a) $\Rightarrow$ (b). This follows from (4.22).

(b) $\Rightarrow$ (c) . To say that G is an inner form of K is the same as to say that G and K define the same band; this implication therefore follows from (4.17a).

(c) $\Rightarrow$ (a) . Let Z be the centre of K (and therefore also of G) and let $K^{ad} = K/Z$. The assumption says that the isomorphism class of G is in the image of

$$H^1(\mathbb{R},\ K^{ad}) \to G^1(\mathbb{R},\ \underline{Aut}(K)) \ .$$

According to Serre [1, III, Thm 6], the canonical map

$${}_2(K^{ad}(\mathbb{R})) = H^1(\mathbb{R}, K^{ad}(\mathbb{R})) \to H^1(\mathbb{R}, K^{ad})$$

is an isomorphism. From the cohomology sequence

$$\begin{tikzcd}
K(\mathbb{R}) \arrow[r] & K^{ad}(\mathbb{R}) \arrow[r] & H^1(\mathbb{R},Z) \arrow[r] \arrow[d, equal] & H^1(\mathbb{R},K) \arrow[d, equal] \\
 & & {}_2Z(\mathbb{R}) \arrow[r, hook] & {}_2K(\mathbb{R})
\end{tikzcd}$$

we see that $K(\mathbb{R}) \twoheadrightarrow K^{ad}(\mathbb{R})$, and so G is the inner form of K defined by an element $C' \in K(\mathbb{R})$ whose square is in $Z(\mathbb{R})$. Let γ be an isomorphism $K_{\mathbb{C}} \to G_{\mathbb{C}}$ such that $\gamma \circ \underline{ad}C' = \bar{\gamma}$, and let $C = \gamma(C')$; then $\bar{C} = \bar{\gamma}(C') = \gamma(C') = C$ and $\bar{\gamma}^{-1} \circ \underline{ad}(C) = \gamma^{-1}$. This shows that $C \in G(\mathbb{R})$ and that K is the form of G defined by C ; the next lemma completes the proof.

Lemma 4.24. An element $C \in G(\mathbb{R})$ such that $C^2 \in Z(\mathbb{R})$ is a Hodge element if and only if the real-form K of G defined by C is a compact real group.

Proof: Identify $K_{\mathbb{C}}$ with $G_{\mathbb{C}}$ and let $\bar{g}$ and g^* respectively be the complex conjugates of $g \in G(\mathbb{C})$ relative to the real structures defined by G and K. Then $g^* = \underline{\mathrm{ad}}(C^{-1})(\bar{g}) = C^{-1}\bar{g}C$. Let ψ be a sesquilinear form on $V \in \mathrm{ob}(\underline{\mathrm{Rep}}_{\mathbb{C}}(G))$. Then ψ is a G-invariant if and only if

$$\psi(gx,\bar{g}y) = \psi(x,y) \ , \ g \in G(\mathbb{C}) \ .$$

On the other hand, ψ^C is K-invariant if and only if

$$\psi^C(gx,g^*y) = \psi^C(x,y) \ , \ g \in G(\mathbb{C}) \ .$$

These conditions are equivalent: V has a C-polarization if and only if V has a K-invariant positive-definite Hermitian form. Thus C is a Hodge element if and only if, for every complex representation V of K, the image of K in $\mathrm{Aut}(V)$ is contained in the unitary group of a positive-definite Hermitian form; this last condition is implied by K being compact and implies that K is contained in a compact real group and so is compact (see (2.33)).

Remark 4.25. (a) The centralizer of a Hodge element C of G is a maximal compact subgroup G, and is the only maximal compact subgroup of G containing C; in particular, if G is compact, then

C is a Hodge element if and only if it is in the centre of G (Saavedra [1, 2.7.3.5]).

(b) If C and C' are Hodge elements of G then there exists a $g \in G(\mathbb{R})$ and a unique $z \in Z(\mathbb{R})$ such that $C' = zgCg^{-1}$ (Saavedra [1, 2.7.4]). As $\pi_{C'} = z\pi_C$, this shows that $\pi_{C'} = \pi_C$ if and only if C and C' are conjugate in $G(\mathbb{R})$.

Remark 4.26. It would perhaps have been more natural to express the above results in terms of bilinear forms (see (4.4), (4.9), (4.14)): a G-invariant bilinear form $\phi : V_o \times V_o \to \mathbb{R}$ on $V_o \in \mathrm{ob}(\underline{\mathrm{Rep}}_{\mathbb{R}}(G))$ is a C-polarization if $\phi^C(x,y) \overset{\mathrm{df}}{=} \phi(x,Cy)$ is a positive-definite symmetric form on V_o; C is a Hodge element if every object of $\underline{\mathrm{Rep}}_{\mathbb{R}}(G)$ has a C-polarization; the positive forms for the (bilinear) polarization defined by C are precisely the C-polarizations.

Symmetric polarizations

A polarization is said to be symmetric if its parity is 1.

Let K be a compact real algebraic group. As 1 is a Hodge element (4.24), $\underline{\mathrm{Rep}}_{\mathbb{R}}(K)$ has a symmetric polarization π for which $\pi(X_o)$, $X_o \in \mathrm{ob}(\underline{\mathrm{Rep}}_{\mathbb{R}}(K))$, consists of the K-invariant positive-definite symmetric bilinear forms on X_o (and $\pi(X)$, $X \in \mathrm{ob}(\underline{\mathrm{Rep}}_{\mathbb{C}}(K))$, consists of the K-invariant positive-definite Hermitian forms on X).

Theorem 4.27. Let $\underline{C}$ be an algebraic Tannakian category over $\mathbb{R}$, and let π be a symmetric polarization on $\underline{C}$. Then $\underline{C}$ has a unique (up to isomorphism) fibre functor ω with values in $\mathbb{R}$ transforming positive bilinear forms for π into positive-definite symmetric bilinear forms; ω defines a tensor equivalence $\underline{C} \xrightarrow{\sim} \underline{\mathrm{Rep}}_{\mathbb{R}}(K)$, where $K = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is a compact real group.

Proof: Let ω_1 be a fibre functor with values in $\mathbb{C}$, and let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega_1)$. Since $\underline{C}$ is polarizable, G has a compact real form K. According to (4.16), $\omega_1' : \underline{C}' \xrightarrow{\sim} \underline{\mathrm{Rep}}_{\mathbb{C}}(G)$ carries the descent datum on $\underline{C}'$ defined by $\underline{C}$ into that on $\underline{\mathrm{Rep}}_{\mathbb{C}}(G)$ defined by $(\sigma_K, 1)$. It therefore defines a tensor equivalence $\omega : \underline{C} \rightarrow \underline{\mathrm{Rep}}_{\mathbb{R}}(K)$ transforming π into the polarization on $\underline{\mathrm{Rep}}_{\mathbb{R}}(K)$ defined by the Hodge element 1. The rest of the proof is now obvious.

Remark 4.28. Let π be a polarization on $\underline{C}$. It follows from (4.20d) that $\underline{C}$ has a symmetric polarization if and only if $\varepsilon(\pi) \in Z(\mathbb{R})^2$.

Polarizations with parity ε of order 2

For $u = \pm 1$, define a real u-space to be a complex vector space V together with a semi-linear automorphism σ such that $\sigma^2 = u$. A bilinear form ϕ on a real u-space is u-symmetric if $\phi(x,y) = u\phi(y,x)$; such a form is positive-definite if $\phi(x,\sigma x) > 0$ for all $x \neq 0$. Thus a 1-symmetric form is symmetric, and a (-1)-symmetric form is skew-symmetric.

Let $\underline{V}_o$ be the category whose objects are pairs (V,σ) where $V = V^o \oplus V^1$ is a $\mathbb{Z}/2\mathbb{Z}$ - graded vector space over $\mathbb{C}$ and

$\sigma : V \xrightarrow{\approx} V$ is a semi-linear automorphism such that $\sigma^2 x=(-1)^{\deg(x)}x$. With the obvious tensor structure, $\underline{V}_o$ becomes a Tannakian category over $\mathbb{R}$ with $\mathbb{C}$-valued fibre functor $(V,\sigma)\longmapsto V$. There is a polarization $\pi = \pi_{can}$ on $\underline{V}_o$ such that, if V is homogeneous, then $\pi(V,\sigma)$ comprises the $(-1)^{\deg(v)}$-symmetric positive-definite forms on V.

Theorem 4.29. Let $\underline{C}$ be an algebraic Tannakian category over $\mathbb{R}$, and let π be a polarization on $\underline{C}$ with parity ε where $\varepsilon^2=1$, $\varepsilon \neq 1$. There exists a unique (up to isomorphism) exact faithful functor $\omega : \underline{C} \to \underline{V}_o$ such that

(a) ω carries the grading on $\underline{C}$ defined by ε into the grading on $\underline{V}_o$, i.e., $\omega(\varepsilon)$ acts as $(-1)^m$ on $\omega(V)^m$;

(b) ω carries π into π_{can}, i.e., $\phi \in \pi(X)$ if and only if $\omega(\phi) \in \pi_{can}(\omega(X))$.

Proof: Note that $\underline{V}_o$ is defined by the triple $(\mu_2,\sigma_o,\varepsilon_o)$ where σ_o is the unique semi-linear automorphism of μ_2 and ε_o is the unique element of $\mu_2(\mathbb{R})$ of order 2. We can assume (by (4.3)) that $\underline{C}$ corresponds to a triple (G,σ,ε). Let G_o be the subgroup of G generated by ε; then $(G_o,\sigma|G_o,\varepsilon) \approx (\mu_2,\sigma_o,\varepsilon_o)$, and so the inclusion $(G_o,\sigma|G_o,\varepsilon) \hookrightarrow (G,\sigma,\varepsilon)$ induces a functor $\underline{C} \to \underline{V}_o$ having the required properties.

Let $\omega,\omega' : \underline{C} \rightrightarrows \underline{V}_o$ be two functors satisfying (a) and (b). It is clear from (3.2a) that there exists an isomorphism $\lambda : \omega \xrightarrow{\approx} \omega'$ from ω to ω' regarded as functors to $\underline{\mathrm{Vec}}_{\mathbb{C}}$. As $\lambda_X : \omega(X) \to \omega'(X)$ commutes with the action of ε, it preserves the gradings; as λ commutes with $\omega(\phi)$, any $\phi \in \pi(X)$, it also commutes with σ; it follows that λ is an isomorphism of ω and ω' as functors to $\underline{V}_o$.

§5. Graded Tannakian categories

Throughout this section, k will be a field of characteristic zero.

Gradings

Let M be set. An M-grading on an object X of an additive category is a decomposition $X = \bigoplus_{m \in M} X^m$; an M-grading on an additive functor $u : \underline{C} \to \underline{C}'$ is an M-grading on each $u(X)$, $X \in \mathrm{ob}(\underline{C})$, that depends functorially on X .

Suppose now that M is an abelian group, and let D be the algebraic group of multiplicative type over k whose character group is M (with trivial Galois action; see (2.32)). In the cases of most interest to us, namely $M = \mathbb{Z}$ or $M = \mathbb{Z}/2\mathbb{Z}$, D equals $\mathbb{G}_m$ or μ_2 $(=\mathbb{Z}/2\mathbb{Z})$. An M-grading on a Tannakian category $\underline{C}$ over k can be variously described as follows:

(5.1a) An M-grading, $X = \oplus X^m$, on each object X of $\underline{C}$ that depends functorially on X and is compatible with tensor products in the sense that $(X \otimes Y)^m = \bigoplus_{r+s=m} X^r \otimes Y^s$;

(5.1b) An M-grading on the identity functor $\mathrm{id}_{\underline{C}}$ of $\underline{C}$ that is compatible with tensor products;

(5.1c) A homomorphism $D \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\underline{C}})$

(5.1d) A central homomorphism $D \to G$, $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, for one (or every) fibre functor ω .

Definitions (a) and (b) are obviously equivalent. By a central homomorphism in (d), we mean a homomorphism from D into the centre of G defined over k ; although G need not be defined over k, its centre is, and equals $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\underline{C}})$, whence follows the equivalence of (c) and (d). Finally, a homomorphism $w : D \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\underline{C}})$ corresponds to the family of gradings $X = \oplus X^m$ for which $w(d)$ acts on $X^m \subset X$ as $m(d) \in k$.

Tate triples

A Tate triple $\underline{T}$ over k is a triple $(\underline{C}, w, T)$ comprising a Tannakian category $\underline{C}$ over k, a $\mathbb{Z}$-grading $w : \mathbb{G}_m \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\underline{C}})$ on $\underline{C}$ (called the weight grading), and an invertible object T (called the Tate object) of weight -2. For any $X \in \mathrm{ob}(\underline{C})$ and $n \in \mathbb{Z}$, we write $X(n) = X \otimes T^{\otimes n}$. A fibre functor on $\underline{T}$ with values in R is a fibre functor $\omega : \underline{C} \to \underline{\mathrm{Mod}}_R$ together with an isomorphism $\omega(T) \xrightarrow{\approx} \omega(T^{\otimes 2})$, i.e., the structure of an identity object on $\omega(T)$. If $\underline{T}$ has a fibre functor with values in k, then $\underline{T}$ is said to be neutral. A morphism of Tate triples $(\underline{C}_1, w_1, T_1) \to (\underline{C}_2, w_2, T_2)$ is a tensor functor $\eta : \underline{C}_1 \to \underline{C}_2$ preserving the gradings together with an isomorphism $\eta(T_1) \xrightarrow{\approx} T_2$.

Example 5.2 (a). The triple $(\underline{\mathrm{Hod}}_{\mathbb{R}}, w, \mathbb{R}(1))$ in which $\underline{\mathrm{Hod}}_{\mathbb{R}}$ is the category of real Hodge structures (see (2.31)), w is the weight grading on $\underline{\mathrm{Hod}}_{\mathbb{R}}$, and $\mathbb{R}(1)$ is the unique real Hodge structure with weight -2 and underlying vector space $2\pi i\mathbb{R}$, is a neutral Tate triple.

(b) The category of $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}$, together with the object $T = \mathbb{Q}_B(1)$ (see I.1), forms a Tate triple $\underline{T}_B$; the category of $\mathbb{Z}$-graded vector spaces over $\mathbb{Q}_\ell$, together with the object $T = \mathbb{Q}_\ell(1)$, forms a Tate triple $\underline{T}_\ell$; the category of $\mathbb{Z}$-graded vector spaces over k, together with the object $T = \mathbb{Q}_{DR}(1)$, forms a Tate triple $\underline{T}_{DR}$.

Example 5.3. Let $\underline{V}$ be the category of $\mathbb{Z}$-graded complex vector spaces V with a semi-linear automorphism a such that $a^2v = (-1)^n v$ if $v \in V^n$. With the obvious tensor structure, $\underline{V}$ becomes a Tannakian category over $\mathbb{R}$, and $\omega : (V,a) \longmapsto V$ is a fibre functor with values in $\mathbb{C}$. Clearly $\mathbb{G}_m = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and $\underline{V}$ corresponds (as in (4.3a)) to the pair $(g \longmapsto \bar{g}, -1)$. Let $w : \mathbb{G}_m \to \mathbb{G}_m$ be the identity map, and let $T = (V,a)$ where V is $\mathbb{C}$ regarded as a homogeneous vector space of weight -2 and a is $z \longmapsto \bar{z}$. Then $(\underline{V}, w, T)$ is a (non-neutral) Tate triple over $\mathbb{R}$.

Example 5.4. Let G be an affine group scheme over k and let $w : \mathbb{G}_m \to G$ be a central homomorphism and $t : G \to \mathbb{G}_m$ a homomorphism such that $t \circ w = -2$ $(\overset{\mathrm{df}}{=} s \mapsto s^{-2})$. Let T be the representation of G on k such that g acts as multiplication by $t(g)$. Then $(\underline{\mathrm{Rep}}_k(G), w, T)$ is a neutral Tate triple over k.

The following proposition is obvious.

Proposition 5.5. Let $\underline{T} = (\underline{C}, w, T)$ be a Tate triple over k, and let ω be a fibre functor on $\underline{T}$ with values in k. Let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, so that w is a homomorphism $\mathbb{G}_m \to Z(G) \subset G$. There is a homomorphism $t : G \to \mathbb{G}_m$ such that g acts on T as multiplication by $t(g)$, and $t \circ w = -2$. The equivalence $\underline{C} \xrightarrow{\approx} \underline{\mathrm{Rep}}_k(G)$ carries w and T into the weight grading and Tate object defined by t and w.

More generally, a Tate triple $\underline{T}$ defines a band B, a homomorphism $w : \mathbb{G}_m \to Z$ into the centre Z of B, and a homomorphism $t : B \to \mathbb{G}_m$ such that $t \circ w = -2$. We say that $\underline{T}$ is <u>bound</u> by (B,w,t).

Let G, w, and t be as in (5.4). Let $G_o = \mathrm{Ker}(t : G \to \mathbb{G}_m)$, and let $\varepsilon : \mu_2 \to G_o$ be the restriction of w to μ_2; we often identify ε with $\varepsilon(-1) = w(-1) \in Z(G_o)(k)$. Note that ε defines a $\mathbb{Z}/2\mathbb{Z}$-grading on $\underline{C}_o = \underline{\mathrm{Rep}}_k(G_o)$. The inclusion $G_o \hookrightarrow G$ defines a tensor functor $Q : \underline{C} \to \underline{C}_o$ with the following properties:

(5.6a) if X is homogeneous of weight n, then $Q(X)$ is homogeneous of weight $n \pmod 2$,

(5.6b) $Q(T) = 1$;

(5.6c) if X and Y are homogeneous of the same weight, then

$$\mathrm{Hom}(X,Y) \xrightarrow{\approx} \mathrm{Hom}(Q(X), Q(Y)) ;$$

(5.6d) if X and Y are homogeneous with weights m and n respectively and $Q(X) \approx Q(Y)$, then $m-n$ is an even integer $2k$ and $X(k) \approx Y$;

(5.6e) Q is essentially surjective.

The first four of these statements are obvious. For the last, note that $G = \mathbb{G}_m \times G/\mu_2$, and so we only have to show that any representation of μ_2 extends to a representation of $\mathbb{G}_m$, but this is obvious.

Remark 5.7 (a) The identity component of G_o is reductive if and only if the identity component of G is reductive; if G_o is connected, so also is G, but the converse statement is false (e.g., $G_o = \mu_2$, $G = \mathbb{G}_m$).

(b) It is possible to reconstruct $(\underline{C},w,T)$ from $(\underline{C}_o,\varepsilon)$ — the following diagram makes it clear how to reconstruct (G,w,t) from (G_o,ε):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbb{G}_m & \xrightarrow{-2} & \mathbb{G}_m & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle w} & & \| & & \\ 1 & \longrightarrow & G_o & \longrightarrow & G & \xrightarrow{t} & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

Proposition 5.8. Let $\underline{T} = (\underline{C},w,T)$ be a Tate triple over k with $\underline{C}$ algebraic. There exists a Tannakian category $\underline{C}_o$ over k, an element ε in $\underline{Aut}^{\otimes}(id_{\underline{C}_o})$ with $\varepsilon^2 = 1$, and a functor $Q : \underline{C} \to \underline{C}_o$ having the properties (5.6).

Proof: For any fibre functor ω on $\underline{C}$ with values in an algebra R, $\underline{Isom}(R,\omega(T))$ regarded as a sheaf on spec R is a torsor for $\mathbb{G}_m$. This association gives rise to a morphism of gerbs $\underline{G} \overset{df}{=} FIB(\underline{C}) \xrightarrow{t} TORS(\mathbb{G}_m)$, and we define $\underline{G}_o$ to be the kernel of t; thus $\underline{G}_o$ is the gerb of pairs (Q,ξ) where $Q \in ob(\underline{G})$ and ξ is an isomorphism $t(Q) \xrightarrow{\approx} \mathbb{G}_{m,X}$, i.e., $\underline{G}_o$ is the gerb of fibre functors on $\underline{T}$. Let $\underline{C}_o$ be the category $\underline{Rep}_k(G_o)$ which (see (3.14)) is Tannakian. If $Z = \underline{Aut}^{\otimes}(id_{\underline{C}})$ and $Z_o = \underline{Aut}^{\otimes}(id_{\underline{C}_o})$, then the homomorphism $Z \to \underline{Aut}(T) = \mathbb{G}_m$, $\alpha \longmapsto \alpha_T$, determined by t has kernel Z_o, and the composite $t \circ w = -2$; we let $\varepsilon = w(-1) \in Z_o$.

There is an obvious (restriction) functor $Q : \underline{C} \to \underline{C}_o$. In showing that Q had the properties (5.6), we can make a finite field extension $k \to k'$. We can therefore assume that $\underline{T}$ is neutral, but this case is covered by (5.5) and (5.6).

Example 5.9. Let $(\underline{V},w,T)$ be the Tate triple defined in (5.3); then $(\underline{V}_o,\varepsilon)$ is the pair defined in the paragraph preceding (4.29).

Example 5.10. Let $\underline{T} = (\underline{C},w,T)$ be a Tate triple over $\mathbb{R}$, and let ω be a fibre functor on $\underline{T}$ with values in $\mathbb{C}$. On combining (4.3) with (5.5) we find that $(\underline{T},\omega)$ corresponds to a quintuple (G,σ,c,w,t) in which

(a) G is an affine group scheme over $\mathbb{C}$;

(b) (σ,c) satisfies (4.2.1);

(c) $w : \mathbb{G}_m \to G$ is a central homomorphism; that the grading is defined over $\mathbb{R}$ means that w is defined over $\mathbb{R}$, i.e., $\sigma(w(g)) = w(\bar{g})$.

(d) $t : G \to \mathbb{G}_m$ is such that $t \circ w = -2$; that T is defined over $\mathbb{R}$ means that $t(\sigma(g)) = \overline{t(g)}$ and there exists $a \in \mathbb{G}_m(\mathbb{C})$ such that $t(c) = \sigma(a)a$.

Let $G_o = \mathrm{Ker}(t)$, and let $m \in G(\mathbb{C})$ be such that $t(m) = a^{-1}$. After replacing (σ,c) with $(\sigma \circ \underline{\mathrm{ad}}\, m, \sigma(m)cm)$ we find that the new c is in G_o. The pair $(\underline{C}_o,\omega|\underline{C}_o)$ corresponds to $(G_o,\sigma|G_o,c)$.

Remark 5.11. As in the neutral case, $\underline{T}$ can be reconstructed from

$(\underline{C}_o, \varepsilon)$. This can be proved by substituting bands for group schemes in the argument used in the neutral case (Saavedra [1, V. 3.14.1]), or by using descent theory to deduce it from the neutral case.

There is a stronger result: $\underline{T} \mapsto (\underline{C}_o, \varepsilon)$ defines an equivalence between the 2-category of Tate triples and that of $\mathbb{Z}/2\mathbb{Z}$-graded Tannakian categories.

Graded polarizations

For the remainder of this section, $\underline{T} = (\underline{C}, w, T)$ will be a Tate triple over $\mathbb{R}$ with $\underline{C}$ algebraic. We use the notations of §4; in particular $\underline{C}' = \underline{C}_{(\mathbb{C})}$. Let U be an invertible object of $\underline{C}'$ that is defined over $\mathbb{R}$, i.e., U is provided with an identification $U \stackrel{\approx}{\to} \bar{U}$; then in the definitions and results in §4 concerning sesquilinear forms and Weil forms, it is possible to replace $\underline{1}$ with U .

For each $X \in \mathrm{ob}(\underline{C}')$ that is homogeneous of degree n , let $\pi(X)$ be an equivalence class of Weil forms $X \otimes \bar{X} \to \underline{1}(-n)$ of parity $(-1)^n$; we say that π is a (graded) <u>polarization</u> on $\underline{T}$ if

(5.12a) for all X , $\bar{\phi} \in \pi(X)$ whenever $\phi \in \pi(\bar{X})$;

(5.12b) for all X and Y that are homogeneous of the same degree, $\phi \oplus \psi \in \pi(X \oplus Y)$ whenever $\phi \in \pi(X)$ and $\psi \in \pi(Y)$;

(5.12c) for all homogeneous X and Y , $\phi \otimes \psi \in \pi(X \otimes Y)$ whenever $\phi \in \pi(X)$ and $\psi \in \pi(Y)$;

(5.12d) the map $T \otimes \bar{T} \to T^{\otimes 2} = \underline{1}(2)$, defined by $T \stackrel{\approx}{\to} \bar{T}$, is in $\pi(T)$.

Proposition 5.13. Let $(\underline{C}_o, \varepsilon)$ be the pair associated with $\underline{T}$ by (5.8). There is a canonical bijection

$$Q : \mathrm{Pol}(\underline{T}) \to \mathrm{Pol}_\varepsilon(\underline{C}_o)$$

from the set of polarizations on $\underline{T}$ to the set of polarizations on $\underline{C}_o$ with parity ε.

Proof: For any $X \in ob(\underline{C}')$ that is homogeneous degree n, (5.6b) and (5.6c) give an isomorphism

$$Q : \mathrm{Hom}\,(X \otimes \bar{X}, \underline{1}(-n)) \xrightarrow{\approx} \mathrm{Hom}(Q(X) \otimes \overline{Q(X)}, \underline{1}) .$$

We define $Q\pi$ to be the polarization such that, for any homogeneous X, $Q\pi(QX) = \{Q\phi \mid \phi \in \pi(X)\}$. It is clear that $\pi \mapsto Q\pi$ is a bijection.

Corollary 5.14. The Tate triple $\underline{T}$ is polarizable if and only if $\underline{C}_o$ has a polarization π with parity $\varepsilon(\pi) \equiv \varepsilon \pmod{Z_o(\mathbb{R})^2}$.

Proof: See (4.20e).

Corollary 5.15. The map $(z,\pi) \mapsto z\pi : {}_2Z_o(\mathbb{R}) \times \mathrm{Pol}(\underline{T}) \to \mathrm{Pol}(\underline{T})$ (where $\phi(x,y) \in z\pi(X) \Longleftrightarrow \phi(x,zy) \in \pi(X)$) makes $\mathrm{Pol}(\underline{T})$ into a pseudo-torsor for ${}_2Z_o(\mathbb{R})$.

Proof: See (4.20d).

Theorem 5.16. Let π be a polarization on $\underline{T}$, and let ω be a fibre functor on $\underline{C}'$ with values in $\mathbb{C}$. Let (G,w,t) correspond to $(\underline{T}_{\mathbb{C}},\omega)$. For any real form K of G such that $K_o = \mathrm{Ker}\,(t)$ is compact, the pair (σ_K,ε) where $\varepsilon = w(-1)$ satisfies (4.2.1), and ω defines an equivalence between $\underline{T}$ and the Tate triple defined by $(G,\sigma_K,\varepsilon,w,t)$. For any simple X in $\underline{C}'$ $\{\omega(\phi)'\,|\,\phi \in \pi(X)\}$ is the set of K_o-invariant positive-definite Hermitian forms on $\omega(X)$.

Proof: See (4.16).

Remark 5.17. From (4.17) one can deduce the following: A triple (B,w,t), where B is an affine algebraic band over $\mathbb{R}$ and $t \circ w = -2$, bounds a polarizable Tate triple if and only if $B_o = \mathrm{Ker}(t{:}B \to \mathbb{G}_m)$ is the band defined by a compact real algebraic group; when this condition holds, the polarizable Tate triple bound by (B,w,t) is unique up to a tensor equivalence preserving the action of B and the polarization, and the equivalence is unique up isomorphism. The Tate triple is neutral if and only if $\varepsilon = w(-1) \in Z_o(\mathbb{R})^2$.

Let (G,w,t) be a triple as in (5.4) defined over $\mathbb{R}$, and let $G_o = \mathrm{Ker}(t)$ and $\varepsilon = w(-1)$. A Hodge element $C \in G_o(\mathbb{R})$ is said to be a Hodge element for (G,w,t) if $C^2 = \varepsilon$. A G-invariant sesquilinear form $\psi : V \times V \to \underline{1}(-n)$ on a homogeneous complex representation V of G of degree n is said to be a C-polarization if

$$\psi^C(x,y) \overset{\mathrm{df}}{=\!=} \psi(x,Cy)$$

is a positive-definite Hermitian form on V. When C is a Hodge element for (G,w,t) there is a polarization π_C on the Tate triple defined by (G,w,t) for which the positive forms are exactly the C-polarizations.

Proposition 5.18. Every polarization on the Tate triple defined by (G,w,t) is of the form π_C for some Hodge element C.

Proof: See (4.22) and (4.23).

Proposition 5.19. Assume that $w(-1)=1$. Then there is a unique (up to isomorphism) fibre functor ω on $\underline{T}$ with values in $\mathbb{R}$ transforming positive bilinear forms for π into positive-definite symmetric bilinear forms.

Proof: See (4.27).

Proposition 5.20. Let $(\underline{V},w,T)$ be the Tate triple defined in (5.3), and let π_{can} be the polarization on $\underline{V}$ such that, if $(V,a) \in ob(\underline{V})$ is homogeneous, then $\pi(V,a)$ comprises the $(-1)^{\deg(V)}$-symmetric positive-definite forms on V. If $w(-1) \neq 1$ for $\underline{T}$ and π is a polarization on $\underline{T}$, then there exists a unique (up to isomorphism) exact faithful functor $\omega : \underline{C} \to \underline{V}$ preserving the Tate-triple structures and carrying π into π_{can}.

Proof: Combine (4.29) and (5.9).

Example 5.21. Let $\underline{T}$ be the Tate triple $(\underline{Hod}_{\mathbb{R}},w,\mathbb{R}(1))$ defined in (5.2). A polarization on a real Hodge structure V of weight n is a bilinear form $\phi : V \times V \to \mathbb{R}(-n)$ such that the real-valued form $(x,y) \mapsto (2\pi i)^n \phi(x,Cy)$, where C denotes the element $i \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, is positive-definite and symmetric. These polarizations are the positive (bilinear) forms for a polarization

π on the Tate triple $\underline{T}$. The functor $\omega : \underline{\mathrm{Hod}}_{\mathbb{R}} \to \underline{V}$ provided by the last proposition is $V \longmapsto (V \otimes \mathbb{C},\ v \mapsto C\bar{v})$. (Note that $(\underline{\mathrm{Hod}}_{\mathbb{R}}, w, \mathbb{R}(1))$ is not quite the Tate triple associated, as in (5.4), with $(\mathbb{S}, w, t)$ because we have chosen a different Tate objects; this difference explains the occurrence of $(2\pi i)^n$ in the above formula; π is essentially the polarization defined by the canonical Hodge element C .)

Filtered Tannakian Categories

For this topic we refer the reader to Saavedra [1, IV.2].

§6. Motives for absolute Hodge cycles

Throughout this section, k will denote a field of characteristic zero with algebraic closure $\bar{k}$ and Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. All varieties will be projective and smooth, and, for X a variety (or motive) over k , $\bar{X}$ denotes $X \otimes_k \bar{k}$. We shall freely use the notations and results of Article I; for example, if $k = \mathbb{C}$ then $H_B(X)$ denotes the graded vector space $\oplus H_B^i(X)$.

Complements on absolute Hodge cycles

For X a variety over k , $C_{AH}^p(X)$ denotes the rational vector space of absolute Hodge cycles on X (see I.2). When X has pure dimension n , we write

$$\mathrm{Mor}^p_{AH}(X,Y) = C^{n+p}_{AH}(X \times Y) \ .$$

Then $\mathrm{Mor}^p_{AH}(X,Y) \subset H^{2n+2p}(X \times Y)(p+n) = \bigoplus_{r+s=2n+2p} H^r(X) \otimes H^s(Y)(p+n)$

$$= \bigoplus_{s=r+2p} H^r(X)^\vee \otimes H^s(Y)(p)$$

$$= \bigoplus_{r} \mathrm{Hom}(H^r(X), H^{r+2p}(Y)(p))$$

The next proposition is obvious from this and the definition of an absolute Hodge cycle.

Proposition 6.1. An element f of $\mathrm{Mor}^p_{AH}(X,Y)$ gives rise to

(a) for each prime ℓ, a homomorphism $f_\ell : H_\ell(\bar{X}) \to H_\ell(\bar{Y})(p)$ of graded vector spaces (meaning that f_ℓ is a family of homomorphisms $f^r_\ell : H^r_\ell(\bar{X}) \to H^{r+2p}_\ell(\bar{Y})(p)$) ;

(b) a homomorphism $f_{DR} : H_{DR}(X) \to H_{DR}(Y)(p)$ of graded vector spaces;

(c) for each $\sigma : k \hookrightarrow \mathbb{C}$, a homomorphism $f_\sigma : H_\sigma(X) \to H_\sigma(Y)(p)$ of graded vector spaces.

These maps satisfy the following conditions:

(d) for all $\gamma \in \Gamma$ and primes ℓ, $\gamma(f_\ell) = f_\ell$;

(e) f_{DR} is compatible with the Hodge filtrations on each homogeneous factor;

(f) for each $\sigma : k \hookrightarrow \mathbb{C}$, the maps f_σ, f_ℓ, and f_{DR} correspond under the comparison isomorphisms (I.1).

Conversely, assume that k is embeddable in $\mathbb{C}$; then any family of maps f_ℓ, f_{DR} as in (a), (b) arises from $f \in \mathrm{Mor}^p_{AH}(X,Y)$ provided (f_ℓ) and f_{DR} satisfy (d) and (e) respectively and for every $\sigma : k \hookrightarrow \mathbb{C}$

there exists an f_σ such that (f_ℓ), f_{DR}, and f_σ satisfy condition (f) ; moreover, f is unique.

Similarly, a $\psi \in C_{AH}^{2n-r}(X \times X)$ gives rise to pairings

$$\psi^s : H^s(X) \times H^{2r-s}(x) \to \mathbb{Q}(-r) .$$

Proposition 6.2. On any variety X (of dimension n) there exists a $\psi \in C_{AH}^{2n-r}(X \times X)$ such that, for every $\sigma : k \hookrightarrow \mathbb{C}$,

$$\psi_\sigma^r : H_\sigma^r(X,\mathbb{R}) \times H_\sigma^r(X,\mathbb{R}) \to \mathbb{R}(-r)$$

is a polarization of real Hodge structures (in the sense of (5.21)).

Proof: Choose a projective embedding of X , and let L be a hyperplane section of X . Let ℓ be the class of L in $H^2(X)(1)$, and write ℓ also for the map $H(X) \to H(X)(1)$ sending a class to its cup-product with ℓ . Assume X is connected and define the primitive cohomology of X by

$$H^r(X)_{prim} = \mathrm{Ker}(\ell^{n-r+1} : H^r(X) \to H^{2n-r+2}(X)(n-r+1)) .$$

The hard Lefschetz theorem states that

$$\ell^{n-r} : H^r(X) \to H^{2n-r}(X)(n-r)$$

is an isomorphism for $r \leq n$; it implies that

$$H^r(X) = \bigoplus_{s \geq r-n,\, s \geq 0} \ell^s H^{r-2s}(X)(-s)_{prim} .$$

Thus any $x \in H^r(X)$ can be written uniquely, $x = \Sigma \ell^s(x_s)$, with $x_s \in H^{r-2s}(X)(-s)_{prim}$; define

$$*x = \sum (-1)^{(r-2s)(r-2s+1)/2} \ell^{n-r+s} x_s \in H^{2n-r}(X)(n-r) .$$

Then $x \mapsto *x : H^r(X) \to H^{2n-r}(X)(n-r)$ is a well-defined map for each of the three cohomology theories, ℓ-adic, de Rham, and Betti. Proposition 6.1 shows that it is defined by an absolute Hodge cycle (rather, the map $H(X) \to H(X)(n-r)$ that is $x \mapsto *x$ on H^r and zero otherwise is so defined). We take ψ^r to be

$$H^r(X) \otimes H^r(X) \xrightarrow{id \otimes *} H^r(X) \otimes H^{2n-r}(X)(n-r) \to H^{2n}(X)(n-r) \xrightarrow{Tr} \mathbb{Q}(-r) .$$

Clearly it is defined by an absolute Hodge cycle, and the Hodge-Riemann bilinear relations (see Wells [1, 5.3]) show that it defines a polarization on the real Hodge structure $H^r_\sigma(X,\mathbb{R})$ for each $\sigma : k \hookrightarrow \mathbb{C}$.

<u>Proposition</u> 6.3. For any $u \in Mor^o_{AH}(Y,X)$ there exists a unique $u' \in Mor^o_{AH}(X,Y)$ such that

$$\psi_X(uy,x) = \psi_Y(y,u'x) \quad , \; x \in H^r(X) , \; y \in H^r(Y)$$

where ψ_X and ψ_Y are the forms defined in (6.2); moreover,

$$\mathrm{Tr}(uu') = \mathrm{Tr}(u'u) \in \mathbb{Q}$$

$$\mathrm{Tr}(uu') > 0 \quad \text{if} \quad u \neq 0$$

Proof: The first part is obvious; the last assertion follows from the fact that the ψ_X and ψ_Y are positive forms for a polarization in $\underline{\mathrm{Hod}}_{\mathbb{R}}$.

Note that the proposition shows that $\mathrm{Mor}^{\circ}_{AH}(X,X)$ is a semisimple $\mathbb{Q}$-algebra (see 4.5).

Construction of the category of motives

Let $\underline{V}_k$ be the category of (smooth projective, not necessarily connected) varieties over k. The category $\underline{CV}_k$ is defined to have as objects symbols $h(X)$, one for each $X \in \mathrm{ob}(\underline{V}_k)$, and as morphisms $\mathrm{Hom}(h(X),h(Y)) = \mathrm{Mor}^{\circ}_{AH}(X,Y)$. There is a map $\mathrm{Hom}(Y,X) \to \mathrm{Hom}(h(X),h(Y))$ sending a homomorphism to the cohomology class of its graph which makes h into contravariant functor $\underline{V}_k \to \underline{CV}_k$.

Clearly $\underline{CV}_k$ is a $\mathbb{Q}$-linear category, and $h(X \coprod Y) = h(X) \oplus h(Y)$. There is a $\mathbb{Q}$-linear tensor structure on $\underline{CV}_k$ for which $h(X) \otimes h(Y) = h(X \times Y)$, the associativity constraint is induced by $(X \times Y) \times Z \to X \times (Y \times Z)$, the commutativity constraint is induced by $Y \times X \to X \times Y$, and the identity object is $h(\text{point})$.

The false category of effective (or positive) motives $\dot{\underline{M}}^{+}_k$ is defined to be the pseudo-abelian (Karoubian) envelope of $\underline{CV}_k$. Thus an object of $\dot{\underline{M}}^{+}_k$ is a pair (M,p) with $M \in \underline{CV}_k$ and p an idempotent in $\mathrm{End}(M)$, and

$$\mathrm{Hom}((M,p),\ (N,q)) = \{f : M \to N \mid f\circ p = q\circ f\}/\sim \qquad (6.3.1)$$

where $f\sim 0$ if $f\circ p = 0 = q\circ f$. The rule

$$(M,p) \otimes (N,q) = (M\otimes N,\ p\otimes q)$$

defines a $\mathbb{Q}$-linear tensor structure on $\underline{\dot{M}}_k^+$, and $M \to (M,\mathrm{id}) : \underline{CV}_k \to \underline{\dot{M}}_k^+$ is a fully faithful functor which we use to identify $\underline{CV}_k$ with a subcategory of $\underline{\dot{M}}_k^+$. With this identification, (M,p) becomes the image of $p : M \to M$. The category $\underline{\dot{M}}_k^+$ is pseudo-abelian: any decomposition of id_M into a sum of pairwise orthogonal idempotents

$$\mathrm{id}_M = e_1 + \cdots + e_m$$

corresponds to a decomposition

$$M = M_1 \oplus \cdots \oplus M_m$$

with $e_i|M_i = \mathrm{id}_{M_i}$. The functor $\underline{CV}_k \to \underline{\dot{M}}_k^+$ is universal for functors from $\underline{CV}_k$ into pseudo-abelian categories.

For any $X \in \mathrm{ob}(\underline{V}_k)$, the projection maps $p^r : H(X) \to H^r(X)$ define an element of $\mathrm{Mor}^o_{AH}(X,X) = \mathrm{End}(h(X))$. Corresponding to the decomposition

$$\mathrm{id}_{h(X)} = p^o + p^1 + p^2 + \cdots$$

there is a decomposition (in $\underline{\dot{M}}_k^+$)

$$h(X) = h^{o}(X) + h^{2}(X) + h^{2}(X) + \cdots .$$

This grading of objects of $\underline{CV}_k$ extends in an obvious way to objects of $\underline{\dot{M}}_k^+$, and the Künneth formulas show that these gradings are compatible with tensor products (and therefore satisfy (5.1a)).

Let L be the Lefschetz motive $h^2(\mathbb{P}^1)$. With the notations of (I.1), $H(L) = \mathbb{Q}(-1)$, whence it follows that $\mathrm{Hom}(M,N) \tilde{\to} \mathrm{Hom}(M \otimes L, N \otimes L)$ for any effective motives M and N. This means that $V \mapsto V \otimes L$ is a fully faithful functor and allows us to invert L. The false category $\dot{M}_k$ of <u>motives</u> is defined as follows:

(6.4a) an object of $\underline{\dot{M}}_k$ is a pair (M,m) with $M \in \mathrm{ob}(\underline{\dot{M}}_k^+)$ and $m \in \mathbb{Z}$;

(6.4b) $\mathrm{Hom}((M,m), (N,n)) = \mathrm{Hom}(M \otimes L^{N-m}, N \otimes L^{N-n}), N \geq m,n$ (for different N, these groups are canonically isomorphic);

(6.4c) composition of morphisms is induced by that in $\underline{\dot{M}}_k^+$.

This category of motives is $\mathbb{Q}$-linear and pseudo-abelian and has a tensor structure

$$(M,m) \otimes (N,n) = (M \otimes N, m+n)$$

and grading

$$(M,m)^r = M^{r-2m}$$

We identify $\dot{\underline{M}}_k^+$ with a subcategory of $\dot{\underline{M}}_k$ by means of $M \mapsto (M,0)$. The Tate motive T is $L^{-1} = (\underline{1},1)$. We abbreviate $M \otimes T^{\otimes m} = (M,m)$ by $M(m)$.

We shall see shortly that $\dot{\underline{M}}_k$ is a rigid abelian tensor category, and $\mathrm{End}(\underline{1}) = \mathbb{Q}$. It is not however a Tannakian category because, for $X \in \mathrm{ob}(\underline{V}_k)$, $\mathrm{rk}(h(X))$ is the Euler-Poincaré characteristic, $\sum (-1)^r \dim H^r(X)$, of X, which is not necessarily positive. To remedy this we modify the commutativity constraint as follows: let

$$\dot{\psi} : M \otimes N \xrightarrow{\approx} N \otimes M, \ \dot{\psi} = \oplus\, \dot{\psi}^{r,s}, \ \dot{\psi}^{r,s} : M^r \otimes N^s \xrightarrow{\approx} N^s \otimes M^r$$

by the commutativity constraint on $\dot{\underline{M}}_k$; define a new commutativity constraint by

$$\psi : M \otimes N \xrightarrow{\approx} N \otimes M\,, \ \psi = \oplus\, \psi^{r,s}, \ \psi^{r,s} = (-1)^{rs}\dot{\psi}^{r,s} \tag{6.4.1}$$

Then $\underline{M}_k$, with $\dot{\psi}$ replaced by ψ, is the true category $\underline{M}_k$ of

Proposition 6.5. The category $\underline{M}_k$ is a semisimple Tannakian category over $\mathbb{Q}$.

Proof: We first need a lemma.

Lemma 6.6. Let C be a $\mathbb{Q}$-linear pseudo-abelian cateogry, and let $\omega : \underline{C} \to \underline{\mathrm{Vec}}_{\mathbb{Q}}$ be a faithful $\mathbb{Q}$-linear functor. If every indecomposable object ov $\underline{C}$ is simple, then $\underline{C}$ is a semisimple abelian category and ω is exact.

Proof: The existence of ω shows that each object of $\underline{C}$ has finite length and hence is a finite direct sum of simple objects. For any map $f : X \to Y$, $\mathrm{Ker}(f)$ is the largest subobject of X on which f is zero, and $\mathrm{Coker}(f)$ is the largest quotient of

Y such that the composite $X \to Y \to \mathrm{Coker}(f)$ is zero. The rest of the proof is easy.

Proof of (6.5): We can replace $\underline{M}_k$ with the tensor subcategory generated by a finite number of objects, and consequently we can assume that there exists an embedding $\sigma : k \hookrightarrow \mathbb{C}$. The functor $H_\sigma : \underline{M}_k \to \underline{\mathrm{Vec}}_{\mathbb{Q}}$ is faithful and $\mathbb{Q}$-linear. Let M be an indecomposable motive, and let $i : N \hookrightarrow M$ be a nonzero simple subobject of M. Clearly M is homogeneous, and after tensoring it with a power of T we can assume that N and M are effective, and therefore

$$M \oplus M' = h^r(X) \quad \text{with} \quad X \in \underline{V}_k \quad \text{and}$$
$$N \oplus N' = h^r(Y) \quad \text{with} \quad Y \in \underline{V}_k .$$

Let $u : h^r(Y) \to h^r(X)$ be the morphism $\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$ and let $u' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be its transpose (see 6.3). As $\mathrm{Tr}(u'u) > 0$, and $\mathrm{Tr}(u'u) = \mathrm{Tr}(ai)$, we see that $ai \neq 0$. It is therefore an automorphism of N, and $(ai)^{-1} a : M \to N$ projects M on N. As M is indecomposable, this shows that $M = N$, and M is simple. The lemma can therefore be applied, and shows that $\underline{M}_k$ is a semisimple $\mathbb{Q}$-linear abelian tensor category. It remains to show that it is rigid. Let X and Y be varieties of pure dimension m and n respectively. Then

$$\begin{aligned}\mathrm{Hom}(h(X),h(Y)) = C^m_{AH}(X\times Y) = C^m_{AH}(Y\times X) &= \mathrm{Hom}(h(Y),h(X)(m-n)) \\ &= \mathrm{Hom}(h(Y)(n),h(X)(m)) .\end{aligned}$$

The functor $h(X) \mapsto h(X)^{\vee} \overset{\mathrm{df}}{=} h(X)(m)$ extends to a fully faithful contravariant functor $M \mapsto M^{\vee} : \underline{M}_k \to \underline{M}_k$, and we set $\underline{\mathrm{Hom}}(M,N) = M^{\vee} \otimes N$. It is straightforward now to check that $\underline{M}_k$ is Tannakian (especially if one applies (1.20)).

The following theorem summarizes what we have (essentially) shown about $\underline{M}_k$.

Theorem 6.7. (a) Let w be the grading on $\underline{M}_k$; then $(\underline{M}_k, w, T)$ is a Tate triple over $\mathbb{Q}$.

(b) There is a contravariant functor $h : \underline{V}_k \to \underline{M}_k$; every effective motive is the image $(h(X), p)$ of an idempotent $p \in \mathrm{End}(h(X))$ for some $X \in \mathrm{ob}(\underline{V}_k)$; every motive is of the form $M(n)$ for some effective M and some $n \in \mathbb{Z}$.

(c) For all varieties X, Y with X of pure dimension m, $C_{AH}^{m+s-r}(X \times Y) = \mathrm{Hom}(h(X)(r), h(Y)(s))$; in particular, $C_{AH}^{m}(X \times Y) = \mathrm{Hom}(h(X), h(Y))$; morphisms of motives can be expressed in terms of absolute Hodge cycles on varieties by means of (6.3.1) and (6.4b).

(d) The constraints on $\underline{M}_k$ have an obvious definition, except that the obvious commutativity constraint has to be modified by (6.4.1).

(e) For varieties X and Y,

$$h(X \coprod Y) = h(X) \oplus h(Y)$$
$$h(X \times Y) = h(X) \otimes h(Y)$$
$$h(X)^{\vee} = h(X)(m), \text{ if } X \text{ is of pure dimension } m.$$

(f) The functors H_{ℓ}, H_{DR}, and H_{σ} define fibre functors on $\underline{M}_k$; these fibre functors define morphisms of Tate triples, $\underline{M}_k \to \underline{T}_{\ell}, T_{DR}, T_B$

(see (5.2b)); in particular $H(T) = \mathbb{Q}(1)$.

(g) When k is embeddable in $\mathbb{C}$, $\mathrm{Hom}(M,N)$ is the vector space of families of maps

$$f_\ell : H_\ell(\bar{M}) \to H_\ell(\bar{N})$$

$$f_{DR} : H_{DR}(M) \to H_{DR}(N)$$

such that f_{DR} preserves the Hodge filtration, $\gamma(f_\ell) = f_\ell$ for all $\gamma \in \Gamma$, and for any $\sigma : k \hookrightarrow \mathbb{C}$ there exists a map $f_\sigma : H_\sigma(M) \to H_\sigma(N)$ agreeing with f_ℓ and f_{DR} under the comparison isomorphisms.

(h) The category $\underline{M}_k$ is semisimple.

(i) There exists a polarization on $\underline{M}_k$ for which $\pi(h^r(X))$ consists of the forms defined in (6.2).

Some calculations

According to (6.7g), to define a map $M \to N$ of motives it suffices to give a procedure for defining a map of cohomology groups $H(M) \to H(N)$ that works (compatibly) for all three theories: Betti, deRham, and ℓ-adic. The map will be an isomorphism if its realization in one theory is an isomorphism.

Let G be a finite group acting on a variety. The group algebra $\mathbb{Q}[G]$ acts on $h(X)$, and we define $h(X)^G$ to be the motive $(h(X),p)$ where p is the idempotent $(\mathrm{ord}\ G)^{-1}\Sigma g$. Note that $H(h(X)^G) = H(X)^G$.

<u>Proposition</u> 6.8. Assume that the finite group G acts freely on X, so that X/G is also smooth; then $h(X/G) = h(X)^G$.

<u>Proof</u>: Since cohomology is functorial, there exists a map $H(X/G) \to H(X)$ whose image lies in $H(X)^G = H(h(X)^G)$. The Hochschild-Serre spectral sequence $H^r(G,H^s(X)) \Rightarrow H^{r+s}(X/G)$ shows that the map $H(X/G) \to H(X)^G$ is an isomorphism for, say the ℓ-adic cohomology, because $H^r(G,V) = 0$, $r>0$, if V is a vector space over a field of characteristic zero.

<u>Remark</u> 6.9. More generally, if $f : Y \to X$ is a map of finite (generic) degree n between connected varieties of the same dimension, then the composite $H(X) \xrightarrow{f^*} H(Y) \xrightarrow{f_*} H(X)$ is multiplication by n; there therefore exist maps $h(X) \to h(Y) \to h(X)$ with composite n, and $h(X)$ is a direct summand of $h(Y)$.

<u>Proposition</u> 6.10. Let E be a vector bundle of rank $m+1$ over a variety X and let $p : \mathbf{P}(E) \to X$ be the associated projective bundle; then $h(\mathbf{P}(E)) = h(X) \oplus h(X)(-1) \oplus \cdots \oplus h(X)(-m)$.

<u>Proof</u>: Let γ be the class in $H^2(\mathbf{P}(E))(1)$ of the canonical line bundle on $\mathbf{P}(E)$, and let $p^* : H(X) \to H(\mathbf{P}(E))$ be the map induced by p. The map

$$(c_o, \ldots, c_m) \mapsto \sum p^*(c_i)\, \gamma^i : H(X) \oplus \cdots \oplus H(X)(-m) \to H(\mathbf{P}(E))$$

has the requisite properties.

<u>Proposition</u> 6.11. Let Y be a smooth closed subvariety of codimension c in the variety X, and let X' be the variety obtained from X

by blowing up Y ; then there is an exact sequence

$$0 \to h(Y)(-c) \to h(X) \oplus h(Y')(-1) \to h(X') \to 0$$

where Y' is the inverse image of Y .

Proof: From the Gysin sequences

$$\begin{array}{ccccccccc} \cdots \to & H^{r-2c}(Y)(-c) & \to & H^r(X) & \to & H^r(X-Y) & \to \cdots \\ & \downarrow & & \downarrow & & \| & \\ \cdots \to & H^{r-2}(Y')(-1) & \to & H^r(X') & \to & H^r(X'-Y') & \to \cdots \end{array}$$

we obtain a long exact sequence

$$\cdots \to H^{r-2c}(Y)(-c) \to H^r(X) \oplus H^{r-2}(Y')(-1) \to H^r(X') \to \cdots .$$

But Y' is a projective bundle over Y , and so $H^{r-2c}(Y)(-c) \to H^{r-2}(Y')(-1)$ is injective. Therefore there are exact sequences

$$0 \to H^{r-2c}(Y)(-c) \to H^r(X) \oplus H^{r-2}(Y')(-1) \to H^r(X') \to 0 ,$$

which can be rewritten as

$$0 \to H(Y)(-c) \to H(X) \oplus H(Y')(-1) \to H(X') \to 0 .$$

We have constructed a sequence of motives, which is exact because the cohomology functors are faithful and exact.

Corollary 6.12. With the notations of the proposition,

$$h(X') = h(X) \oplus \bigoplus_{r=1}^{c-1} h(Y)(-r) .$$

Proof: (6.10) shows that $h(Y') = \bigoplus_{r=0}^{c-1} h(Y)(r)$.

Proposition 6.13. If X is an abelian variety, then $h(X) = \Lambda(h^1(X))$.

Proof: Cup-product defines a map $\Lambda(H^1(X)) \to H(X)$ which, for the Betti cohomology say, is known to be an isomorphism. (See Mumford [1,I.1].)

Proposition 6.14. If X is a curve with Jacobian J , then

$$h(X) = \underline{1} \oplus h^1(J) \oplus L .$$

Proof: The map $X \to J$ (well-defined up to translation) defines an isomorphism $H^1(J) \to H^1(X)$.

Proposition 6.15. Let X be a unirational variety of dimension $d \leq 3$ over an algebraically closed field; then

(d=1) $h(X) = \underline{1} \oplus L$;

(d=2) $h(X) = 1 \oplus rL \oplus L^2$, some $r \in \mathbb{N}$;

(d=3) $h(X) = \underline{1} \oplus rL \oplus h^1(A)(-1) \oplus rL^2 \oplus L^3$, some $r \in \mathbb{N}$,

where A is an abelian variety.

Proof: We prove the proposition only for $d=3$. According to the resolution theorem of Abhyankar [1], there exist maps

$$\mathbb{P}^3 \xleftarrow{u} X' \xrightarrow{v} X$$

with v surjective of finite degree and u a composite of blowing-ups. We know $h(\mathbb{P}^3) = \underline{1} \oplus L \oplus L^2 \oplus L^3$ (special case of (6.10)). When a point is blown up, a motive $L \oplus L^2$ is added, and when a curve Y is blown up, a motive $L \oplus h^1(Y)(-1) \oplus L^2$ is added. Therefore

$$h(X') = \underline{1} \oplus sL \oplus M(-1) \oplus sL^2 \oplus L^3$$

where M is a sum of motives of the form $h^1(Y)$, Y a curve. A direct summand of such an M is of the form $h^1(A)$ for A an abelian variety (see (6.21) below). As $h(X)$ is a direct summand of $h(X')$ (see (6.9)) and Poincaré duality shows that the multiples of L^2 and L^3 occurring in $h(X)$ are the same as those of L and $\underline{1}$ respectively the proof is complete.

Proposition 6.16. Let X_d^n denote the Fermat hypersurface of dimension n and degree d:

$$T_o^d + T_1^d + \cdots + T_{n+1}^d = 0 .$$

Then

$$h^n(X_d^n) \oplus d\, h^n(\mathbb{P}^n) = h^n(X_d^{n-1} \times X_d^1)^{\mu_d} \oplus (d-1)h^{n-2}(X_d^{n-2})(-1)$$

where μ_d, the group of d^{th} roots of 1, acts on $X_d^{n-1} \times X_d^1$ according to

$$\zeta(t_o:\cdots:t_n;s_o:s_1:s_2) = (t_o:\cdots:t_{n-1}:\zeta t_n;s_o:s_1:\zeta s_2)$$

Proof: See Shioda-Katsura [1, 2.5].

Artin Motives

Let $\underline{V}_k^o$ be the category of zero-dimensional varieties over k, and let $\underline{CV}_k^o$ be the image of $\underline{V}_k^o$ in $\underline{M}_k$. The Tannakian subcategory $\underline{M}_k^o$ of $\underline{M}_k$ generated by the objects of $\underline{CV}_k^o$ is called the category of (E.) Artin motives.

For any X in $\mathrm{ob}(\underline{V}_k^o)$, $X(\bar{k})$ is a finite set on which Γ acts continuously. Thus $\mathbb{Q}^{X(\bar{k})}$ is a finite-dimensional continuous representation of Γ. If we regard Γ, in the obvious way, as a (constant, pro-finite) affine group scheme over k, $\mathbb{Q}^{X(\bar{k})} \in \underline{\mathrm{Rep}}_{\mathbb{Q}}(\Gamma)$. For $X, Y \in \mathrm{ob}(\underline{V}_k^o)$,

$$\mathrm{Hom}(h(X),h(Y)) \stackrel{\mathrm{df}}{=} C_{AH}^o(X \times Y) = (\mathbb{Q}^{X(\bar{k})\times Y(\bar{k})})^{\Gamma} = \mathrm{Hom}_{\Gamma}(\mathbb{Q}^{X(\bar{k})}, \mathbb{Q}^{Y(\bar{k})}).$$

Thus $h(X) \mapsto \mathbb{Q}^{X(\bar{k})} : \underline{CV}_k^o \to \underline{\mathrm{Rep}}_{\mathbb{Q}}(\Gamma)$ is fully faithful, and Grothendieck's formulation of Galois theory shows that it is essentially surjective. Therefore $\underline{CV}_k^o$ is abelian and $\underline{M}_k^o = \underline{CV}_k^o$. We have shown:

Proposition 6.17. The category of Artin motives $\underline{M}_k^o = \underline{CV}_k^o$; the functor $h(X) \mapsto \mathbb{Q}^{X(\bar{k})}$ defines an equivalence of tensor categories $\underline{M}_k^o \xrightarrow{\sim} \underline{\mathrm{Rep}}_{\mathbb{Q}}(\Gamma)$.

Remark 6.18. Let M be an Artin motive, and regard M as an object of $\underline{\mathrm{Rep}}_{\mathbb{Q}}(\Gamma)$. Then

$H_\sigma(M) = M$ (underlying vector space) for any $\sigma : k \hookrightarrow \mathbb{C}$;

$H_\ell(\bar{M}) = M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, as a Γ-module;

$H_{DR}(M) = (M \otimes_{\mathbb{Q}} \bar{k})^\Gamma$.

Note that, if $M = h(X)$ where $X = \operatorname{spec} A$, then

$$H_{DR}(M) = (\mathbb{Q}^{X(\bar{k})} \otimes_{\mathbb{Q}} \bar{k})^\Gamma = (A \otimes_k \bar{k})^\Gamma = A .$$

Remark 6.19. The proposition shows that $\underline{M}_k^o$ is equivalent to the category of sheaves of finite-dimensional $\mathbb{Q}$ vector spaces on the étale site $\operatorname{spec}(k)_{et}$.

Effective motives of degree 1

A $\mathbb{Q}$-rational Hodge structure is a finite-dimensional vector space V over $\mathbb{Q}$ together with a real Hodge structure on $V \otimes \mathbb{R}$ whose weight filtration is defined over $\mathbb{Q}$. Let $\underline{\mathrm{Hod}}_{\mathbb{Q}}$ be the category of $\mathbb{Q}$-rational Hodge structures. A polarization on an object V of $\underline{\mathrm{Hod}}_{\mathbb{Q}}$ is bilinear pairing $\psi : V \times V \to \mathbb{Q}(-n)$ such that $\psi \otimes \mathbb{R}$ is a polarization on the real Hodge structure $V \otimes \mathbb{R}$.

Let $\underline{\mathrm{Isab}}_k$ be the category of abelian varieties up to isogeny over k . The following theorem summarizes part of the analytic theory of abelian varieties.

Theorem 6.20. (Riemann) The functor $H_B^1 : \underline{\mathrm{Isab}}_{\mathbb{C}} \to \underline{\mathrm{Hod}}_{\mathbb{Q}}$ is fully faithful; the essential image consists of polarizable Hodge structures of weight 1.

Let $\underline{M}_k^{+1}$ be the pseudo-abelian subcategory of $\underline{M}_k$ generated by motives of the form $h^1(X)$ for X a geometrically connected curve; according to (6.14), $\underline{M}_k^{+1}$ can also be described as the category generated by motives of the form $h^1(J)$ for J a Jacobian.

Proposition 6.21. (a) The functor $h^1 : \underline{\text{Isab}}_k \to \underline{M}_k$ factors through $\underline{M}_k^{+1}$ and defines an equivalence of categories, $\underline{\text{Isab}}_k \xrightarrow{\approx} \underline{M}_k^{+1}$.

(b) The functor $H^1 : \underline{M}_{\mathbb{C}}^{+1} \to \underline{\text{Hod}}_{\mathbb{Q}}$ is fully faithful; its essential image consists of polarizable Hodge structures of weight 1.

Proof: Every object of $\underline{\text{Isab}}_k$ is a direct summand of a Jacobian, which shows that h^1 factors through $\underline{M}_k^{+1}$. Assume, for simplicity, that k is algebraically closed. Then, for any $A, B \in \text{ob}(\underline{\text{Isab}}_k)$,

$$\text{Hom}(B,A) \subset \text{Hom}(h^1(A),\ h^1(B)) \subset \text{Hom}(H_\sigma(A), H_\sigma(B)) \ ,$$

and (6.20) shows that $\text{Hom}(B,A) = \text{Hom}(H_\sigma(A), H_\sigma(B))$. Thus h^1 is fully faithful and (as $\underline{\text{Isab}}_k$ is abelian) essentially surjective. This proves (a), and (b) follows from (a) and (6.20).

The motivic Galois group

Let k be a field that is embeddable in $\mathbb{C}$. For any $\sigma : k \hookrightarrow \mathbb{C}$, we define $G(\sigma) = \underline{\text{Aut}}^{\otimes}(H_B)$. Thus $G(\sigma)$ is an affine group scheme over $\mathbb{Q}$, and H_B defines an equivalence of tensor categories $\underline{M}_k \xrightarrow{\approx} \underline{\text{Rep}}_{\mathbb{Q}}(G(\sigma))$. Because $G(\sigma)$ plays the same role for $\underline{M}_k$ as $\Gamma = \text{Gal}(\bar{k}/k)$ plays for $\underline{M}_k^{\text{o}}$, it is called the motivic Galois group.

Proposition 6.22. (a) If k is algebraically closed, then $G(\sigma)$ is a connected pro-reductive affine group scheme over $\mathbb{Q}$.
(b) Let $k \subset k'$ be algebraically closed fields, let $\sigma' : k' \hookrightarrow \mathbb{C}$, and let $\sigma = \sigma'|k$. The homomorphism $G(\sigma') \to G(\sigma)$ induced by $\underline{M}_k \to \underline{M}_{k'}$ is faithfully flat; if k has infinite transcendence degree over $\mathbb{Q}$, then $G(\sigma') \to G(\sigma)$ is an isomorphism.

Proof: (a) Let $X \in ob(\underline{M}_k)$, and let $\underline{C}_X$ be the abelian tensor subcategory of $\underline{M}_k$ generated by X, $X^\vee$, T, and $T^\vee$. According to (I 3.4), $G_X \overset{df}{=} \underline{Aut}^\otimes(H_\sigma|\underline{C}_X)$ is the smallest subgroup of $Aut(H_\sigma(X)) \times \mathbb{G}_m$ such that $(G_X)_\mathbb{C}$ contains the image of the homomorphism $\mu : \mathbb{G}_{m\mathbb{C}} \to Aut(H_\sigma(X,\mathbb{C})) \times \mathbb{G}_{m\mathbb{C}}$ defined by the Hodge structure on $H_\sigma(X)$. As $Im(\mu)$ is connected, so also is G_X. As $\underline{C}_X$ is semisimple (see (6.5)), G_X is a reductive group (2.23). Therefore $G = \varprojlim G_X$ is connected and pro-reductive.

(b) According to (I 2.9), $\underline{M}_k \to \underline{M}_{k'}$ is fully faithful, and so (2.29) shows that $G(\sigma') \twoheadrightarrow G(\sigma)$. When k has infinite transcendence degree over $\mathbb{Q}$, $\underline{M}_k \to \underline{M}_k{}'$ is essentially surjective, and so $G(\sigma') \xrightarrow{\approx} G(\sigma)$.

Now let k be arbitrary, and fix an embedding $\sigma : \bar{k} \hookrightarrow \mathbb{C}$. The inclusion $\underline{M}_k^o \to \underline{M}_k$ defines a homomorphism $\pi : G(\sigma) \to \Gamma$ because $\Gamma = \underline{Aut}^\otimes(H_\sigma|\underline{M}_k^o)$ (see (6.17)), and the functor $\underline{M}_k \to \underline{M}_{\bar{k}}$ defines a homomorphism $i : G^o(\sigma) \to G(\sigma)$ where $G^o(\sigma) \overset{df}{=} \underline{Aut}^\otimes(H_\sigma|\underline{M}_{\bar{k}})$.

Proposition 6.23. (a) The sequence

$$1 \to G^o(\sigma) \xrightarrow{i} G(\sigma) \xrightarrow{\pi} \Gamma \to 1$$

is exact.
(b) The identity component of $G(\sigma)$ is $G^o(\sigma)$.

(c) For any $\tau \in \Gamma$, $\pi^{-1}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(H_\sigma, H_{\tau\sigma})$, regarding H_σ and $H_{\tau\sigma}$ as functors on $\underline{M}_{\bar{k}}$.

(d) For any prime ℓ, there is a canonical continuous homomorphism $sp_\ell : \Gamma \to G(\sigma)(\mathbb{Q}_\ell)$ such that $\pi \circ sp_\ell = \mathrm{id}$.

Proof: (a) As $\underline{M}_k^o \to \underline{M}_k$ is fully faithful, π is surjective (2.29). To show that i is injective, it suffices to show that every motive $h(X)$, $X \in \underline{V}_{\bar{k}}$, is a subquotient of a motive $h(\bar{X}')$ for some $X' \in \underline{V}_k$; but X has a model X_o over a finite extension k' of k, and we can take $X' = \mathrm{Res}_{k'/k} X_o$. The exactness of $G(\sigma)$ is a special case of (c).

(b) This is an immediate consequence of (6.22a) and (a).

(c) Let $M, N \in \mathrm{ob}(\underline{M}_k)$. Then $\mathrm{Hom}(\bar{M},\bar{N}) \in \mathrm{ob}(\underline{\mathrm{Rep}}_{\mathbb{Q}}(\Gamma))$, and so we can regard it as an Artin motive over k. There is a canonical map of motives $\mathrm{Hom}(\bar{M},\bar{N}) \hookrightarrow \underline{\mathrm{Hom}}(M,N)$ giving rise to

$$H_\sigma(\mathrm{Hom}(\bar{M},\bar{N})) = \mathrm{Hom}(\bar{M},\bar{N}) \xrightarrow{H_\sigma} \mathrm{Hom}(H_\sigma(\bar{M}), H_\sigma(\bar{N})) = H_\sigma(\underline{\mathrm{Hom}}(M,N)) .$$

Let $\tau \in \Gamma$; then $H_\sigma(\bar{M}) = H_\sigma(M) = H_{\tau\sigma}(M) = H_{\tau\sigma}(\bar{M})$ and, for $f \in \mathrm{Hom}(\bar{M},\bar{N})$, $H_\sigma(f) = H_{\tau\sigma}(\tau f)$.

Let $g \in G(R)$; for any $f : M \to N$ in $\underline{M}_k$, there is a commutative diagram

$$\begin{array}{ccc} H_\sigma(M,R) & \xrightarrow{g_M} & H_\sigma(M,R) \\ \downarrow{\scriptstyle H_\sigma(f)} & & \downarrow{\scriptstyle H_\sigma(f)} \\ H_\sigma(N,R) & \xrightarrow{g_N} & H_\sigma(N,R) \end{array}$$

Let $\tau = \pi(g)$, so that g acts on $\mathrm{Hom}(\bar{M},\bar{N}) \subset \underline{\mathrm{Hom}}(M,N)$ as τ. Then for any $f : \bar{M} \to \bar{N}$ in $\underline{M}_{\bar{k}}$

$$\begin{array}{ccccc}
H_\sigma(\bar{M},R) & \xrightarrow{g_M} & H_\sigma(\bar{M},R) & = & H_{\tau\sigma}(\bar{M},R) \\
\downarrow H_\sigma(f) & & \downarrow H_\sigma(\tau^{-1}f) & & \downarrow H_{\tau\sigma}(f) \\
H_\sigma(\bar{N},R) & \xrightarrow{g_N} & H_\sigma(\bar{N},R) & = & H_{\tau\sigma}(\bar{N},R)
\end{array}$$

commutes. The diagram shows that $g_M : H_\sigma(\bar{M},R) \to H_{\tau\sigma}(\bar{M},R)$ depends only on M as an object of $\underline{M}_{\bar{k}}$. We observed in the proof of (a) above that $\underline{M}_{\bar{k}}$ is generated by motives of the form $\bar{M}$, $M \in \underline{M}_k$. Thus g defines an element of $\underline{\mathrm{Hom}}^{\otimes}(H_\sigma, H_{\tau\sigma})(R)$, where H_σ and $H_{\tau\sigma}$ are to be regarded as functors on $\underline{M}_{\bar{k}}$. We have defined a map $\pi^{-1}(\tau) \to \underline{\mathrm{Hom}}^{\otimes}(H_\sigma, H_{\tau\sigma})$, and it is easy to see that it is surjective.

(d) After (c), we have to find a canonical element of $\mathrm{Hom}^{\otimes}(H_\ell(\sigma M), H_\ell(\tau\sigma M))$ depending functorially on $M \in \underline{M}_{\bar{k}}$. Extend τ to an automorphism $\bar{\tau}$ of $\mathbb{C}$. For any variety X over $\bar{k}$, there is a $\bar{\tau}^{-1}$-linear isomorphism $\sigma X \leftarrow \tau\sigma X$ which induces an isomorphism $\tau : H_\ell(\sigma X) \xrightarrow{\approx} H_\ell(\tau\sigma X)$.

The "espoir" (Deligne [2, 0.10) that every Hodge cycle is absolutely Hodge has a particularly elegant formulation in terms of motives.

<u>Conjecture</u> 6.24. For any algebraically closed field k and embedding $\sigma : k \hookrightarrow \mathbb{C}$, the functor $H_\sigma : \underline{M}_k \to \underline{\mathrm{Hod}}_{\mathbb{Q}}$ is fully faithful.

The functor is obviously faithful. There is no description, not even conjectural, for the essential image of H_σ.

Motives of abelian varieties

Let $\underline{M}_k^{av}$ be the Tannakian subcategory of $\underline{M}_k$ generated by motives of abelian varieties and Artin motives. The main theorem of I has the following restatement.

Theorem 6.25. For any algebraically closed field k and embedding $\sigma : k \hookrightarrow \mathbb{C}$, the functor $H_\sigma : \underline{M}_k^{av} \to \underline{Hod}_{\mathbb{Q}}$ is fully faithful.

Proposition 6.26. The motive $h(X) \in ob(\underline{M}_k^{av})$ if

(a) X is a curve;

(b) X is a unirational variety of dimension ≤ 3;

(c) X is a Fermat hypersurface;

(d) X is a K3-surface

Before proving this, we note the following consequence.

Corollary 6.27. Every Hodge cycle on a variety that is a product of abelian varieties, zero-dimensional varieties, and varieties of type (a), (b), (c) and (d), is absolutely Hodge.

Proof of 6.26. Cases (a) and (b) follow immediately from (6.14) and (6.15), and (c) follows by induction (on n) from (6.16). In fact one does not need the full strength of (6.16). There is a rational map

$$X_d^r \times X_d^s \dashrightarrow X_d^{r+s}$$

$$(x_o:\cdots:x_{r+1}),\ (y_o:\cdots:y_{s+1}) \longmapsto (x_o y_{s+1}:\cdots:x_r y_{s+1}:\varepsilon x_{r+1} y_o:\cdots:\varepsilon x_{r+1} y_s)$$

where ε is a primitive $2m^{th}$ root of 1. The map is not defined on the subvariety $Y : x_{r+1} = y_{s+1} = 0$. On blowing up $X_d^r \times X_d^s$

along the nonsingular centre Y , one obtains maps

$$\begin{tikzcd} Z_d^{r,s} \arrow[d] \arrow[dr] & \\ X_d^r \times X_d^s \arrow[r, dashed] & X_d^{r+s} \end{tikzcd} .$$

By induction, we can assume that the motives of X_d^r, X_d^s, and $Y(=X_d^{r-1} \times X_d^{s-1})$ are in $\underline{M}_k^{av}$. Corollary (6.12) now shows that $h(Z_d^{r,s}) \in \mathrm{ob}(\underline{M}_k^{av})$ and (6.9) that $h(X_d^{r+s}) \in \mathrm{ob}(\underline{M}_k^{av})$.

For (d), we first note that the proposition is obvious if X is a Kummer surface, for then $X = \tilde{A}/\langle\sigma\rangle$ where $\tilde{A}$ is an abelian variety A with its 16 points of order ≤ 2 blown up and σ induces $a \longmapsto -a$ on A .

Next consider an arbitrary K3-surface X , and fix a projective embedding of X . Then

$$h(X) = h(\mathbb{P}^2) \oplus h^2(X)_{prim}$$

and so it suffices to show that $h^2(X)_{prim}$ is in $\underline{M}^{av}$. We can assume $k = \mathbb{C}$. It is known (Kuga-Shimura [1], Deligne [1, 6.5]) that there is a smooth connected variety S over $\mathbb{C}$ and families $f : Y \to S$, $a : A \to S$ of polarized K3-surfaces and abelian varieties respectively parametrized by S having the following properties:

(a) for some $o \in S$, $Y_o \overset{df}{=} f^{-1}(o)$ is X together with its given polarization;

(b) for some $1 \in S$, Y_1 is a polarized Kummer surface;

(c) there is an inclusion $u : R^2 f_* \mathbb{Q}(1)_{prim} \hookrightarrow \underline{\mathrm{End}}(R^1 a_* \mathbb{Q})$

compatible with the Hodge filtrations.

The map $u_o : H^2_B(X)(1)_{prim} \hookrightarrow \mathrm{End}(H^1(A_o,\mathbb{Q}))$ is therefore defined by a Hodge cycle, and it remains to show that it is defined by an absolutely Hodge cycle. But the initial remark shows that u_1, being a Hodge cycle on a product of Kummer and abelian surfaces, is absolutely Hodge, and principle B (I2.12) completes the proof.

Motives of abelian varieties of potential CM-type

An abelian variety A over k is said to be of potential CM-type if it becomes of CM-type over an extension of k. Let A be such an abelian variety defined over $\mathbb{Q}$, and let $MT(A)$ be the Mumford-Tate group of $A_{\mathbb{C}}$ (see I.5). Since $A_{\mathbb{C}}$ is of CM-type, $MT(A)$ is a torus, and we let $L \subset \mathbb{C}$ be a finite Galois extension of $\mathbb{Q}$ splitting $MT(A)$. Let $\underline{M}^{A,L}_{\mathbb{Q}}$ be the Tannakian subcategory of $\underline{M}_{\mathbb{Q}}$ generated by A, the Tate motive, and the Artin motives split by L^{ab}, and let G^A be affine group scheme associated with this Tannakian category and the fibre functor H_B.

Proposition 6.28. There is an exact sequence of affine group schemes

$$1 \to MT(A) \xrightarrow{i} G^A \xrightarrow{\pi} \mathrm{Gal}(L^{ab}/\mathbb{Q}) \to 1 .$$

Proof: Let $\underline{M}^A_{\mathbb{C}}$ be the image of $\underline{M}^{A,L}_{\mathbb{Q}}$ in $\underline{M}_{\mathbb{C}}$; then $MT(A)$ is the affine group scheme associated with $\underline{M}^A_{\mathbb{C}}$, and so the above sequence is a subsequence of the sequence in (6.23a).

Remark 6.29. If we identify $MT(A)$ with the subgroup of $\mathrm{Aut}(H^1_B(A))$, then (as in 6.23c) $\pi^{-1}(\tau)$ becomes identified with the $MT(A)$-torsor whose R-points, for any $\mathbb{Q}$-algebra R, are the

R-linear isomorphisms $a : H^1(A_{\mathbb{C}},R) \to H^1(\tau A_{\mathbb{C}},R)$ such that $a(s) = \tau s$ for all (absolute) Hodge cycles on $A_{\bar{\mathbb{Q}}}$. We can also identify $MT(A)$ with a subgroup of $\mathrm{Aut}(H_1^B(A))$ and then it becomes more natural to identify $\pi^{-1}(\tau)$ with the torsor of R-linear isomorphisms $a^\vee : H_1(A_{\mathbb{C}},R) \to H_1(\tau A_{\mathbb{C}},R)$ preserving Hodge cycles.

On passing to the inverse limit over all A and L, we obtain an exact sequence

$$1 \to S^o \to S \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1$$

with S^o and S respectively the connected Serre group and the Serre group. This sequence plays an important role in the next three articles.

Appendix: Terminology from non-abelian cohomology

We review some definitions from Giraud [1].

Fibred categories

Let $\alpha : \underline{F} \to \underline{A}$ be a functor. For any object U of $\underline{A}$ we write $\underline{F}_U$ for the category whose objects are those F in $\underline{F}$ such that $\alpha(\underline{F}) = U$ and whose morphisms are those f such that $\alpha(f) = \mathrm{id}_U$. For any morphism $a : \alpha(F_1) \to \alpha(F_2)$, we write $\mathrm{Hom}_a(F_1,F_2)$ for the set of $f : F_1 \to F_2$ such that $\alpha(f) = a$. A morphism $f : F_1 \to F_2$ in $\underline{F}$ is said to be cartesian, and F_1 is said to be the inverse image $\alpha(f)^*F_2$ of F_2 relative to $\alpha(f)$, if, for any $F' \in \underline{F}_{\alpha(F_1)}$ and $h \in \mathrm{Hom}_{\alpha(f)}(F',F_2)$, there is a unique $g \in \mathrm{Hom}_{id}(F', F_1)$ such that $fg=h$:

$$\begin{tikzcd} F' \arrow[d, dashed, "g"'] \arrow[dr, "h"] & \\ F_1 \arrow[r, "f"] & F_2 \\ \alpha(F_1) \arrow[r, "\alpha(f)"] & \alpha(F_2) \end{tikzcd} .$$

We say that $\alpha : \underline{F} \to \underline{A}$ is a <u>fibered category</u> if

(a) for any morphism $a : U_1 \to U_2$ in $\underline{A}$ and $F_2 \in ob(\underline{F}_{U_2})$, the inverse image $a^*(F_2)$ of F_2 exists;

(b) the composite of two cartesian morphisms is cartesian. (Existence and transitivity of inverse images.) Then a^* can be made into a functor $\underline{F}_{U_2} \to \underline{F}_{U_1}$, and $(ab)^*$ is canonically isomorphic to b^*a^*.

Let $\alpha : \underline{F} \to \underline{A}$ and $\alpha' : \underline{F}' \to \underline{A}$ be fibred categories over $\underline{A}$, and let $\beta : \underline{F} \to \underline{F}'$ be a functor such that $\alpha' \circ \beta = \alpha$; one says β is <u>cartesian</u> if it maps cartesian morphisms to cartesian morphisms.

<u>Stacks</u> (Champs)

Let $\alpha : \underline{F} \to \underline{Aff}_S$ be a fibred category where $\underline{Aff}_S$ is the category of affine schemes over $S = \mathrm{spec}\ R$. We endow $\underline{Aff}_S$ with the f.p.q.c. topology. Let $a : T' \to T$ be a faithfully flat map of affine S-schemes and let $F \in ob(\underline{F}_{T'})$; a <u>descent datum</u> on F relative to a is an isomorphism $\phi : p_1^*(F) \to p_2^*(F)$ such that $p_{31}^*(\phi) = p_{32}^*(\phi)\, p_{21}^*(\phi)$ where p_1 and p_2 are the projections $T'' = T' \times_T T' \rightrightarrows T'$ and the p_{ij} are the projections $T''' = T' \times_T T' \times_T T' \rightrightarrows T' \times_T T'$. With an obvious notion of morphism, the pairs (F,ϕ) form a category $\underline{Des}(T'/T)$. There is a functor $\underline{F}_T \to \underline{Des}(T'/T)$ under which $F \in ob(\underline{F}_T)$ maps to $(a^*(F),\phi)$ where ϕ is the canonical morphism $p_1^*a^*(F) \approx (p_1a)^*F = (p_2a)^*F \approx p_2^*a^*F$. The fibred category $\alpha : \underline{F} \to \underline{Aff}_S$ is a <u>stack</u> if, for all faithfully flat maps $a : T' \to T$, $\underline{F}_T \to \underline{Des}(T'/T)$ is an equivalence of categories.

For example, let $\alpha : MOD \to \underline{Aff}_S$ be the fibred category such that MOD_T is the category of finitely presented $\Gamma(T,O_T)$-modules; descent theory shows that this is a stack (Waterhouse [1,17.2], Bourbaki [2,I.3.6]). Similarly, there is a stack $PROJ \to \underline{Aff}_S$ for which $PROJ_T$ is the category of finitely generated projective $\Gamma(T,O_T)$-modules (ibid.) and a stack $AFF \to \underline{Aff}_S$ for which $AFF_T = \underline{Aff}_T$.

<u>Gerbs</u> (Gerbes)

A stack $\underline{G} \to \underline{Aff}_S$ is a <u>gerb</u> if

(a) each fibre $\underline{G}_T$ is a groupoid (i.e., all morphisms in $\underline{G}_T$ are isomorphisms);

(b) there is a faithfully flat map $T \to S$ such that $\underline{G}_T$ is nonempty;

(c) any two objects of a fibre $\underline{G}_T$ are locally isomorphic (i.e., their inverse images relative to some faithfully flat map $T' \to T$ are isomorphic).

By a <u>morphism of gerbs</u> over $\underline{Aff}_S$ we mean a cartesian functor. A gerb $\underline{G} \to \underline{Aff}_S$ is said to be <u>neutral</u> (or trivial) if $\underline{G}_S$ is nonempty.

Let F be a sheaf of groups on S for the f.p.q.c. topology. The fibred category $TORS(F) \to \underline{Aff}_S$ for which $TORS(F)_T$ is the category $\underline{Tors}_T(F)$ of right F-torsors on T is a neutral gerb. Conversely, let $\underline{G}$ be a neutral gerb and let $Q \in ob(\underline{G}_S)$; then $F = \underline{Aut}(Q)$ is a sheaf of groups on $\underline{Aff}_S$ and $\underline{G} \to TORS(F)$, $P \mapsto \underline{Isom}_T(a^*Q, P)$ (for $a : T \to S$) is an equivalence of gerbs.

Bands (Liens)

Let F and G be sheaves of groups for the f.p.q.c. topology on S , and let G^{ad} be the quotient sheaf G/Z where Z is the centre of G . The action of G^{ad} on G by conjugation induces an action of G^{ad} on the sheaf $\underline{Isom}(F,G)$ and we set $Isex(F,G) = \Gamma(S, G^{ad} \backslash \underline{Isom}(F,G))$. As G^{ad} acts faithfully on $\underline{Isom}(F,G)$,

$$Isex(F,G) = \varinjlim Ker(G^{ad}(T)\backslash Isom(F|T, G|T) \rightrightarrows G^{ad}(T\times T)\backslash Isom(F|T\times T, G|T\times T))$$

where the limit is over all $T \to S$ faithfully flat and affine.

A band B on S is defined by a triple (S', G, ϕ) where S' is an affine S-scheme, faithfully flat over S , G is a sheaf of groups on S' , and $\phi \in Isex(p_1^*G, p_2^*G)$ is such that $p_{31}^*(\phi) = p_{32}^*(\phi)p_{21}^*(\phi)$. (As before, the p_i and p_{ij} are the various projection maps $S'' \rightrightarrows S'$ and $S''' \rightrightarrows S''$). If T is also a faithfully flat affine S-scheme, and $a : T \to S'$ is an S-morphism, then we do not distinguish between the bands defined by (S', G, ϕ) and $(T, a^*(G), (a \times a)^*(\phi))$. Let B_1 and B_2 be the bands defined by (S', G_1, ϕ_1) and (S', G_2, ϕ_2); an isomorphism $B_1 \xrightarrow{\approx} B_2$ is an element $\psi \in \mathrm{Isex}(G_1,G_2)$ such that $p_2^*(\psi)\circ\phi_1 = \phi_2\circ p_1^*(\psi)$.

If G is a sheaf of groups on S , we write $B(G)$ for the band defined by (S,G,id). One shows that $\mathrm{Isom}(B(G_1),B(G_2)) = \mathrm{Isex}(G_1,G_2)$. Thus $B(G_1)$ and $B(G_2)$ are isomorphic if and only if G_2 is an inner form of G_1, i.e. G_2 becomes isomorphic to G_1 on some faithfully flat S-scheme T , and the class of G_2 in $H^1(S,\underline{\mathrm{Aut}}(G_1))$ comes from $H^1(S,G_1^{ad})$. When G_2 is commutative, then $\mathrm{Isom}(B(G_1),B(G_2)) = \mathrm{Isex}(G_1,G_2) = \mathrm{Isom}(G_1,G_2)$, and we usually do not distinguish $B(G_2)$ from G_2 .

The <u>centre</u> $Z(B)$ of the band B defined by (S',G,ϕ) is defined by $(S',Z,\phi|p_1^*Z)$ where Z is the centre of G . The above remark shows that $\phi|p_1^*Z$ lifts to an element $\phi_1 \in \mathrm{Isom}(p_1^*Z,\ p_2^*Z)$, and one checks immediately that $p_{31}^*(\phi_1) = p_{32}^*(\phi_1)\ p_{21}^*(\phi_1)$. Thus $(S',Z,\phi|p_1^*Z)$ arises from a sheaf of groups on S , which we identify with $Z(B)$.

Let $\underline{G}$ be a gerb on $\underline{\mathrm{Aff}}_S$. By definition, there exists an object $Q \in \underline{G}_{S'}$, for some $S' \to S$ faithfully flat and affine. Let $G = \underline{\mathrm{Aut}}(Q)$; it is a sheaf of groups on S' . Again by definition, p_1^*Q and p_2^*Q are locally isomorphic on S'', and the locally-defined isomorphisms determine an element $\phi \in \mathrm{Isex}(p_1^*(G),p_2^*(G))$. The triple (S',G,ϕ) defines a band B which is uniquely determined up to a unique isomorphism. This band B is called the <u>band associated with the gerb</u> $\underline{G}$, and $\underline{G}$ is said to be bound by B . For example, the gerb $\mathrm{TORS}(G)$ is bound by $B(G)$.

A band B is said to be <u>affine</u> (or <u>algebraic</u>) if it can be defined by a triple (S',G,ϕ) with G an affine (or algebraic) group scheme over S'. A gerb is said to be <u>affine</u> (or <u>algebraic</u>) if it is bound by an affine (or algebraic) band.

<u>Cohomology</u>

Let B be a band. Two gerbs $\underline{G}_1$ and $\underline{G}_2$ bound by B are said to be <u>B-equivalent</u> if there is an isomorphism $m : \underline{G}_1 \to \underline{G}_2$ with the following property: for some triple (S',G,ϕ) defining B there is an object $Q \in \underline{G}_{1S'}$ such that the automorphism $G \approx \underline{\mathrm{Aut}}(Q) \xrightarrow{\approx} \underline{\mathrm{Aut}}(m(Q)) \approx G$ defined by m is equal to id in $\mathrm{Isex}(G,G)$. The cohomology set $H^2(S,B)$ is defined to be the set of B-equivalence classes of gerbs bound by B. If Z is the centre of B, then $H^2(S,Z)$ is equal to the cohomology group of Z in the usual sense of the f.p.q.c. topology on S, and either $H^2(S,B)$ is empty or $H^2(S,Z)$ acts simply transitively on it (Giraud [1, IV. 3.3.3]).

<u>Proposition</u>: Let $S = \mathrm{spec}\, k$, k a field, and let $\underline{G}$ be an affine algebraic gerb on S; then there is a finite field extension k' of k such that $\underline{G}_{S'}$, $S' = \mathrm{spec}\, k'$, is nonempty.

<u>Proof</u>: By assumption, the band B of $\underline{G}$ is defined by a triple (S',G,ϕ) with G of finite type over S'. Let $S' = \mathrm{spec}\, R'$; R' can be replaced by a finitely generated subalgebra, and then by a quotient modulo a maximal ideal, and so we may suppose $S' = \mathrm{spec}\, k'$ where k' is a finite field extension of k. We shall

show that the gerbs $\underline{G}$ and TORS(G) become B-equivalent over some finite field extension of k' . The statement preceding the proposition shows we have to prove that an element of $H^2(S',Z)$, Z the centre of B , is killed by a finite field extension of k' . But this assertion is obvious for elements of $H^1(S',Z)$ and is easy to prove for elements of the Čech groups $\check{H}^r(S',Z)$, and so the exact sequence

$$0 \to \check{H}^2(S',Z) \to H^2(S',Z) \to \check{H}^1(S',\underline{H}^1(Z))$$

completes the proof. (See Saavedra [1, III 3.1] for more details.)

Note: (Added July, 1981): It seems likely that the final question in (3.5) can be shown to have a positive answer when k has characteristic zero. In particular this would show that any rigid abelian tensor category $\underline{C}$ with $\mathrm{End}(\underline{1}) = k$ having a fibre functor with values in some extension of k is Tannakian, provided k is a field of characteristic zero.

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