Shimura Varieties and Moduli

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December 1, 2010, v1.00

Abstract

Connected Shimura varieties are the quotients of hermitian symmetric domains by discrete groups defined by congruence conditions. We examine how to interprete them as moduli varieties.

Contents

Introduction

The hermitian symmetric domains are the complex manifolds isomorphic to bounded symmetric domains. Each is a product of indecomposable hermitian symmetric domains, which themselves are classified by the "special" nodes on Dynkin diagrams.

The Griffiths period domains are the parameter spaces for polarized rational Hodge structures. A period domain is a hermitian symmetric domain if the universal family of Hodge structures on it is a variation of Hodge structures, i.e., satisfies Griffiths transversality. This rarely happens, but, as Deligne showed, every hermitian symmetric domain can be realized as the subdomain of a period domain on which certain tensors for the universal family are of type (p, p) (i.e., are Hodge tensors).

In particular, every hermitian symmetric domain can be realized as a moduli space for Hodge structures plus tensors. This all takes place in the analytic realm, because hermitian symmetric domains are not algebraic varieties. To obtain an algebraic variety, we must pass to the quotient by an arithmetic group. In fact, in order to obtain a moduli variety, we should assume that the arithmetic group is defined by congruence conditions. The algebraic varieties obtained in this way are the connected Shimura varieties.

The arithmetic subgroup lives in a semisimple algebraic group over $\mathbb{Q}$, and the variations of Hodge structures on the connected Shimura variety are classified in terms of auxiliary reductive algebraic groups. In order to realize the connected Shimura variety as a moduli variety, we must choose the additional data so that the variation of Hodge structures is of geometric origin. The main result of the article classifies the connected Shimura varieties for which this is known to be possible. Briefly, in a small number of cases, the connected Shimura variety is a moduli variety for abelian varieties with polarization, endomorphism, and level structure (the PEL case); for a much larger class, the variety is a moduli variety for abelian varieties with polarization, Hodge class, and level structure (the PHL case); for all connected Shimura varieties except those of type E_6, E_7, and certain types D, the variety is a moduli variety for abelian *motives* with additional structure. In the remaining cases, the connected Shimura variety is not a moduli variety for abelian motives, and it is not known whether it is a moduli variety at all.

We now summarize the contents of the article.

§1. As an introduction to the general theory, we review the case of elliptic modular curves. In particular, we prove that the modular curve constructed analytically coincides with the modular curve constructed algebraically using geometric invariant theory.

§2. To give a hermitian symmetric domain amounts to giving a real semisimple algebraic group H with trivial centre and a homomorphism $u: U^1 \to H(\mathbb{R})$ satisfying certain conditions. We briefly review the theory of hermitian symmetric domains and their classification in terms of Dynkin diagrams and special nodes.

§3. The group of holomorphic automorphisms of a hermitian symmetric domain is a real Lie group, and the algebraic varieties we are concerned with are quotients of hermitian symmetric domains by discrete subgroups of this Lie group. In this section we review the fundamental theorems of Borel, Harish-Chandra, Margulis, Mostow, Selberg, Tamagawa, and others concerning discrete subgroups of Lie groups.

§4. The arithmetic locally symmetric varieties (resp. connected Shimura varieties) are the quotients of hermitian symmetric domains by arithmetic (resp. congruence) groups. We explain the fundamental theorems of Baily and Borel on the algebraicity of these varieties and of the maps into them.

§5. We review the definition of Hodge structures and of their variations, and state the fundamental theorem of Griffiths that motivates their definition.

§6. We define the Mumford-Tate group of a rational Hodge structure, and we prove the basic results concerning their behaviour in families.

§7. We review the theory of period domains, and explain Deligne's interpretation of hermitian symmetric domains as period subdomains.

§8. We classify certain variations of Hodge structures on locally symmetric varieties in terms of group-theoretic data.

§9. In order to be able to realize all but a handful of locally symmetric varieties as moduli varieties, we shall need to replace algebraic varieties and algebraic classes by more general objects. In this section, we prove Deligne's theorem that all Hodge classes on abelian varieties are absolutely Hodge, and so make sense algebraically.

§10. Following Satake and Deligne, we classify the symplectic embeddings of an algebraic group that give rise to an embedding of the associated hermitian symmetric domain into a Siegel upper half space.

§11. We use the results of the preceding sections to determine which Shimura varieties can be realized as moduli varieties for abelian varieties (or abelian motives) plus absolute Hodge classes and level structure.

Although the expert will find little that is new in this article, there is much that is not well explained in the literature. As far as possible, complete proofs have been included.

Notations

We use k to denote the base field (always of characteristic zero), and k^{al} to denote an algebraic closure of k. "Algebraic group" means "affine algebraic group scheme" and "algebraic variety" means "geometrically reduced scheme of finite type over a field". For a smooth algebraic variety X over $\mathbb{C}$, we let X^{an} denote the set $X(\mathbb{C})$ endowed with its natural structure of a complex manifold. The tangent space at a point p of space X is denoted by $T_p(X)$.

Vector spaces and representations are finite dimensional unless indicated otherwise. The dual of a vector space V is denoted by $V^{\vee}$. For a k-vector space V and commutative k-algebra R,

$V_R = R \otimes_k V$. Similarly, we write V_S for the constant sheaf (or free sheaf $\mathcal{O}_S \otimes V$ of $\mathcal{O}_S$-modules) on a space S defined by a vector space V.

A *vector sheaf* on a complex manifold (or scheme) S is a locally free sheaf of $\mathcal{O}_S$-modules of finite rank. In order for $\mathcal{W}$ to be a vector subsheaf of a vector sheaf $\mathcal{V}$, we require that the maps on the fibres $\mathcal{W}_s \to \mathcal{V}_s$ be injective. With these definitions, vector sheaves correspond to vector bundles and vector subsheaves to vector subbundles.

The quotient of a Lie group or algebraic group G by its centre $Z(G)$ is denoted by G^{ad}. A Lie group or algebraic group is said to be *adjoint* if it is semisimple (in particular, connected) with trivial centre. An algebraic group G is *simply connected* if it is semisimple and every surjective homomorphism $G' \to G$ (of algebraic groups) with finite kernel is an isomorphism. The inner automorphism of G or G^{ad} defined by an element g is denoted by $\mathrm{inn}(g)$. Let $\mathrm{ad}\colon G \to G^{\mathrm{ad}}$ be the quotient map. There is an action of G^{ad} on G such that $\mathrm{ad}(g)$ acts as $\mathrm{inn}(g)$ for all $g \in G(k^{\mathrm{al}})$. For an algebraic group G over $\mathbb{R}$, $G(\mathbb{R})^+$ is the identity component of $G(\mathbb{R})$ for the real topology. For a finite extension of fields L/k and an algebraic group G over L, we write $(G)_{L/k}$ for algebraic group over k obtained by (Weil) restriction of scalars. We sometimes use the term "simple" for representations as well as modules.

A *prime*[1] of a number field k is a prime ideal in $\mathcal{O}_k$ (a finite prime), an embedding of k into $\mathbb{R}$ (a real prime), or a conjugate pair of embeddings of k into $\mathbb{C}$ (a complex prime). The ring of finite adèles of $\mathbb{Q}$ is $\mathbb{A}_f = \mathbb{Q} \otimes \left(\prod_p \mathbb{Z}_p\right)$.

We use ι or $z \mapsto \bar{z}$ to denote complex conjugation on $\mathbb{C}$ or on a subfield of $\mathbb{C}$.

We use the language of modern algebraic geometry, not Weil's Foundations. For example, if G and G' are algebraic groups over a field k, then by a homomorphism $G \to G'$ we mean a homomorphism defined over k, not over some universal domain. Similarly, a simple algebraic group need not be geometrically (i.e., absolutely) simple.

1. Elliptic modular curves

The first Shimura varieties, and the first moduli varieties, were the elliptic modular curves. In this section, we review the theory of elliptic modular curves as an introduction to the general theory.

Definition

Let D be the complex upper half plane,

$$D = \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on D by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup if, for some integer $N \geq 1$, Γ contains the principal congruence subgroup of level N,

$$\Gamma(N) \stackrel{\mathrm{def}}{=} \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv I \text{ modulo } N\}.$$

[1] Some authors say "place".

An elliptic modular curve is the quotient $\Gamma\backslash D$ of D by a congruence group Γ. Initially this is a one-dimensional complex manifold, but it can be compactified by adding a finite number of "cusps", and so it has a unique structure of an algebraic curve. This curve can be realized as a moduli variety for elliptic curves with level structure, from which it is possible deduce many beautiful properties of the curve, for example, that it has a canonical model over a specific number field, and that the coordinates of the special points on the model generate class fields.

Elliptic modular curves as moduli varieties

For an elliptic curve E over $\mathbb{C}$, the exponential map defines an exact sequence

$$0 \to \Lambda \to T_0(E^{\mathrm{an}}) \xrightarrow{\exp} E^{\mathrm{an}} \to 0 \tag{1}$$

with

$$\Lambda \simeq \pi_1(E^{\mathrm{an}}, 0) \simeq H_1(E^{\mathrm{an}}, \mathbb{Z}).$$

The functor $E \rightsquigarrow (T_0 E, \Lambda)$ is an equivalence from the category of complex elliptic curves to the category of pairs consisting of a one-dimensional $\mathbb{C}$-vector space and a lattice. Thus, to give an elliptic curve over $\mathbb{C}$ amounts to giving a two-dimensional $\mathbb{R}$-vector space V, a complex structure on V, and a lattice in V. It is known that D parametrizes elliptic curves plus additional data. Traditionally, to a point τ of D one attaches the quotient of $\mathbb{C}$ by the lattice spanned by 1 and τ. In other words, one fixes the real vector space and the complex structure, and varies the lattice. From the point of view of period domains and Shimura varieties, it is more natural to fix the real vector space and the lattice, and vary the complex structure.

Thus, let V be a two-dimensional vector space over $\mathbb{R}$. Let J be a complex structure on V (i.e., an endomorphism of V such that $J^2 = -1$), and let $V_{\mathbb{C}} = V_J^+ \oplus V_J^-$ be the decomposition of $V_{\mathbb{C}}$ into its $+1$ and -1 eigenspaces. The isomorphism $V \to V_{\mathbb{C}}/V_J^- \simeq V_J^+$ carries the complex structure J on V to the natural complex structure on V_J^+. The map $J \mapsto V_J^+$ identifies the set of complex structures on V with the set of nonreal one-dimensional quotients of $V_{\mathbb{C}}$, i.e., with $\mathbb{P}(V_{\mathbb{C}}) \smallsetminus \mathbb{P}(V)$. This set has two connected components.

Now choose a basis for V, and identify it with $\mathbb{R}^2$. Let $\psi\colon V \times V \to \mathbb{R}$ be the alternating form

$$\psi\left(\binom{a}{b}, \binom{c}{d}\right) = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc.$$

On one of the connected components, which we denote D, the symmetric bilinear form

$$(x, y) \mapsto \psi_J(x, y) \overset{\text{def}}{=} \psi(x, Jy)\colon V \times V \to \mathbb{R}$$

is positive definite and on the other it is negative definite. Thus D is the set of complex structures on V for which $+\psi$ (rather than $-\psi$) is a Riemann form. Our choice of a basis for V identifies $\mathbb{P}(V_{\mathbb{C}}) \smallsetminus \mathbb{P}(V)$ with $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$ and D with the complex upper half plane.

Now let Λ be the lattice $\mathbb{Z}^2$ in V. For each $J \in D$, the quotient V_J^+/Λ is an elliptic curve E with $\Lambda \simeq H_1(E^{\mathrm{an}}, \mathbb{Z})$. In this way, D classifies the isomorphism classes of pairs consisting of an elliptic curve E over $\mathbb{C}$ and a basis for $H_1(E^{\mathrm{an}}, \mathbb{Z})$.

We let E_N denote the kernel of multiplication by N on an elliptic curve E. Thus, for the curve $E = V_J^+/\Lambda$,

$$E_N(\mathbb{C}) = \tfrac{1}{N}\Lambda/\Lambda \simeq \Lambda/N\Lambda.$$

A level-N structure on an elliptic curve E is a pair of points $\eta = (t_1, t_2)$ in $E(\mathbb{C})$ that form a basis for $E_N(\mathbb{C})$. In the following, we always require that $e_N(t_1, t_2) = \zeta$ where e_N is the Weil pairing $E_N \times E_N \to \mu_N$ and ζ is a fixed Nth root of 1 in $\mathbb{C}$.

Identify $\Gamma(N)$ with the subgroup of $\mathrm{SL}(V)$ whose elements map Λ into itself and act as the identity on $\Lambda/N\Lambda$. On passing to the quotient by $\Gamma(N)$, we obtain a one-to-one correspondence between the points of $\Gamma(N)\backslash D$ and the isomorphism classes of pairs consisting of an elliptic curve E over $\mathbb{C}$ and a level-N structure η on E. Let Y_N denote the algebraic curve over $\mathbb{C}$ with $Y_N^{\mathrm{an}} = \Gamma(N)\backslash D$.

Let $f\colon E \to S$ be a family of elliptic curves over a scheme S. A level N structure on E/S is a pair of sections to f that give a level-N structure on E_s for each point s of S.

PROPOSITION 1.1. *Let $f\colon E \to S$ be a family of elliptic curves on a smooth algebraic curve S over $\mathbb{C}$, and let η be a level-N structure on E/S. The map $\gamma\colon S(\mathbb{C}) \to Y_N(\mathbb{C})$ sending $s \in S(\mathbb{C})$ to the point of $\Gamma(N)\backslash D$ corresponding to (E_s, η_s) is regular, i.e., defined by a morphism of algebraic curves.*

PROOF. We first show that γ is holomorphic. For this, we regard $\mathbb{P}(V_{\mathbb{C}})$ as the Grassmann manifold classifying the one-dimensional quotients of $V_{\mathbb{C}}$. Thus, for any surjective homomorphism $\mathcal{O}_M \otimes_{\mathbb{R}} V \xrightarrow{\alpha} \mathcal{W}$ of vector sheaves on a complex manifold M such that $\mathcal{W}_m$ has dimension 1 for all $m \in M$, the map sending $m \in M$ to the point of $\mathbb{P}(V_{\mathbb{C}})$ corresponding to the quotient $V_{\mathbb{C}} \xrightarrow{\alpha_s} \mathcal{W}_m$ is holomorphic.

Let $f\colon E \to S$ be a family of elliptic curves on a connected smooth algebraic variety S (not necessary of dimension one). The exponential map defines an exact sequence of sheaves on S^{an}

$$0 \longrightarrow R_1 f_* \mathbb{Z} \longrightarrow \mathcal{T}_0(E^{\mathrm{an}}/S^{\mathrm{an}}) \longrightarrow E^{\mathrm{an}} \longrightarrow 0$$

whose fibre at a point $s \in S^{\mathrm{an}}$ is the sequence (1) for E_s. From the first map in the sequence we get a surjective map

$$\mathcal{O}_{S^{\mathrm{an}}} \otimes_{\mathbb{Z}} R_1 f_* \mathbb{Z} \twoheadrightarrow \mathcal{T}_0(E^{\mathrm{an}}/S^{\mathrm{an}}). \tag{2}$$

Each point of S^{an} has an open neighbourhood U for which there exists an isomorphism

$$\mathbb{Z}_U^2 \to R_1 f_* \mathbb{Z}|_U$$

compatible with the level-N structure. On tensoring such an isomorphism with $\mathcal{O}_{U^{\mathrm{an}}}$,

$$\mathcal{O}_{U^{\mathrm{an}}} \otimes_{\mathbb{Z}} \mathbb{Z}_U^2 \to \mathcal{O}_{U^{\mathrm{an}}} \otimes R_1 f_* \mathbb{Z}|_U$$

and composing with (2), we get a surjective map

$$\mathcal{O}_{U^{\mathrm{an}}} \otimes_{\mathbb{R}} V \twoheadrightarrow \mathcal{T}_0(E^{\mathrm{an}}/S^{\mathrm{an}})|U,$$

which defines a holomorphic map $U \to \mathbb{P}(V_{\mathbb{C}})$. With the correct choice of (2), this will map into D (rather than $-D$), and its composite with the quotient map $D \to \Gamma(N)\backslash D$ will be the map γ. Therefore γ is holomorphic.

It remains to show that γ is algebraic. We now assume that S has dimension 1. After passing to a finite covering, we may suppose that N is even. Let $\overline{Y}_N$ (resp. $\overline{S}$) be the completion of Y_N (resp. S) to a smooth complete algebraic curve. We have a holomorphic map

$$S^{\mathrm{an}} \xrightarrow{\gamma} Y_N^{\mathrm{an}} \subset \overline{Y}_N^{\mathrm{an}}$$

which we wish to show to be regular. The curve Y_2 is isomorphic to the projective line minus three points and is a quotient of Y. The composite

$$S^{\mathrm{an}} \xrightarrow{\gamma} Y_N^{\mathrm{an}} \xrightarrow{\text{onto}} Y_2^{\mathrm{an}} \simeq \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$$

does not have an essential singularity at any of the (finitely many) points of $\overline{S}^{\mathrm{an}} \smallsetminus S^{\mathrm{an}}$ because this would violate the big Picard theorem. Therefore, it extends to a holomorphic map $\overline{S}^{\mathrm{an}} \to \mathbb{P}^1(\mathbb{C})$, which implies that γ extends to a holomorphic map $\overline{\gamma}\colon \overline{S}^{\mathrm{an}} \to \overline{Y}_N^{\mathrm{an}}$. As $\overline{Y}_N$ and $\overline{S}$ are complete algebraic curves, $\overline{\gamma}$ is regular. □

Let $\mathcal{F}$ be the functor sending a scheme S of finite type over $\mathbb{C}$ to the set of isomorphism classes of pairs consisting of a family elliptic curves $f\colon E \to S$ over S and a level-N structure on E. When $3|N$, Mumford (1965, Chapter 7) proves that $\mathcal{F}$ is representable by a smooth algebraic curve S_N over $\mathbb{C}$. This means that there exists a (universal) family of elliptic curves E/S_N over S_N and a level-N structure η on E/S_N such that, for any similar pair $(E'/S,\eta')$ over a scheme S, there exists a unique morphism $\alpha\colon S \to S_N$ for which $\alpha^*(E/S_N,\eta) \approx (E'/S',\eta')$.

THEOREM 1.2. *There is a canonical isomorphism $\gamma\colon S_N \to Y_N$.*

PROOF. According to Proposition 1.1, the universal family of elliptic curves with level-N structure on S_N defines a morphism of algebraic curves $\gamma\colon S_N \to Y_N$. Both sets $S_N(\mathbb{C})$ and $Y_N(\mathbb{C})$ are in natural one-to-one correspondence with the set of isomorphism classes of complex elliptic curves with level-N structure, and γ sends the point in $S_N(\mathbb{C})$ corresponding to a pair (E,η) to the point in $Y_N(\mathbb{C})$ corresponding to the same pair. Therefore, $\gamma(\mathbb{C})$ is bijective, which implies that γ is an isomorphism. □

In particular, we have shown that the curve S_N, constructed by Mumford purely in terms of algebraic geometry, is isomorphic by the obvious map to the curve Y_N, constructed analytically. Of course, this is well known, but it is difficult to find a proof of it in the literature.[2]

2. Hermitian symmetric domains

The natural generalization of the complex upper half plane is a hermitian symmetric domain.

Preliminaries on Cartan involutions and polarizations

Let G be a connected algebraic group over $\mathbb{R}$, and let $g \mapsto \overline{g}$ denote complex conjugation on $G(\mathbb{C})$. An involution θ of G (as an algebraic group over $\mathbb{R}$) is said to be ***Cartan*** if the group

$$G^{(\theta)}(\mathbb{R}) = \{g \in G(\mathbb{C}) \mid g = \theta(\overline{g})\}$$

is compact.

THEOREM 2.1. *There exists a Cartan involution if and only if G is reductive, in which case any two are conjugate by an element of $G(\mathbb{R})$ (Satake 1980, I 4.3, 4.4).*

EXAMPLE 2.2. Consider the algebraic group GL_V attached to a real vector space V. The choice of a basis for V determines a transpose operator $g \mapsto g^t$, and $g \mapsto (g^t)^{-1}$ is obviously a Cartan involution. The theorem says that all Cartan involutions of GL_V arise in this way. An algebraic subgroup G of GL_V is reductive if and only if it is stable under $g \mapsto g^t$ for some basis of V, in which case the restriction of $g \mapsto (g^t)^{-1}$ to G is a Cartan involution. All Cartan involutions of G arise in this way for a suitable choice of basis for V.

[2]For example, Brian Conrad has pointed out that it is not proved in Katz and Mazur 1985 (mathoverflow.net, 21755).

Let C be an element of $G(\mathbb{R})$ whose square is central (so $\mathrm{inn}(C)$ is an involution). A ***C-polarization*** on a real representation V of G is a G-invariant bilinear form $\varphi: V \times V \to \mathbb{R}$ such that the form $\varphi_C: (x, y) \mapsto \varphi(x, Cy)$ is symmetric and positive definite.

THEOREM 2.3. *If $\mathrm{inn}(C)$ is a Cartan involution of G, then every finite dimensional real representation of G carries a C-polarization; conversely, if one* faithful *finite dimensional real representation of G carries a C-polarization, then $\mathrm{inn}(C)$ is a Cartan involution.*

PROOF. Let $G \to \mathrm{GL}_V$ be a faithful representation of G, and let φ be a C-polarization. Let ϕ_C be the sesquilinear form

$$(u, v) \mapsto \varphi_{\mathbb{C}}(u, C\bar{v}): V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$$

defined by φ_C. Because φ_C is symmetric and positive definite, ϕ_C is hermitian and positive definite; because φ is G-invariant, ϕ_C is $G^{(\mathrm{inn}\,C)}$-invariant. Therefore $G^{(\mathrm{inn}\,C)}(\mathbb{R})$ is compact and so $\mathrm{inn}(C)$ is a Cartan involution.

Conversely, if $G^{(\mathrm{inn}\,C)}(\mathbb{R})$ is compact, every real representation $G \to \mathrm{GL}_V$ of G carries a $G^{(\mathrm{inn}\,C)}(\mathbb{R})$-invariant positive definitive symmetric bilinear form φ.[3] For such a φ, the bilinear form $\varphi_{C^{-1}}$ is a C-polarization on V. □

2.4. VARIANT. Let G be an algebraic group over $\mathbb{Q}$, and let C be an element of $G(\mathbb{R})$ whose square is central. A ***C-polarization*** on a $\mathbb{Q}$-representation V of G is a G-invariant bilinear form $\varphi: V \times V \to \mathbb{Q}$ such that $\varphi_{\mathbb{R}}$ is a C-polarization on $V_{\mathbb{R}}$. In order to show that a $\mathbb{Q}$-representation V of G is polarizable, it suffices to check that $V_{\mathbb{R}}$ is polarizable. Consider, for example, the case that C^2 acts as $+1$ or -1 on V. Let $P(\mathbb{Q})$ (resp. $P(\mathbb{R})$) denote the space of G-invariant bilinear forms on V (resp. on $V_{\mathbb{R}}$) that are symmetric when C^2 acts as $+1$ or skew-symmetric when it acts as -1. Then $P(\mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Q}} P(\mathbb{Q})$. The C-polarizations of $V_{\mathbb{R}}$ form an open subset of $P(\mathbb{R})$, whose intersection with $P(\mathbb{Q})$ consists of the C-polarizations of V.

Definition

Let M be a complex manifold, and let $J_p: T_pM \to T_pM$ denote the action of $\sqrt{-1}$ on the tangent space at a point p of M. A ***hermitian metric*** on M is a riemannian metric g on the underlying smooth manifold of M such that J_p is an isometry for all p.[4] A ***hermitian manifold*** is a complex manifold equipped with a hermitian metric g, and a ***hermitian symmetric space*** is a connected hermitian manifold M that admits a symmetry at each point p, i.e., an involution s_p having p as an isolated fixed point. The group $\mathrm{Hol}(M)$ of holomorphic automorphisms of a hermitian symmetric space M is a real Lie group whose identity component $\mathrm{Hol}(M)^+$ acts transitively on M.

Every hermitian symmetric space M is a product of hermitian symmetric spaces of the following types:

- ◇ Noncompact type — the curvature is negative[5] and $\mathrm{Hol}(M)^+$ is a noncompact adjoint Lie group; example, the complex upper half plane.
- ◇ Compact type — the curvature is positive and $\mathrm{Hol}(M)^+$ is a compact adjoint Lie group; example, the Riemann sphere.

[3] Let V be a real representation of a compact group U; for any positive definite symmetric bilinear form $\varphi: V \times V \to \mathbb{R}$, the form $(x, y) \mapsto \int_{u \in U} \varphi(ux, uy)du$ is positive definite, symmetric, bilinear, *and* U-invariant.

[4] Then g_p is the real part of a unique hermitian form on T_pM, which explains the name.

[5] This means that the sectional curvature $K(p, E)$ is < 0 for every $p \in M$ and every two-dimensional subspace E of T_pM.

◇ Euclidean type — the curvature is zero; M is isomorphic to a quotient of a space $\mathbb{C}^n$ by a discrete group of translations.

In the first two cases, the space is simply connected. A hermitian symmetric space is ***indecomposable*** if it is not a product of two hermitian symmetric spaces of lower dimension. For an indecomposable hermitian symmetric space M of compact or noncompact type, the Lie group $\mathrm{Hol}(M)^+$ is simple.

A ***hermitian symmetric domain*** is a connected complex manifold that admits a hermitian metric for which it is a hermitian symmetric space of noncompact type.[6] The hermitian symmetric domains are exactly the complex manifolds isomorphic to bounded symmetric domains (via the Harish-Chandra embedding; Satake 1980, II §4). Thus a connected complex manifold M is a hermitian symmetric domain if and only if

(a) it is isomorphic to a bounded open subset of $\mathbb{C}^n$ for some n, and
(b) each point p of M admits a symmetry, i.e, a holomorphic involution having p as an isolated fixed point.

For example, the bounded domain $\{z \in \mathbb{C} \mid |z| < 1\}$ is symmetric because it is homogeneous and admits a symmetry at the origin (rotation through 180 degrees). The map $z \mapsto \frac{z-i}{z+i}$ is an isomorphism from the complex upper half plane D onto the open unit disk, and so D is a hermitian symmetric domain. Its automorphism group is

$$\mathrm{Hol}(D) \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm I\} \simeq \mathrm{PGL}_2(\mathbb{R})^+.$$

NOTES. The standard reference for hermitian symmetric spaces is Helgason 1978, Chapter VIII.

Classification

Let U^1 be the circle group $U^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. For each point o of a hermitian symmetric domain D, there is a unique homomorphism $u_o : U^1 \to \mathrm{Hol}(D)$ such that $u_o(z)$ fixes o and acts on $T_o D$ as multiplication by z ($z \in U^1$).[7] In particular, $u_o(-1)$ is the symmetry at o.

EXAMPLE 2.5. Let D be the complex upper half plane and let $o = i$. Let $h : U^1 \to \mathrm{SL}_2(\mathbb{R})$ be the homomorphism $a + bi \mapsto \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. Then $h(z)$ fixes o, and it acts as z^2 on $T_o(D)$. For $z \in U^1$, choose a square root $\sqrt{z}$ in U^1, and let $u_o(z) = h(\sqrt{z}) \bmod \pm I$. Then $u_o(z)$ is independent of the choice of $\sqrt{z}$ because $h(-1) = -I$. The homomorphism $u_o : U^1 \to \mathrm{SL}_2(\mathbb{Z})/\{\pm I\} = \mathrm{Hol}(D)$ has the correct properties.

Now let D be a hermitian symmetric domain. Because $\mathrm{Hol}(D)$ is an adjoint Lie group, there is a unique real algebraic group H such that $H(\mathbb{R})^+ = \mathrm{Hol}(D)^+$. Similarly, U^1 is the group of $\mathbb{R}$-points of the algebraic torus $\mathbb{S}^1$ defined by the equation $X^2 + Y^2 = 1$. A point $o \in D$ defines a homomorphism $u : \mathbb{S}^1 \to H$ of real algebraic groups.

THEOREM 2.6. *The homomorphism $u : \mathbb{S}^1 \to H$ has the following properties:*

SU1: *only the characters $z, 1, z^{-1}$ occur in the representation of $\mathbb{S}^1$ on $\mathrm{Lie}(H)_{\mathbb{C}}$ defined by u;*[8]

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[6] Usually a hermitian symmetric domain is defined to be a complex manifold *equipped* with a hermitian metric etc.. However, a hermitian symmetric domain in our sense satisfies conditions (A.1) and (A.2) of Kobayashi 1959, and so has a canonical Bergman metric, invariant under all holomorphic automorphisms.

[7] See, for example, Milne 2005, Theorem 1.9.

[8] The maps $\mathbb{S}^1 \xrightarrow{u} H_{\mathbb{R}} \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathrm{Lie}(H))$ define an action of $\mathbb{S}^1$ on $\mathrm{Lie}(H)$, and hence on $\mathrm{Lie}(H)_{\mathbb{C}}$. The condition means that $\mathrm{Lie}(H)_{\mathbb{C}}$ is a direct sum of subspaces on which $u(z)$ acts as z, 1, or z^{-1}.

SU2: $\mathrm{inn}(u(-1))$ *is a Cartan involution.*

Conversely, if H is a real adjoint algebraic group with no compact factor and $u\colon \mathbb{S}^1 \to H$ satisfies the conditions (SU1,2), then the set D of conjugates of u by elements of $H(\mathbb{R})^+$ has a natural structure of a hermitian symmetric domain for which $u(z)$ acts on $T_u D$ as multiplication by z; moreover, $H(\mathbb{R})^+ = \mathrm{Hol}(D)^+$.

PROOF. See Satake 1980, II, Proposition 3.2; cf. also Milne 2005, 1.21. □

Now assume that D is indecomposable. Then H is simple, and $H_{\mathbb{C}}$ is also simple because H is an inner form of its compact form (by SU2).[9] Thus, from D and a point o, we get a simple adjoint algebraic group $H_{\mathbb{C}}$ over $\mathbb{C}$ and a nontrivial cocharacter $\mu \overset{\text{def}}{=} u_{\mathbb{C}}\colon \mathbb{G}_m \to H_{\mathbb{C}}$ satisfying the condition:

(*) $\mathbb{G}_m$ acts on $\mathrm{Lie}(H_{\mathbb{C}})$ through the characters z, 1, z^{-1}.

Moreover, we can recover (H,u) from $(H_{\mathbb{C}},\mu)$: the real group H is the twist of the (unique) compact real form of $H_{\mathbb{C}}$ defined by the involution $\mathrm{inn}(\mu(-1))$, and u is the restriction of μ to $U^1 \subset \mathbb{C}^\times$. Changing o replaces μ by a conjugate.

In sum, the indecomposable hermitian symmetric domains are classified by the pairs (G,M) consisting of a simple adjoint algebraic group over $\mathbb{C}$ and a conjugacy class of nontrivial cocharacters of G satisfying (*) (Deligne 1979b, 1.2.2). It remains to classify the pairs (G,M).

Fix a maximal torus T of G and a base S for the root system $R = R(G,T)$, and let R^+ be the corresponding set of positive roots. As each μ in M factors through some maximal torus, and all maximal tori are conjugate, we may choose $\mu \in M$ to factor through T. Among the μ in M factoring through T, there is exactly one such that $\langle \alpha,\mu\rangle \geq 0$ for all $\alpha \in R^+$ (because the Weyl group acts simply transitively on the Weyl chambers). The condition (*) says that $\langle \alpha,\mu\rangle \in \{1,0,-1\}$ for all roots α. Since μ is nontrivial, not all of the $\langle \alpha,\mu\rangle$ can be zero, and so $\langle \tilde{\alpha},\mu\rangle = 1$ where $\tilde{\alpha}$ is the highest root. Recall that the highest root $\tilde{\alpha} = \sum_{\alpha\in S} n_\alpha \alpha$ has the property that $n_\alpha \geq m_\alpha$ for any other root $\sum_{\alpha\in S} m_\alpha \alpha$; in particular, $n_\alpha \geq 1$. It follows that $\langle \alpha,\mu\rangle = 0$ for all but one simple root α, and that for that simple root $\langle \alpha,\mu\rangle = 1$ and $n_\alpha = 1$. Thus, the pairs (G,M) are classified by the simple roots α for which $n_\alpha = 1$ — these are called the ***special*** simple roots. On examining the tables, one finds that the special simple roots are as in the following table:

type	$\tilde{\alpha}$	special roots	#
A_n	$\alpha_1+\alpha_2+\cdots+\alpha_n$	$\alpha_1,\ldots,\alpha_n$	n
B_n	$\alpha_1+2\alpha_2+\cdots+2\alpha_n$	α_1	1
C_n	$2\alpha_1+\cdots+2\alpha_{n-1}+\alpha_n$	α_n	1
D_n	$\alpha_1+2\alpha_2+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$	$\alpha_1,\alpha_{n-1},\alpha_n$	3
E_6	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	α_1,α_6	2
E_7	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	α_7	1
E_8,F_4,G_2		none	0

[9] If $H_{\mathbb{C}}$ is not simple, say, $H_{\mathbb{C}} = H_1 \times H_2$, then $H = (H_1)_{\mathbb{C}/\mathbb{R}}$, and every inner form of H is isomorphic to H itself, which is not compact because $H(\mathbb{R}) = H_1(\mathbb{C})$.

Thus there are indecomposable hermitian symmetric domains of all possible types except E_8, F_4, and G_2. Mnemonic: the number of special simple roots is one less than the connection index $(P(R){:}\,Q(R))$ of the root system.[10]

Example: the Siegel upper half space

A ***symplectic space*** (V,ψ) over a field k is a finite dimensional vector space V over k together with a nondegenerate alternating form ψ on V. The ***symplectic group*** $S(\psi)$ is the algebraic subgroup of GL_V of elements fixing ψ. It is an almost simple simply connected group of type C_{n-1} where $n = \frac{1}{2}\dim_k V$.

Now let $k=\mathbb{R}$, and let $H=S(\psi)$. Let D be the space of complex structures J on V such that $(x,y)\mapsto \psi_J(x,y) \overset{\text{def}}{=} \psi(x,Jy)$ is symmetric and positive definite. The symmetry is equivalent to J lying in $S(\psi)$. Therefore, D is the set of complex structures J on V for which $J\in H(\mathbb{R})$ and ψ is a J-polarization for H.

The action,

$$g, J \mapsto gJg^{-1}\colon H(\mathbb{R})\times D\to D,$$

of $H(\mathbb{R})$ on D is transitive.[11] Each $J\in D$ defines an action of $\mathbb{C}$ on V, and

$$\psi(Jx,Jy)=\psi(x,y) \text{ all } x,y\in V \implies \psi(zx,zy)=|z|^2\psi(x,y) \text{ all } x,y\in V.$$

Let $h_J\colon U^1\to H(\mathbb{R})$ be the homomorphism such that $h_J(z)$ acts on V as multiplication by z. As

$$\mathrm{Lie}(H)\subset \mathrm{End}(V)\simeq V^\vee\otimes V,$$

$h_J(z)$ acts on $\mathrm{Lie}(H)_{\mathbb{C}}$ through the characters $z/\overline{z}=z^2$, 1, and $\overline{z}/z=z^{-2}$.

For $z\in U^1$, choose a square root $\sqrt{z}$ in U^1, and let $u_J(z)=h_J(\sqrt{z}) \bmod \pm 1$. Then u_J is a well-defined homomorphism $U^1\to H^{\mathrm{ad}}(\mathbb{R})$, and it satisfies the conditions (SU1,2) of Theorem 2.6. Therefore, there is a unique complex structure on D such that $z\in U^1$ acts on $T_J(D)$ as multiplication by z, and, relative to this structure, D is the (unique) indecomposable hermitian symmetric domain of type C_{n-1}. It is called the ***Siegel upper half space*** (of degree, or genus, n).

3. Discrete subgroups of Lie groups

The algebraic varieties we are concerned with are quotients of hermitian symmetric domains by the action of discrete groups. In this section, we describe the discrete groups of interest to us.

[10] It is possible to prove this directly. Let $S^+ = S\cup\{\alpha_0\}$ where α_0 is the negative of the highest root — the elements of S^+ correspond to the nodes of the completed Dynkin diagram (Bourbaki Lie, VI 4, 3). The group P/Q acts on S^+, and it acts simply transitively on the set $\{\text{simple roots}\}\cup\{\alpha_0\}$ (Deligne 1979b, 1.2.5).

[11] Recall that a basis $(e_{\pm i})_{1\le i\le n}$ for V is symplectic if $\psi(e_i,e_{-i})=1$ $(1\le i\le n)$, $\psi(e_{-i},e_i)=-1$ $(1\le i\le n)$, and $\psi(e_i,e_j)=0$ $(i+j\neq 0)$. The group $H(\mathbb{R})$ acts simply transitively on the set of symplectic bases for V: for any bases $(e_{\pm i})$ and $(f_{\pm i})$ for V, there is a unique $g\in\mathrm{GL}(V)$ such that $ge_{\pm i}=f_{\pm i}$; clearly, $g\in H(k)$ if the bases are symplectic. A symplectic basis $e=(e_{\pm i})$ of V defines a complex structure J_e on V by $J_e e_{\pm i}=\pm e_{\mp i}$, i.e.,

$$e_i \xrightarrow{J_e} e_{-i} \xrightarrow{J_e} -e_i,\quad 1\le i\le n.$$

Then $J_e\in D$ — in fact, e is an orthonormal basis for ψ_{J_e}. Conversely, if $J\in D$, then $J=J_e$ for any orthonormal basis e for ψ_J. As the map $e\mapsto J_e$ is equivariant, this shows that $H(\mathbb{R})$ acts transitively on D.

Lattices in Lie groups

Let H be a real Lie group. A ***lattice*** in H is a discrete subgroup Γ of finite covolume, i.e., such that $\Gamma C = H$ for some Borel subset C of H with finite measure relative to a Haar measure on H. For example, the lattices in $\mathbb{R}^n$ are exactly the $\mathbb{Z}$-submodules generated by bases for $\mathbb{R}^n$, and two such lattices are commensurable[12] if and only if they generate the same $\mathbb{Q}$-vector space. Every discrete subgroup commensurable with a lattice is itself a lattice.

Now assume that H is semisimple with finite centre. A lattice Γ is ***irreducible*** if ΓN is dense in H for every noncompact closed normal subgroup N of H.

THEOREM 3.1. *Let H be an adjoint Lie group with no compact factors, and let Γ be a lattice H. Then H can be written (uniquely) as a direct product $H = H_1 \times \cdots \times H_r$ of Lie subgroups H_i such that $\Gamma_i \overset{\text{def}}{=} \Gamma \cap H_i$ is an irreducible lattice in H_i and $\Gamma_1 \times \cdots \times \Gamma_r$ has finite index in Γ*

PROOF. See Morris 2008, 4.24. □

THEOREM 3.2. *Let D be a hermitian symmetric domain, and let $H = \mathrm{Hol}(D)^+$. A discrete subgroup Γ of H is a lattice if and only if $\Gamma \backslash D$ has finite volume. Let Γ be a lattice in H; then there is a (unique) decomposition $D = D_1 \times \cdots \times D_r$ of D into a product of hermitian symmetric domains such that $\Gamma_i \overset{\text{def}}{=} \Gamma \cap \mathrm{Hol}(D_i)^+$ is an irreducible lattice in $\mathrm{Hol}(D_i)^+$ and $\Gamma_1 \backslash D_1 \times \cdots \times \Gamma_r \backslash D_r$ is a finite covering of $\Gamma \backslash D$.*

PROOF. The first statement of (3.2) follows from the fact that D is a quotient of H by a *compact* subgroup. The second statement follows from (3.1) and §2. □

Arithmetic subgroups of algebraic groups

Arithmetic is the main source for lattices in Lie groups.

Let G be an algebraic group over $\mathbb{Q}$. When $r: G \to \mathrm{GL}_n$ is an injective homomorphism, we let

$$G(\mathbb{Z})_r = \{g \in G(\mathbb{Q}) \mid r(g) \in \mathrm{GL}_n(\mathbb{Z})\}.$$

Then $G(\mathbb{Z})_r$ is independent of r up to commensurability (Borel 1969, 7.13), and we sometimes omit r from the notation. A subgroup Γ of $G(\mathbb{Q})$ is ***arithmetic*** if it is commensurable with $G(\mathbb{Z})_r$ for some r.

THEOREM 3.3. *Let $\rho: G \to G'$ be a surjective homomorphism of algebraic groups over $\mathbb{Q}$. If $\Gamma \subset G(\mathbb{Q})$ is arithmetic, then so also is $\rho(\Gamma) \subset G'(\mathbb{Q})$.*

PROOF. See Borel 1969, 8.9, 8.11. □

An arithmetic subgroup Γ of $G(\mathbb{Q})$ is obviously discrete in $G(\mathbb{R})$, but it need not have finite covolume. For example, $\mathbb{G}_m(\mathbb{Z}) = \{\pm 1\}$ is an arithmetic subgroup of $\mathbb{G}_m(\mathbb{Q})$ of infinite covolume in $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$. More generally, if G has $\mathbb{G}_m$ as a quotient, then it may have arithmetic subgroups of infinite covolume.

THEOREM 3.4. *Let G be a reductive algebraic group over $\mathbb{Q}$, and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$.*

[12] Recall that two subgroup S_1 and S_2 of a group are ***commensurable*** if $S_1 \cap S_2$ has finite index in both S_1 and S_2. Commensurability is an equivalence relation.

(a) *The quotient* $\Gamma\backslash G(\mathbb{R})$ *has finite volume if and only if* $\mathrm{Hom}(G,\mathbb{G}_m)=0$; *in particular,* Γ *is a lattice if* G *is semisimple.*[13]

(b) *(Godement compactness criterion) The quotient* $\Gamma\backslash G(\mathbb{R})$ *is compact if and only if* $\mathrm{Hom}(G,\mathbb{G}_m)=0$ *and* $G(\mathbb{Q})$ *contains no unipotent element other than* 1.[14]

PROOF. See Borel 1969, 13.2, 8.4. □

Let k be a subfield of $\mathbb{C}$. An automorphism α of a k-vector space V is said to be ***neat*** if its eigenvalues in $\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}^{\times}$. Let G be an algebraic group over $\mathbb{Q}$. An element $g\in G(\mathbb{Q})$ is ***neat*** if $\rho(g)$ is neat for one faithful representation $G\hookrightarrow\mathrm{GL}(V)$, in which case $\rho(g)$ is neat for every representation ρ of G defined over a subfield of $\mathbb{C}$. A subgroup of $G(\mathbb{Q})$ is ***neat*** if all its elements are.

THEOREM 3.5. *Let* G *be an algebraic group over* $\mathbb{Q}$, *and let* Γ *be an arithmetic subgroup of* $G(\mathbb{Q})$. *Then,* Γ *contains a neat subgroup of finite index. In particular,* Γ *contains a torsion free subgroup of finite index.*

PROOF. In fact, the neat subgroup can be defined by congruence conditions. See Borel 1969, 17.4. □

DEFINITION 3.6. A semisimple group G over $\mathbb{Q}$ is said to be of ***compact type*** if $G(\mathbb{R})$ is compact, and it is said to be of ***noncompact type*** if it does not contain a nontrivial connected normal algebraic subgroup of compact type.

Thus a simply connected[15] or adjoint group over $\mathbb{Q}$ is of compact type if all of its almost simple factors are of compact type, and it is of noncompact type if *none* of its almost simple factors is of compact type. In particular, a group may fail to be of compact type without being of noncompact type.

THEOREM 3.7 (BOREL DENSITY THEOREM). *Let* G *be a semisimple algebraic group over* $\mathbb{Q}$. *If* G *is of noncompact type, then every arithmetic subgroup of* $G(\mathbb{Q})$ *is dense in the Zariski topology.*

PROOF. See Borel 1969, 15.12. □

THEOREM 3.8 (MARGULIS SUPERRIGIDITY THEOREM). *Let* G *and* H *be algebraic groups over* $\mathbb{Q}$ *with* G *simply connected and almost simple. Let* Γ *be an arithmetic subgroup of* $G(\mathbb{Q})$, *and let* $\delta\colon\Gamma\to H(\mathbb{Q})$ *a homomorphism. If* $\mathrm{rank}\,G_{\mathbb{R}}\geq 2$, *then the Zariski closure of* $\delta(\Gamma)$ *in* H *is a semisimple algebraic group (not necessarily connected), and there is a unique homomorphism* $\varphi\colon G\to H$ *of algebraic groups such that* $\varphi(\gamma)=\delta(\gamma)$ *for all* γ *in a subgroup of finite index in* Γ.

PROOF. The conditions on G imply that $G=(G')_{F/\mathbb{Q}}$ for some simply connected geometrically almost simple algebraic group G' over a number field F. Thus, the statement is a special case of Margulis 1991, Theorem B, p. 258. □

[13] This was proved in particular cases by Siegel and others, and in general by Borel and Harish-Chandra (1962).

[14] This was conjectured by Godement, and proved independently by Mostow and Tamagawa (1962) and by Borel and Harish-Chandra (1962).

[15] When G is simply connected, a theorem of Cartan says that $G(\mathbb{R})$ is connected.

Arithmetic subgroups of Lie groups

Let H be a semisimple real Lie group with finite centre. A lattice Γ in H is ***arithmetic*** if there exists a simply connected algebraic group G over $\mathbb{Q}$ and a surjective homomorphism $\varphi\colon G(\mathbb{R}) \to H$ with compact kernel such that Γ is commensurable with $\varphi(G(\mathbb{Z}))$. The arithmetic group Γ is irreducible if and only if the algebraic group G is almost simple.[16]

EXAMPLE 3.9. For example, let $H = \mathrm{SL}_2(\mathbb{R})$. Let B be a quaternion algebra over a totally real number field F such that $H \otimes_{F,v} \mathbb{R} \approx M_2(\mathbb{R})$ for exactly one real prime v. Let G be the algebraic group over $\mathbb{Q}$ such that $G(\mathbb{Q}) = \{b \in B \mid \mathrm{Norm}_{B/\mathbb{Q}}(b) = 1\}$. Then $H \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R}) \times \mathbb{H} \times \mathbb{H} \times \cdots$ where $\mathbb{H}$ is usual quaternion algebra, and so there exists a surjective homomorphism $\varphi\colon G(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ with compact kernel. The image under φ of any arithmetic subgroup of $G(\mathbb{Q})$ is an arithmetic subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, and every arithmetic subgroup of $\mathrm{SL}_2(\mathbb{R})$ is commensurable with one of this form. If $F = \mathbb{Q}$ and $B = M_2(\mathbb{Q})$, then $G = \mathrm{SL}_{2\mathbb{Q}}$ and $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ is noncompact (see §1), and otherwise B is a division algebra, and $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ is compact by Godement's criterion (3.4b).

THEOREM 3.10. *If H admits a faithful finite dimensional representation, then every lattice in H contains a torsion free subgroup of finite index.*

PROOF. This is a variant of Theorem 3.5 (the condition on H is necessary). See Morris 2008, §4I. □

There are many nonarithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$. However, except in a few groups of low rank like $\mathrm{SL}_2(\mathbb{R})$, no one was able to find a lattice that was not arithmetic. Eventually Selberg conjectured that there are none, and this was proved by Margulis.

THEOREM 3.11 (MARGULIS ARITHMETICITY THEOREM). *Every irreducible lattice in a semisimple Lie group is arithmetic unless the group is isogenous to* $\mathrm{SO}(1,n) \times$ *(compact) or* $\mathrm{SU}(1,n) \times$ *(compact).*

For a discussion of the theorem, see Morris 2008, §5B.

THEOREM 3.12. *Let H be the identity component of the group of automorphisms of a hermitian symmetric domain (see 2.6), and let Γ be a lattice in H. If* $\mathrm{rank}\, H_i \geq 2$ *for each factor H_i in (3.1), then there exists a simply connected algebraic group G of noncompact type over $\mathbb{Q}$ and a surjective homomorphism $\varphi\colon G(\mathbb{R}) \to H$ with compact kernel such that Γ is commensurable with $\varphi(G(\mathbb{Z}))$. Moreover, the pair (G,φ) is unique up to a unique isomorphism.*

PROOF. Each factor H_i is again the identity component of the group of automorphisms of a hermitian symmetric domain, and so we may suppose that Γ is irreducible. The existence of the pair (G,φ) just means that Γ is arithmetic, which follows from the Margulis arithmeticity theorem (3.10).

The group G is a product of its almost simple factors (because it is simply connected), and because Γ is irreducible, it is almost simple. Therefore $G = (G^s)_{F/\mathbb{Q}}$ where F is a number field and G^s is a geometrically almost simple algebraic group over F. Recall that $G_{\mathbb{R}}$ is an inner form of its compact form (by SU2). If F had a complex prime, $G_{\mathbb{R}}$ would have a factor $(G')_{\mathbb{C}/\mathbb{R}}$, but every inner form of $(G')_{\mathbb{C}/\mathbb{R}}$ is isomorphic to $(G')_{\mathbb{C}/\mathbb{R}}$, which is not compact. Therefore F is totally real.

[16] Let $G = G_1 \times G_2$; if Γ is an arithmetic subgroup of $G(\mathbb{Q})$, then $\Gamma_1 \overset{\text{def}}{=} \Gamma \cap G_1(\mathbb{Q})$ and $\Gamma_2 \overset{\text{def}}{=} \Gamma \cap G_2(\mathbb{Q})$ are arithmetic, and Γ is commensurable with $\Gamma_1 \times \Gamma_2$ (and $\Gamma \cdot G_2(\mathbb{R}) = \Gamma_2 \cdot G_2(\mathbb{R})$ is not dense in $G(\mathbb{R})$). This proves the implication "irreducible implies almost simple", and the converse follows from (3.1).

Let (G_1, φ_1) be a second pair. Because the kernel of φ_1 is compact, its intersection with $G_1(\mathbb{Z})$ is finite, and so there exists an arithmetic subgroup Γ_1 of $G_1(\mathbb{Q})$ such $\varphi_1|\Gamma_1$ is injective. Because $\varphi(G(\mathbb{Z}))$ and $\varphi_1(\Gamma_1)$ are commensurable, there exists an arithmetic subgroup Γ' of $G(\mathbb{Q})$ such that $\varphi(\Gamma') \subset \varphi_1(\Gamma_1)$. Now the Margulis superrigidity theorem 3.8 shows that there exists a homomorphism $\alpha\colon G \to G_1$ such that

$$\varphi_1(\alpha(\gamma)) = \varphi(\gamma) \tag{3}$$

for all γ in a subgroup Γ'' of Γ' of finite index. The subgroup Γ'' of $G(\mathbb{Q})$ is Zariski-dense in G (Borel density theorem 3.7), and so (3) implies that

$$\varphi_1 \circ \alpha = \varphi. \tag{4}$$

Because G and G_1 are almost simple, (4) implies that α is an isogeny, and because G_1 is simply connected, this implies that α is an isomorphism. It is unique because it is uniquely determined on an arithmetic subgroup of G. □

Congruence subgroups of algebraic groups

As in the case of elliptic modular curves, we shall need to consider a special class of arithmetic subgroups, namely, the congruence subgroups.

Let G be an algebraic group over $\mathbb{Q}$. Choose an embedding of G into GL_n, and define

$$\Gamma(N) = G(\mathbb{Q}) \cap \{A \in \mathrm{GL}_n(\mathbb{Z}) \mid A \equiv 1 \bmod N\}.$$

A ***congruence subgroup***[17] of $G(\mathbb{Q})$ is any subgroup containing $\Gamma(N)$ as a subgroup of finite index. Although $\Gamma(N)$ depends on the choice of the embedding, this definition does not — in fact, the congruence subgroups are exactly those of the form $K \cap G(\mathbb{Q})$ for K a compact open subgroup of $G(\mathbb{A}_f)$.

For a surjective homomorphism $G \to G'$ of algebraic groups over $\mathbb{Q}$, the homomorphism $G(\mathbb{Q}) \to G'(\mathbb{Q})$ need not send congruence subgroups to congruence subgroups. For example, the image in $\mathrm{PGL}_2(\mathbb{Q})$ of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Q})$ is an arithmetic subgroup (see 3.3) but not necessarily a congruence subgroup.

Every congruence subgroup is an arithmetic subgroup, and for a simply connected group the converse is often, but not always, true. For a survey of what is known about the relation of congruence subgroups to arithmetic groups (the congruence subgroup problem), see Prasad and Rapinchuk 2008.

ASIDE 3.13. Let H be a connected adjoint real Lie group without compact factors. The pairs (G, φ) consisting of a simply connected algebraic group over $\mathbb{Q}$ and a surjective homomorphism $\varphi\colon G(\mathbb{R}) \to H$ with compact kernel have been classified (this requires class field theory). Therefore the arithmetic subgroups of H have been classified up to commensurability. When all arithmetic subgroups are congruence, there is even a classification of the groups themselves in terms of congruence conditions or, equivalently, in terms of compact open subgroups of $G(\mathbb{A}_f)$.

4. Locally symmetric varieties

A hermitian symmetric domain is never an algebraic variety. To obtain an algebraic variety, we must pass to the quotient by an arithmetic group.

[17] Subgroup defined by congruence conditions.

Quotients of hermitian symmetric domains

Let D be a hermitian symmetric domain, and let Γ be a discrete subgroup of $\mathrm{Hol}(D)^+$. If Γ is torsion free, then Γ acts freely on D, and there is a unique complex structure on $\Gamma\backslash D$ for which the quotient map $\pi\colon D \to \Gamma\backslash D$ is a local isomorphism. Relative to this structure, a map φ from $\Gamma\backslash D$ to a second complex manifold is holomorphic if and only if $\varphi\circ\pi$ is holomorphic.

When Γ is torsion free, we often write $D(\Gamma)$ for $\Gamma\backslash D$ regarded as a complex manifold. In this case, D is the universal covering space of $D(\Gamma)$ and Γ is the group of covering transformations. The choice of a point $p \in D$ determines an isomorphism of Γ with the fundamental group $\pi_1(D(\Gamma), \pi p)$.[18]

The complex manifold $D(\Gamma)$ is locally symmetric in the sense that, for each $p \in D(\Gamma)$, there is an involution s_p defined on a neighbourhood of p having p as an isolated fixed point.

The algebraic structure on the quotient

Recall that X^{an} denotes the complex manifold attached to a smooth complex algebraic variety X. The functor $X \rightsquigarrow X^{\mathrm{an}}$ is faithful, but it is far from being surjective on arrows or on objects. For example, $\left(\mathbb{A}^1\right)^{\mathrm{an}} = \mathbb{C}$ and the exponential function is a nonpolynomial holomorphic map $\mathbb{C} \to \mathbb{C}$. A Riemann surface arises from an algebraic curve if and only if it can be compactified by adding a finite number of points. In particular, if a Riemann surface is an algebraic curve, then every bounded function on it is constant and so the complex upper half plane is not an algebraic curve.[19]

CHOW'S THEOREM

An algebraic variety (resp. complex manifold) is ***projective*** if it can be realized as a closed subvariety of $\mathbb{P}^n$ for some n (resp. closed submanifold of $(\mathbb{P}^n)^{\mathrm{an}}$).

THEOREM 4.1 (CHOW 1949). *The functor $X \rightsquigarrow X^{\mathrm{an}}$ from smooth projective complex algebraic varieties to projective complex manifolds is an equivalence of categories.*

In other words, a projective complex manifold has a unique structure of a smooth projective algebraic variety, and every holomorphic map of projective complex manifolds is regular for these structures. See Taylor 2002, 13.6, for the proof.

Chow's theorem remains true when singularities are allowed.

THE BAILY-BOREL THEOREM

THEOREM 4.2 (BAILY AND BOREL 1966). *Every quotient $D(\Gamma)$ of a hermitian symmetric domain D by a torsion-free arithmetic subgroup Γ of $\mathrm{Hol}(D)^+$ has a canonical structure of an algebraic variety.*

Let G be the algebraic group attached to (D, Γ) by Theorem 3.12. Assume, for simplicity, that G has no normal algebraic subgroup of dimension 3, and let A_n be the vector space of automorphic forms on D for the nth power of the canonical automorphy factor. Then $A = \bigoplus_{n\geq 0} A_n$ is a finitely generated graded $\mathbb{C}$-algebra, and the canonical map

$$D(\Gamma) \to D(\Gamma)^* \stackrel{\mathrm{def}}{=} \mathrm{Proj}(A)$$

[18]Let $\gamma \in \Gamma$, and choose a path from p to γp; the image of this in $\Gamma\backslash D$ is a loop whose homotopy class does not depend on the choice of the path.

[19]For example, $\frac{z-i}{z+i}$ is bounded on the complex upper half plane.

realizes $D(\Gamma)$ as a Zariski-open subvariety of the projective algebraic variety $D(\Gamma)^*$ (Baily and Borel 1966, §10).

BOREL'S THEOREM

THEOREM 4.3 (BOREL 1972). *Let $D(\Gamma)$ be the quotient $\Gamma\backslash D$ in (4.2) endowed with its canonical algebraic structure, and let V be a smooth complex algebraic variety. Every holomorphic map $f: V^{an} \to D(\Gamma)^{an}$ is regular.*

In the proof of Proposition 1.1, we saw that for curves this theorem follows from the big Picard theorem. Recall that this theorem says every holomorphic map from a punctured disk to $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{\text{three points}\}$ extends to a holomorphic map from the whole disk to $\mathbb{P}^1(\mathbb{C})$. Following earlier work of Kwack and others, Borel generalized the big Picard theorem in two respects: the punctured disk is replaced by a product of punctured disks and disks, and the target space is allowed to be any quotient of a hermitian symmetric domain by a torsion-free arithmetic group.

Resolution of singularities (Hironaka 1964) shows that every open affine subvariety U of an algebraic variety V can be embedded in a smooth projective variety $\bar{U}$ as the complement of a divisor with normal crossings. This means that $\bar{U}^{\text{an}} \smallsetminus U^{\text{an}}$ is locally a product of disks and punctured disks. Therefore $f|U^{\text{an}}$ extends to a holomorphic map $\bar{U}^{\text{an}} \to D(\Gamma)^*$ (by Borel) and so is a regular map (by Chow).

Locally symmetric varieties

A ***locally symmetric variety*** is a smooth algebraic variety X over $\mathbb{C}$ such that X^{an} is isomorphic to $\Gamma\backslash D$ for some hermitian symmetric domain D and torsion-free subgroup Γ of $\text{Hol}(D)$.[20] In other words, X is a locally symmetric variety if the universal covering space D of X^{an} is a hermitian symmetric domain and the group of covering transformations of D over X^{an} is a torsion-free subgroup Γ of $\text{Hol}(D)$. When Γ is an arithmetic subgroup of $\text{Hol}(D)^+$, X is called an ***arithmetic locally symmetric variety***. The group Γ is automatically a lattice, and so the Margulis arithmeticity theorem (3.11) shows that nonarithmetic locally symmetric varieties can occur only when there are factors of low dimension.

A nonsingular projective curve over $\mathbb{C}$ has a model over $\mathbb{Q}^{\text{al}}$ if and only if it contains an arithmetic locally symmetric curve as the complement of a finite set (Belyi; see Serre 1990, p. 71). This suggests that there are too many arithmetic locally symmetric varieties for us to be able to say much about their arithmetic.

Let $D(\Gamma)$ be an arithmetic locally symmetric variety. Recall that Γ is arithmetic if there is a simply connected algebraic group G over $\mathbb{Q}$ and a surjective homomorphism $\varphi: G(\mathbb{R}) \to \text{Hol}(D)^+$ with compact kernel such that Γ is commensurable with $\varphi(G(\mathbb{Z}))$. If there exists a *congruence subgroup* Γ_0 of $G(\mathbb{Z})$ such that Γ contains $\varphi(\Gamma_0)$ as a subgroup of finite index, then we call $D(\Gamma)$ a ***connected Shimura variety***. Only for Shimura varieties do we have a rich arithmetic theory (see Deligne 1971b, Deligne 1979b, and the many articles of Shimura, especially, Shimura 1964, 1966, 1967a,b, 1970).

[20]As $\text{Hol}(D)$ has only finitely many components, $\Gamma \cap \text{Hol}(D)^+$ has finite index in Γ. Sometimes we only allow discrete subgroups of $\text{Hol}(D)$ contained in $\text{Hol}(D)^+$. In the theory of Shimura varieties, we generally consider only "sufficiently small" discrete subgroups, and we regard the remainder as "noise". Algebraic geometers do the opposite.

Example: Siegel modular varieties

For an abelian variety A over $\mathbb{C}$, the exponential map defines an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow T_0(A^{\mathrm{an}}) \xrightarrow{\exp} A^{\mathrm{an}} \longrightarrow 0$$

with $T_0(A^{\mathrm{an}})$ a complex vector space and Λ a lattice in $T_0(A^{\mathrm{an}})$ (isomorphic to $H_1(A^{\mathrm{an}}, \mathbb{Z})$).

THEOREM 4.4 (RIEMANN'S THEOREM). *The functor $A \rightsquigarrow (T_0(A), \Lambda)$ is an equivalence from the category of abelian varieties over $\mathbb{C}$ to the category of pairs consisting of a $\mathbb{C}$-vector space V and a lattice Λ in V that admits a Riemann form.*

PROOF. See, for example, Mumford 1970, Chapter I. □

A Riemann form for a pair (V, Λ) is an alternating form $\psi: \Lambda \times \Lambda \to \mathbb{Z}$ such that the pairing $(x, y) \mapsto \psi_{\mathbb{R}}(x, \sqrt{-1}y): V \times V \to \mathbb{R}$ is symmetric and positive definite. Here $\psi_{\mathbb{R}}$ denotes the linear extension of ψ to $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \simeq V$. A principal polarization on an abelian variety A is Riemann form for $(T_0(A), \Lambda)$ whose discriminant is ± 1. A level-N structure on an abelian variety is defined similarly to an elliptic curve.

Let D denote the Siegel upper half space of degree n, and let D_N be the quotient of D by the principal congruence subgroup $\Gamma(N)$ of level N in the corresponding symplectic group.

PROPOSITION 4.5. *Let $f: A \to S$ be a family of principally polarized abelian varieties on a smooth algebraic variety S over $\mathbb{C}$, and let η be a level-N structure on A/S. The map $\gamma: S(\mathbb{C}) \to D_N(\mathbb{C})$ sending $s \in S(\mathbb{C})$ to the point of $\Gamma(N)\backslash D$ corresponding to (A_s, η_s) is regular.*

PROOF. The holomorphicity of γ can be proved by the same argument as in the proof of Proposition 1.1. Its algebraicity then follows from Borel's theorem 4.3. □

Let $\mathcal{F}$ be the functor sending a scheme S of finite type over $\mathbb{C}$ to the set of isomorphism classes of pairs consisting of a family of principally polarized abelian varieties $f: A \to S$ over S and a level-N structure on A. When $3|N$, Mumford (1965, Chapter 7) proves that $\mathcal{F}$ is representable by a smooth algebraic variety S_N over $\mathbb{C}$. This means that there exists a (universal) family of principally polarized abelian varieties A/S_N and a level-N structure η on A/S_N such that, for any similar pair $(A'/S, \eta')$ over a scheme S, there exists a unique morphism $\alpha: S \to S_N$ for which $\alpha^*(A/S_N, \eta) \approx (A'/S', \eta')$.

THEOREM 4.6. *There is a canonical isomorphism $\gamma: S_N \to D_N$.*

PROOF. The proof is the same as that of Theorem 1.2. □

COROLLARY 4.7. *The universal family of complex tori on D_N is algebraic.*

5. Variations of Hodge structures

We review the definitions.

The Deligne torus

The ***Deligne torus*** is the algebraic torus $\mathbb{S}$ over $\mathbb{R}$ obtained from $\mathbb{G}_m$ over $\mathbb{C}$ by restriction of the base field; thus

$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times, \quad \mathbb{S}_\mathbb{C} \simeq \mathbb{G}_m \times \mathbb{G}_m.$$

The map $\mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C})$ induced by $\mathbb{R} \to \mathbb{C}$ is $z \mapsto (z, \bar{z})$. There are homomorphisms

$$\begin{array}{ccccccc} \mathbb{G}_m & \xrightarrow{w} & \mathbb{S} & \xrightarrow{t} & \mathbb{G}_m, & & t \circ w = -2, \\ \mathbb{R}^\times & \xrightarrow{a \mapsto a^{-1}} & \mathbb{C}^\times & \xrightarrow{z \mapsto z\bar{z}} & \mathbb{R}^\times. & & \end{array}$$

The kernel of t is $\mathbb{S}^1$. A homomorphism $h\colon \mathbb{S} \to G$ of real algebraic groups gives rise to cocharacters

$$\begin{array}{lll} \mu_h\colon \mathbb{G}_m \to G_\mathbb{C}, & z \mapsto h_\mathbb{C}(z,1), & z \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times, \\ w_h\colon \mathbb{G}_m \to G, & w_h = h \circ w & (\textit{\textbf{weight homomorphism}}). \end{array}$$

The following formulas are useful ($\mu = \mu_h$):

$$h_\mathbb{C}(z_1, z_2) = \mu(z_1) \cdot \bar{\mu}(z_2); \quad h(z) = \mu(z) \cdot \overline{\mu(z)} \tag{5}$$

$$h(i) = \mu(-1) \cdot w_h(i). \tag{6}$$

Real Hodge structures

A ***real Hodge structure*** is a representation $h\colon \mathbb{S} \to \mathrm{GL}_V$ of $\mathbb{S}$ on a real vector space V. Equivalently, it is a real vector space V together with a (Hodge) decomposition,

$$V_\mathbb{C} = \bigoplus\nolimits_{p,q \in \mathbb{Z}} V^{p,q} \text{ such that } \overline{V^{p,q}} = V^{q,p} \text{ for all } p, q.$$

To pass from one description to the other, use the rule (Deligne 1973, 1979b):

$$v \in V^{p,q} \iff h(z)v = z^{-p}\bar{z}^{-q}v, \text{ all } z \in \mathbb{C}^\times.$$

The integers $h^{p,q} \overset{\text{def}}{=} \dim_\mathbb{C} V^{p,q}$ are called the ***Hodge numbers*** of the Hodge structure. A real Hodge structure defines a (weight) gradation on V,

$$V = \bigoplus\nolimits_{m \in \mathbb{Z}} V_m, \quad V_m = V \cap \Big(\bigoplus\nolimits_{p+q=m} V^{p,q} \Big),$$

and a descending (Hodge) filtration,

$$V_\mathbb{C} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset 0, \quad F^p = \bigoplus\nolimits_{p' \geq p} V^{p',q'}.$$

The weight gradation and Hodge filtration together determine the Hodge structure because

$$V^{p,q} = \left(V_{p+q}\right)_\mathbb{C} \cap F^p \cap \overline{F^q}.$$

Note that the weight gradation is defined by w_h. A filtration F on $V_\mathbb{C}$ arises from a Hodge structure of weight m on V if and only if

$$V = F^p \oplus \overline{F^q} \text{ whenever } p + q = m + 1.$$

The $\mathbb{R}$-linear map $C = h(i)$ is called the ***Weil operator***. It acts as i^{q-p} on $V^{p,q}$, and C^2 acts as $(-1)^m$ on V_m.

5.1. Let V be a real vector space. To give a Hodge structure h on V of type $\{(-1,0), (0,-1)\}$ is the same as giving a complex structure on V: given h we get a complex structure from $V \simeq V_\mathbb{C}/F^0$; given a complex structure, we let $h(z)$ act on V as multiplication by z.

Rational Hodge structures

A ***rational Hodge structure*** is a $\mathbb{Q}$-vector space V together with a real Hodge structure on $V_{\mathbb{R}}$ such that the weight gradation is defined over $\mathbb{Q}$. Thus to give a rational Hodge structure on V is the same as giving

- ◇ a gradation $V = \bigoplus_m V_m$ on V together with a real Hodge structure of weight m on $V_{m\mathbb{R}}$ for each m, *or*
- ◇ a homomorphism $h: \mathbb{S} \to \mathrm{GL}_{V_{\mathbb{R}}}$ such that $w_h: \mathbb{G}_m \to \mathrm{GL}_{V_{\mathbb{R}}}$ is defined over $\mathbb{Q}$.

The ***Tate Hodge structure*** $\mathbb{Q}(m)$ is defined to be the $\mathbb{Q}$-subspace $(2\pi i)^m\mathbb{Q}$ of $\mathbb{C}$ with $h(z)$ acting as multiplication by $\mathrm{Norm}_{\mathbb{C}/\mathbb{R}}(z)^m = (z\bar{z})^m$. It has weight $-2m$ and type $(-m,-m)$.

Polarizations

A ***polarization*** of a real Hodge structure (V,h) of weight m is a morphism of Hodge structures

$$\psi: V \otimes V \to \mathbb{R}(-m), \quad m \in \mathbb{Z}, \tag{7}$$

such that

$$(x,y) \mapsto (2\pi i)^m \psi(x, Cy): V \times V \to \mathbb{R} \tag{8}$$

is symmetric and positive definite. The condition (8) means that ψ is symmetric if m is even and skew-symmetric if it is odd, and that $(2\pi i)^m \cdot i^{p-q} \psi_{\mathbb{C}}(x,\bar{x}) > 0$ for $x \in V^{p,q}$.

A ***polarization*** of a rational Hodge structure (V,h) of weight m is a morphism of rational Hodge structures $\psi: V \otimes V \to \mathbb{Q}(-m)$ such that $\psi_{\mathbb{R}}$ is a polarization of $(V_{\mathbb{R}},h)$. A rational Hodge structure (V,h) is polarizable if and only if $(V_{\mathbb{R}},h)$ is polarizable (cf. 2.4).

The rational Hodge structures form a tannakian category over $\mathbb{Q}$, and the polarizable rational Hodge structures form a *semisimple* tannakian category, which we denote $\mathsf{Hdg}_{\mathbb{Q}}$.

Local systems and vector sheaves with connection

Let S be a complex manifold. A ***connection*** on a vector sheaf $\mathcal{V}$ on S is a $\mathbb{C}$-linear homomorphism $\nabla: \mathcal{V} \to \Omega^1_S \otimes \mathcal{V}$ satisfying the Leibniz condition

$$\nabla(fv) = df \otimes v + f \cdot \nabla v$$

for all local sections f of $\mathcal{O}_S$ and v of $\mathcal{V}$. The ***curvature*** of ∇ is the composite of ∇ with the map

$$\begin{aligned} \nabla_1: \Omega^1_S \otimes \mathcal{V} &\to \Omega^2_S \otimes \mathcal{V} \\ \omega \otimes v &\mapsto d\omega \otimes v - \omega \wedge \nabla(v). \end{aligned}$$

A connection ∇ is said to be ***flat*** if its curvature is zero. In this case, the kernel $\mathcal{V}^{\nabla}$ of ∇ is a local system of complex vector spaces on S such that $\mathcal{O}_S \otimes \mathcal{V}^{\nabla} \simeq \mathcal{V}$.

Conversely, let V be a local system of complex vector spaces on S. The vector sheaf $\mathcal{V} = \mathcal{O}_S \otimes \mathsf{V}$ has a canonical connection ∇: on any open set where V is trivial, say $\mathsf{V} \approx \mathbb{C}^n$, the connection is the map $(f_i) \mapsto (df_i): (\mathcal{O}_S)^n \to \left(\Omega^1_S\right)^n$. This connection is flat because $d \circ d = 0$. Obviously for this connection, $\mathcal{V}^{\nabla} \simeq \mathsf{V}$.

In this way, we obtain an equivalence between the category of vector sheaves on S equipped with a flat connection and the category of local systems of complex vector spaces.

Variations of Hodge structures

Let S be a complex manifold. By a ***family of real Hodge structures on*** S we mean a holomorphic family. Thus a family of real Hodge structures on S is a local system V of $\mathbb{R}$-vector spaces on S together with a filtration F on $\mathcal{V} \overset{\text{def}}{=} \mathcal{O}_S \otimes_{\mathbb{R}} \mathsf{V}$ by (holomorphic) vector subsheaves that gives a Hodge filtration at each point. For example, for a family of weight m, the last condition means that

$$F^p\mathcal{V}_s \oplus \overline{F^{m+1-p}\mathcal{V}_s} \simeq \mathcal{V}_s, \quad \text{all } s \in S,\ p \in \mathbb{Z}.$$

For the notion of a ***family of rational Hodge structures***, replace $\mathbb{R}$ with $\mathbb{Q}$.

A ***polarization*** of a family of real Hodge structures of weight m is a bilinear pairing of local systems

$$\psi\colon \mathsf{V} \times \mathsf{V} \to \mathbb{R}(-m)$$

that gives a polarization at each point s of S. For rational Hodge structures, replace $\mathbb{R}$ with $\mathbb{Q}$.

Let ∇ be connection on a vector sheaf $\mathcal{V}$. A holomorphic vector field Z on S defines a map $\Omega_S^1 \to \mathcal{O}_S$ and hence a map $\nabla_Z\colon \mathcal{V} \to \mathcal{V}$. A family of rational Hodge structures V on S is a ***variation*** of rational Hodge structures on S if it satisfies the following axiom (***Griffiths transversality***):

$$\nabla_Z(F^p\mathcal{V}) \subset F^{p-1}\mathcal{V} \text{ for all } p \text{ and } Z.$$

Equivalently,

$$\nabla(F^p\mathcal{V}) \subset \Omega_S^1 \otimes F^{p-1}\mathcal{V} \text{ for all } p.$$

Here ∇ is the flat connection on $\mathcal{V} \overset{\text{def}}{=} \mathcal{O}_S \otimes_{\mathbb{Q}} \mathsf{V}$ defined by V.

These definitions are motivated by the following theorem.

THEOREM 5.2 (GRIFFITHS 1968). *Let $f\colon X \to S$ be a smooth projective map of smooth algebraic varieties over $\mathbb{C}$. For each m, the local system $R^m f_*\mathbb{Q}$ of $\mathbb{Q}$-vector spaces on S^{an} together with the de Rham filtration on $\mathcal{O}_S \otimes_{\mathbb{Q}} Rf_*\mathbb{Q} \simeq Rf_*(\Omega^\bullet_{X/\mathbb{C}})$ is a polarizable variation of rational Hodge structures of weight m on S^{an}.*

This theorem suggests that the first step in realizing an algebraic variety as a moduli variety should be to show that it carries a polarized variation of rational Hodge structures.

6. Mumford-Tate groups and their variation in families

We define Mumford-Tate groups, and we study their variation in families. Throughout this section, "Hodge structure" means "rational Hodge structure".

The conditions (SV)

We list some conditions on a homomorphism $h\colon \mathbb{S} \to G$ of real connected algebraic groups:

SV1: the Hodge structure on the Lie algebra of G defined by $\mathrm{Ad}\circ h\colon \mathbb{S} \to \mathrm{GL}_{\mathrm{Lie}(G)}$ is of type $\{(1,-1),(0,0),(-1,1)\}$;

SV2: $\mathrm{inn}(h(i))$ is a Cartan involution of G^{ad}.

In particular, (SV2) says that the Cartan involutions of G^{ad} are inner, and so G^{ad} is an inner form of its compact form. This implies that the simple factors of G^{ad} are geometrically simple (see footnote 9).

Condition (SV1) implies that the Hodge structure on $\mathrm{Lie}(G)$ defined by h has weight 0, and so $w_h(\mathbb{G}_m) \subset Z(G)$. In the presence of this condition, we sometimes need to consider a stronger form of (SV2):

SV2*: $\mathrm{inn}(h(i))$ is a Cartan involution of $G/w_h(\mathbb{G}_m)$.

Note that (SV2*) implies that G is reductive.

Let G be an algebraic group over $\mathbb{Q}$, and let h be a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$. We say that (G,h) satisfies the condition (SV1) or (SV2) if $(G_{\mathbb{R}},h)$ does. When w_h is defined over $\mathbb{Q}$, we say that (G,h) satisfies (SV2*) if $(G_{\mathbb{R}},h)$ does. We shall also need to consider the condition:

SV3: G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

In the presence of (SV1,2), the condition (SV3) is equivalent to G^{ad} being of noncompact type (apply Lemma 4.7 of Milne 2005).

Each condition holds for a homomorphism h if and only if it holds for a conjugate of h by an element of $G(\mathbb{R})$.

Let G be a reductive group over $\mathbb{Q}$. Let h be a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$, and let $\bar{h}\colon \mathbb{S} \to G_{\mathbb{R}}^{\mathrm{ad}}$ be $\mathrm{ad} \circ h$. Then (G,h) satisfies (SV1,2,3) if and only if $(G^{\mathrm{ad}},\bar{h})$ satisfies the same conditions.[21]

NOTES. Conditions (SV1), (SV2), and (SV3) are respectively the conditions (2.1.1.1), (2.1.1.2), and (2.1.1.3) of Deligne 1979b, and (SV2*) is the condition (2.1.1.5).

Definition

Let (V,h) be a rational Hodge structure. Following Deligne 1972, 7.1, we define the ***Mumford-Tate group*** of (V,h) to be the smallest algebraic subgroup G of GL_V such that $G_{\mathbb{R}} \supset h(\mathbb{S})$. It is also the smallest algebraic subgroup G of GL_V such that $G_{\mathbb{C}} \supset \mu_h(\mathbb{G}_m)$ (apply (5), p. 20). We usually regard the Mumford-Tate group as a pair (G,h), and we sometimes denote it by MT_V. Note that G is connected (otherwise we could replace it with its identity component), and that $w_h\colon \mathbb{G}_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$ and maps into the centre of G.[22]

For $m,n \in \mathbb{N}$, let $T^{m,n}$ denote the Hodge structure $V^{\otimes m} \otimes V^{\vee \otimes n}$. By a ***Hodge class*** of V, we mean an element of V of type $(0,0)$, and by a ***Hodge tensor*** of V, we mean a Hodge class of some $T^{m,n}$. The elements of $T^{m,n}$ fixed by the Mumford-Tate group G are exactly the Hodge tensors, and G is the largest algebraic subgroup of GL_V fixing all the Hodge tensors of V (apply Deligne 1982, 3.1c)[23].

The Hodge structures form a tannakian category over $\mathbb{Q}$ with a canonical fibre functor, namely, the forgetful functor. The Mumford-Tate group of (V,h) is the algebraic group attached to the tannakian subcategory $\langle V,h\rangle^{\otimes}$ generated by (V,h).

Let G and G^e respectively denote the Mumford-Tate groups of V and $V \oplus \mathbb{Q}(1)$. The action of G^e on V defines a homomorphism $G^e \to G$, which is an isogeny unless V has weight 0, in which case $G^e \simeq G \times \mathbb{G}_m$. The action of G^e on $\mathbb{Q}(1)$ defines a homomorphism $G^e \to \mathrm{GL}_{\mathbb{Q}(1)}$ whose kernel we denote G^1 and call the ***special Mumford-Tate group*** of V. Thus $G^1 \subset \mathrm{GL}_V$, and it is the smallest algebraic subgroup of GL_V such that $G^1_{\mathbb{R}} \supset h(\mathbb{S}^1)$. Clearly $G^1 \subset G$ and $G = G^1 \cdot w_h(\mathbb{G}_m)$.

PROPOSITION 6.1. *Let G be a connected algebraic group over $\mathbb{Q}$, and let h be a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$. The pair (G,h) is the Mumford-Tate group of a Hodge structure if and only if the weight*

[21] For (SV1), note that $\mathrm{Ad}(h(z))\colon \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is the derivative of $\mathrm{ad}(h(z))\colon G \to G$. The latter is trivial on $Z(G)$, and so the former is trivial on $\mathrm{Lie}(Z(G))$.

[22] Let $Z(w_h)$ be the centralizer of w_h in G. For any $a \in \mathbb{R}^{\times}$, $w_h(a)\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a morphism of real Hodge structures, and so it commutes with the action of $h(\mathbb{S})$. Hence $h(\mathbb{S}) \subset Z(w_h)_{\mathbb{R}}$. As h generates G, this implies that $Z(w_h) = G$.

[23] The argument in the proof of Lemma 3.5 of Deligne 1982 shows that some multiple of every $\mathbb{Q}$-character of G extends to GL_V.

homomorphism $w_h\colon \mathbb{G}_m \to G_{\mathbb{R}}$ *is defined over* $\mathbb{Q}$ *and* G *is generated by* h *(i.e., any algebraic subgroup* H *of* G *such that* $h(\mathbb{S}) \subset H_{\mathbb{R}}$ *equals* G*).*

PROOF. If (G,h) is the Mumford-Tate group of a Hodge structure (V,h), then certainly h generates G. The weight homomorphism w_h is defined over $\mathbb{Q}$ because (V,h) is a rational Hodge structure.

Conversely, suppose that (G,h) satisfy the conditions. For any faithful representation $\rho\colon G \to \mathrm{GL}_V$ of G, the pair $(V, h\circ\rho)$ is a rational Hodge structure, and (G,h) is its Mumford-Tate group. □

PROPOSITION 6.2. *Let* (G,h) *be the Mumford-Tate group of a Hodge structure* (V,h)*. Then* (V,h) *is polarizable if and only if* (G,h) *satisfies (SV2*).*

PROOF. Let $C = h(i)$. For simplicity, assume that (V,h) has a single weight m. Let G^1 be the special Mumford-Tate group of (V,h). Then $C \in G^1(\mathbb{R})$, and a pairing $\psi\colon V\times V \to \mathbb{Q}(-m)$ is a polarization of (V,h) if and only if $(2\pi i)^m\psi$ is a C-polarization of V for G^1 in the sense of §2. It follows from (2.3) and (2.4) that a polarization ψ for (V,h) exists if and only if $\mathrm{inn}(C)$ is a Cartan involution of $G^1_{\mathbb{R}}$. Now $G^1 \subset G$ and the quotient map $G^1 \to G/w_h(\mathbb{G}_m)$ is an isogeny, and so $\mathrm{inn}(C)$ is a Cartan involution of G^1 if and only if it is a Cartan involution of $G/w_h(\mathbb{G}_m)$. □

COROLLARY 6.3. *The Mumford-Tate group of a polarizable Hodge structure is reductive.*

PROOF. An algebraic group G over $\mathbb{Q}$ is reductive if and only if $G_{\mathbb{R}}$ is reductive, and we have already observed that (SV2*) implies that a group is reductive. Alternatively, polarizable Hodge structures are semisimple (obviously), and an algebraic group in characteristic zero is reductive if its representations are semisimple (e.g., Deligne and Milne 1982, 2.23). □

REMARK 6.4. Note that (6.2) implies the following statement: let (V,h) be a Hodge structure; if there exists an algebraic group $G \subset \mathrm{GL}_V$ such that $h(\mathbb{S}) \subset G_{\mathbb{R}}$ and (G,h) satisfies (SV2*), then (V,h) is polarizable.

NOTES. The Mumford-Tate group of a complex abelian variety A is defined to be the Mumford-Tate group of the Hodge structure $H_1(A^{\mathrm{an}},\mathbb{Q})$. In this context, they were first introduced in the talk of Mumford (1966).

Special Hodge structures

A rational Hodge structure is ***special***[24] if its Mumford-Tate group satisfies (SV1,2*) or, equivalently, if it is polarizable and its Mumford-Tate group satisfies (SV1).

PROPOSITION 6.5. *The special Hodge structures form a tannakian subcategory of* $\mathsf{Hdg}_{\mathbb{Q}}$.

PROOF. Let (V,h) be a special Hodge structure. The Mumford-Tate group of any object in the tannakian subcategory of $\mathsf{Hdg}_{\mathbb{Q}}$ generated by (V,h) is a quotient of MT_V, and hence satisfies (SV1,2*). □

Recall that the ***level*** of a Hodge structure (V,h) is the maximum value of $|p-q|$ as (p,q) runs over the pairs (p,q) with $V^{p,q} \neq 0$. It has the same parity as the weight of (V,h).

EXAMPLE 6.6. Let $V_n(a_1,\ldots,a_d)$ denote a complete intersection of d smooth hypersurfaces of degrees $a_1,\ldots,a_d$ in general position in $\mathbb{P}^{n+d}$ over $\mathbb{C}$. Then $H^n(V_n,\mathbb{Q})$ has level ≤ 1 only for the varieties $V_n(2)$, $V_n(2,2)$, $V_2(3)$, $V_n(2,2,2)$ (n odd), $V_3(3)$, $V_3(2,3)$, $V_5(3)$, $V_3(4)$ (Rapoport 1972).

[24] Poor choice of name, since "special" is overused and special points on Shimura varieties don't correspond to special Hodge structures, but I can't think of a better one. Perhaps an "SV Hodge structure"?

PROPOSITION 6.7. *Every polarizable Hodge structure of level* ≤ 1 *is special.*

PROOF. A Hodge structure of level 0 is direct sum of copies of $\mathbb{Q}(m)$ for some m, and so its Mumford-Tate group is $\mathbb{G}_m$. A Hodge structure (V,h) of level 1 is of type $\{(p,p+1),(p+1,p)\}$ for some p. Then

$$\mathrm{Lie}(\mathrm{MT}_V) \subset \mathrm{End}(V) = V^\vee \otimes V,$$

which is of type $\{(-1,1),(0,0),(1,-1)\}$. □

EXAMPLE 6.8. Let A be an abelian variety over $\mathbb{C}$. The Hodge structures $H^n_B(A)$ are special for all n. To see this, note that $H^1_B(A)$ is of level 1, and hence is special by (6.7), and that

$$H^n_B(A) \simeq \bigwedge^n H^1_B(A) \subset H^1_B(A)^{\otimes n},$$

and hence $H^n_B(A)$ is special by (6.5).

It follows that a nonspecial Hodge structure does not lie in the tannakian subcategory of $\mathsf{Hdg}_{\mathbb{Q}}$ generated by the cohomology groups of abelian varieties.

PROPOSITION 6.9. *A pair* (G,h) *is the Mumford-Tate group of a special Hodge structure if and only if* h *satisfies (SV1,2*), the weight* w_h *is defined over* $\mathbb{Q}$*, and* G *is generated by* h*.*

PROOF. Immediate consequence of Proposition 6.1, and of the definition of a special Hodge structure. □

Note that, because h generates G, it also satisfies (SV3).

EXAMPLE 6.10. Let $f\colon X \to S$ be the universal family of smooth hypersurfaces of a fixed degree δ and of a fixed odd dimension n. For s outside a meagre subset of S, the Mumford-Tate group of $H^n(X_s,\mathbb{Q})$ is the full group of symplectic similitudes (see 6.20 below). This implies that $H^n(X_s,\mathbb{Q})$ is not special unless it has level ≤ 1. According to (6.6), this rarely happens.

The generic Mumford-Tate group

Throughout this subsection, (V,F) is a family of Hodge structures on a connected complex manifold S. Recall that "family" means "holomorphic family".

LEMMA 6.11. *For any* $t \in \Gamma(S,\mathsf{V})$*, the set*

$$Z(t) = \{s \in S \mid t_s \text{ is of type } (0,0) \text{ in } \mathsf{V}_s\}$$

is an analytic subset of S*.*

PROOF. An element of V_s is of type $(0,0)$ if and only if it lies in $F^0\mathsf{V}_s$. On S, we have an exact sequence

$$0 \to F^0\mathcal{V} \to \mathcal{V} \to \mathcal{Q} \to 0$$

of locally free sheaves of $\mathcal{O}_S$-modules. Let U be an open subset of S such that $\mathcal{Q}$ is free over U. Choose an isomorphism $\mathcal{Q} \simeq \mathcal{O}_U^r$, and let $t|U$ map to $(t_1,\ldots,t_r)$ in $\mathcal{O}_U^r$. Then

$$Z(t) \cap U = \{s \in U \mid t_1(s) = \cdots = t_r(s) = 0\}.$$

□

For $m,n \in \mathbb{N}$, let $\mathsf{T}^{m,n} = \mathsf{T}^{m,n}\mathsf{V}$ be the family of Hodge structures $\mathsf{V}^{\otimes m} \otimes \mathsf{V}^{\vee\otimes n}$ on S. Let $\pi\colon \tilde{S} \to S$ be a universal covering space of S, and define

$$\mathring{S} = S \setminus \bigcup\nolimits_t \pi_*(Z(t)) \tag{9}$$

where t runs over the global sections of the local systems $\pi^*\mathsf{T}^{m,n}$ ($m,n \in \mathbb{N}$) such that $\pi_*(Z(t)) \neq S$. Thus $\mathring{S}$ is the complement in S of a countable union of proper analytic subvarieties (in particular, $S \setminus \mathring{S}$ is meagre).

EXAMPLE 6.12. For a "general" abelian variety of dimension g over $\mathbb{C}$, it is known that the $\mathbb{Q}$-algebra of Hodge classes is generated by the class of an ample divisor class (Comessatti 1938, Mattuck 1958). It follows that the same is true for all abelian varieties in the subset $\mathring{S}$ of the moduli space S. The Hodge conjecture obviously holds for these abelian varieties.

Let t be a section of $\mathsf{T}^{m,n}$ over an open subset U of $\mathring{S}$; if t is a Hodge class in $\mathsf{T}^{m,n}_s$ for one $s \in U$, then it is Hodge tensor for every $s \in U$. Thus, there exists a local system of $\mathbb{Q}$-subspaces $H\mathsf{T}^{m,n}$ on $\mathring{S}$ such that $(H\mathsf{T}^{m,n})_s$ is the space of Hodge classes in $\mathsf{T}^{m,n}_s$ for each s. Since the Mumford-Tate group of (V_s, F_s) is the largest algebraic subgroup of GL_{V_s} fixing the Hodge tensors in the spaces $\mathsf{T}^{m,n}_s$, we have the following result.

PROPOSITION 6.13. *Let G_s be the Mumford-Tate group of (V_s, F_s). Then G_s is locally constant on $\mathring{S}$.*

More precisely:

> Let U be an open subset of S on which V is constant[25], say, $\mathsf{V} = V_U$; identify the stalk V_s ($s \in U$) with V, so that G_s is a subgroup of GL_V; then G_s is constant for $s \in U \cap \mathring{S}$, say $G_s = G$, and $G \supset G_s$ for all $s \in U \setminus (U \cap \mathring{S})$.

6.14. We say that G_s is ***generic*** if $s \in \mathring{S}$. Suppose that V is constant, say $\mathsf{V} = V_S$, and let $G = G_{s_0} \subset \mathrm{GL}_V$ be generic. By definition, G is the smallest algebraic subgroup of GL_V such that $G_{\mathbb{R}}$ contains $h_{s_0}(\mathbb{S})$. As $G \supset G_s$ for all $s \in S$, the generic Mumford-Tate group of (V, F) is the smallest algebraic subgroup G of GL_V such that $G_{\mathbb{R}}$ contains $h_s(\mathbb{S})$ for *all* $s \in S$.

Let $\pi\colon \tilde{S} \to S$ be a universal covering of S, and fix a trivialization $\pi^*\mathsf{V} \simeq V_S$ of V. Then, for each $s \in S$, there are given isomorphisms

$$V \simeq (\pi^*\mathsf{V})_s \simeq \mathsf{V}_{\pi s}. \tag{10}$$

There is an algebraic subgroup G of GL_V such that, for each $s \in \pi^{-1}(\mathring{S})$, G maps isomorphically onto G_s under the isomorphism $\mathrm{GL}_V \simeq \mathrm{GL}_{\mathsf{V}_{\pi s}}$ defined by (10). It is the smallest algebraic subgroup of GL_V such that $G_{\mathbb{R}}$ contains the image of $h_s\colon \mathbb{S} \to \mathrm{GL}_{V_{\mathbb{R}}}$ for all $s \in \tilde{S}$.

QUESTION 6.15. Let $f\colon X \to S$ be a projective map of smooth algebraic varieties over the algebraic closure $\mathbb{Q}^{\mathrm{al}}$ of $\mathbb{Q}$ in $\mathbb{C}$, and let $(\mathsf{V}, F) = R^m(f_{\mathbb{C}})_*\mathbb{Q}$. Does there exist an $s \in S(\mathbb{Q}^{\mathrm{al}})$ such that the Mumford-Tate group of $H^m(X_{s\mathbb{C}}, \mathbb{Q}^{\mathrm{al}})$ is generic? In other words, is $\mathring{S} \cap S(\mathbb{Q})$ nonempty? I expect that the answer is yes in general.

ASIDE 6.16. For a polarizable integral variation of Hodge structures on a smooth algebraic variety S, Cattani, Deligne, and Kaplan (1995, Corollary 1.3) show that the sets $\pi_*(Z(t))$ in (9) are algebraic subvarieties of S. This answers a question of Weil 1977.

[25] For example, U can be any simply connected subset of S.

Variation of Mumford-Tate groups in families

DEFINITION 6.17. Let (V, F) be a family of Hodge structures on a connected complex manifold S.

(a) An ***integral structure*** on (V, F) is a local system of $\mathbb{Z}$-modules $\Lambda \subset \mathsf{V}$ such that $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda \simeq \mathsf{V}$.
(b) The family (V, F) is said to ***satisfy the theorem of the fixed part*** if, for every finite covering $a: S' \to S$ of S, there is a Hodge structure on the $\mathbb{Q}$-vector space $\Gamma(S', a^*\mathsf{V})$ such that, for all $s \in S'$, the canonical map $\Gamma(S', a^*\mathsf{V}) \to a^*\mathsf{V}_s$ is a morphism of Hodge structures, or, in other words, if the largest constant local subsystem V^f of $a^*\mathsf{V}$ is a constant family of Hodge substructures of $a^*\mathsf{V}$.
(c) The ***algebraic monodromy group*** at point $s \in S$ is the smallest algebraic subgroup of $\mathrm{GL}_{\mathsf{V}_s}$ containing the image of the monodromy homomorphism $\pi_1(S, s) \to \mathrm{GL}(\mathsf{V}_s)$. Its identity connected component is called the ***connected monodromy group*** M_s at s. In other words, M_s is the smallest connected algebraic subgroup of GL_{V_s} such that $M_s(\mathbb{Q})$ contains the image of a subgroup of $\pi_1(S, s)$ of finite index.

6.18. Let $\pi: \widetilde{S} \to S$ be the universal covering of S, and let Γ be the group of covering transformations of $\widetilde{S}/S$. The choice of a point $s \in \widetilde{S}$ determines an isomorphism $\Gamma \simeq \pi_1(S, \pi s)$. Now choose a trivialization $\pi^*\mathsf{V} \approx V_{\widetilde{S}}$. The choice of a point $s \in \widetilde{S}$ determines an isomorphism $V \simeq V_{\pi(s)}$. There is an action of Γ on V such that, for each $s \in \widetilde{S}$, the diagram

$$\begin{array}{ccccc} \Gamma & \times & V & \to & V \\ \wr| & & \wr| & & \wr| \\ \pi_1(S, \pi s) & \times & V_s & \to & V_s \end{array}$$

commutes. Let M be the smallest connected algebraic subgroup of GL_V such $M(\mathbb{Q})$ contains a subgroup of Γ of finite index. Under the isomorphism $V \simeq V_{\pi s}$ defined by $s \in S$, M maps isomorphically onto M_s.

THEOREM 6.19. *Let (V, F) be a polarizable family of Hodge structures on a connected complex manifold S, and assume that (V, F) admits an integral structure. Let G_s (resp. M_s) denote the Mumford-Tate (resp. the connected monodromy group) at $s \in S$.*

(a) *For all $s \in \mathring{S}$, $M_s \subset G_s^{\mathrm{der}}$.*
(b) *If $\mathsf{T}^{m,n}$ satisfies the theorem of the fixed part for all m, n, then M_s is normal in G_s^{der} for all $s \in \mathring{S}$; moreover, if $G_{s'}$ is commutative for some $s' \in S$, then $M_s = G_s^{\mathrm{der}}$ for all $s \in \mathring{S}$.*

EXAMPLE 6.20. Let $f: X \to \mathbb{P}^1$ be a Lefschetz pencil over $\mathbb{C}$ of hypersurfaces of fixed degree and odd dimension n, and let S be the open subset of $\mathbb{P}^1$ where X_s is smooth. Let (V, F) be the variation of Hodge structures $R^n f_*\mathbb{Q}$ on S. The action of $\pi_1(S, s)$ on $\mathsf{V}_s = H^n(X_s^{\mathrm{an}}, \mathbb{Q})$ preserves the cup-product form on V_s, and a theorem of Kazhdan and Margulis (Deligne 1974, 5.10) says that the image of $\pi_1(S, s)$ is Zariski-dense in the symplectic group. It follows that the generic Mumford-Tate group G_s is the full group of symplectic similitudes. This implies that, for $s \in \mathring{S}$, the Hodge structure V_s is not special unless it has level ≤ 1.

The proof of Theorem 6.19 will occupy the rest of this subsection.

PROOF OF (a) OF THEOREM 6.19

We first show that $M_s \subset G_s$ for $s \in \mathring{S}$. Recall that on $\mathring{S}$ there is a local system of $\mathbb{Q}$-vector spaces $H\mathsf{T}^{m,n} \subset \mathsf{T}^{m,n}$ such that $H\mathsf{T}_s^{m,n}$ is the space of Hodge tensors in $\mathsf{T}_s^{m,n}$. The fundamental group $\pi_1(S,s)$ acts on $H\mathsf{T}_s^{m,n}$ through a discrete subgroup of $\mathrm{GL}(H\mathsf{T}_s^{m,n})$ (because it preserves a lattice in $\mathsf{T}_s^{m,n}$), and it preserves a positive definite quadratic form on $H\mathsf{T}_s^{m,n}$. It therefore acts on $H\mathsf{T}_s^{m,n}$ through a finite quotient. As G_s is the algebraic subgroup of $\mathrm{GL}_{\mathsf{V}_s}$ fixing the Hodge tensors in some finite direct sum of spaces $\mathsf{T}_s^{m,n}$, this shows that the image of some finite index subgroup of $\pi_1(S,s)$ is contained in $G_s(\mathbb{Q})$. Hence $M_s \subset G_s$.

We next show that M_s is contained in the special Mumford-Tate group G_s^1 at s. Consider the family of Hodge structures $\mathsf{V} \oplus \mathbb{Q}(1)$, and let G_s^e be its Mumford-Tate group at s. As $\mathsf{V} \oplus \mathbb{Q}(1)$ is polarizable and admits an integral structure, its connected monodromy group M_s^e at s is contained in G_s^e. As $\mathbb{Q}(1)$ is a constant family, $M_s^e \subset \mathrm{Ker}(G_s^e \to \mathrm{GL}_{\mathbb{Q}(1)}) = G_s^1$. Therefore $M_s = M_s^e \subset G_s^1$.

There exists an object W in $\mathsf{Rep}_{\mathbb{Q}} G_s \simeq \langle V_s \rangle^{\otimes} \subset \mathsf{Hdg}_{\mathbb{Q}}$ such that $G_s^{\mathrm{der}} \cdot w_{h_s}(\mathbb{G}_m)$ is the kernel of $G_s \to \mathrm{GL}_W$. The Hodge structure W admits an integral structure, and its Mumford-Tate group is $G' \simeq G_s / \left(G_s^{\mathrm{der}} \cdot w_{h_s}(\mathbb{G}_m)\right)$. As W has weight 0 and G' is commutative, we find from (6.2) that $G'(\mathbb{R})$ is compact. As the action of $\pi_1(S,s)$ on W preserves a lattice, its image in $G'(\mathbb{R})$ must be discrete, and hence finite. This shows that

$$M_s \subset \left(G_s^{\mathrm{der}} \cdot w_{h_s}(\mathbb{G}_m)\right) \cap G_s^1 = G_s^{\mathrm{der}}.$$

PROOF OF THE FIRST STATEMENT OF (b) OF THEOREM 6.19

We first prove two lemmas.

LEMMA 6.21. *Let V be a $\mathbb{Q}$-vector space, and let $H \subset G$ be algebraic subgroups of GL_V. Assume:*

(a) *the action of H on any H-stable line in a finite direct sum of spaces $T^{m,n}$ is trivial;*
(b) *$(T^{m,n})^H$ is G-stable for all $m,n \in \mathbb{N}$.*

Then H is normal in G.

PROOF. There exists a line L in some finite direct sum T of spaces $T^{m,n}$ such that $H = \mathrm{Stab}(L)$ (Chevalley's theorem, Deligne 1982, 3.1a,b). According to (a), H acts trivially on L. Let W be the intersection of the G-stable subspaces of T containing L. Then $W \subset T^H$ because T^H is G-stable by (b). Let φ be the homomorphism $G \to \mathrm{GL}_{W^\vee \otimes W}$ defined by the action of G on W. As H acts trivially on W, it is contained in the kernel of φ. On the other hand, the elements of the kernel of φ act as scalars on W, and so stabilize L. Therefore $\mathrm{Ker}(\varphi) \subset H$. □

LEMMA 6.22. *Let (V, F) be a polarizable family of Hodge structures on a connected complex manifold S. Let L be a local system of $\mathbb{Q}$-vector spaces on S contained in a finite direct sum of local systems $\mathsf{T}^{m,n}$. If (V, F) admits an integral structure and L has dimension 1, then M_s acts trivially on L_s.*

PROOF. The hypotheses imply that L also admits an integral structure, and so $\pi_1(S,s)$ acts through the finite subgroup $\{\pm 1\}$ of $\mathrm{GL}_{\mathsf{L}_s}$. This implies that M_s acts trivially on L_s. □

We now prove the first part of (b) of the theorem. Let $s \in \mathring{S}$; we shall apply Lemma 6.21 to $M_s \subset G_s \subset \mathrm{GL}_{\mathsf{V}_s}$. After passing to a finite covering of S, we may suppose that $\pi_1(S,s) \subset M_s(\mathbb{Q})$. Any M_s-stable line in $\bigoplus_{m,n} \mathsf{T}_s^{m,n}$ is of the form L_s for a local subsystem L of $\bigoplus_{m,n} \mathsf{T}_s^{m,n}$, and so hypothesis (a) of Lemma 6.21 follows from (6.22). It remains to show $(\mathsf{T}_s^{m,n})^{M_s}$ is stable under G_s.

Let H be the largest algebraic subgroup of $\mathrm{GL}_{\mathsf{T}_s^{m,n}}$ stabilizing $(\mathsf{T}_s^{m,n})^{M_s}$.[26] Because $\mathsf{T}^{m,n}$ satisfies the theorem of the fixed part, $(\mathsf{T}_s^{m,n})^{M_s}$ is a Hodge substructure of $\mathsf{T}_s^{m,n}$, and so $(\mathsf{T}_s^{m,n})_{\mathbb{R}}^{M_s}$ is stable under $h(\mathbb{S})$. Therefore $h(\mathbb{S}) \subset G_{\mathbb{R}}$, and this implies that $G_s \subset G$.

PROOF OF THE SECOND STATEMENT OF (b) OF THEOREM 6.19

We first prove a lemma.

LEMMA 6.23. *Let (V, F) be a variation of polarizable Hodge structures on a connected complex manifold S. Assume:*

(a) *M_s is normal in G_s for all $s \in \mathring{S}$;*
(b) *$\pi_1(S,s) \subset M_s(\mathbb{Q})$ for one (hence every) $s \in S$;*
(c) *(V, F) satisfies the theorem of the fixed part.*

Then the subspace $\Gamma(S,\mathsf{V})$ of V_s is stable under G_s, and the image of G_s in $\mathrm{GL}_{\Gamma(S,\mathsf{V})}$ is independent of $s \in S$.

In fact, (c) implies that $\Gamma(S,\mathsf{V})$ has a well-defined Hodge structure, and we shall show that the image of G_s in $\mathrm{GL}_{\Gamma(S,\mathsf{V})}$ is the Mumford-Tate group of $\Gamma(S,\mathsf{V})$.

PROOF. We shall apply the following statement:

> (*) For any polarizable Hodge structure (V,h) and Hodge structure W in $\langle V,h\rangle^{\otimes}$, the action of MT_V on W is described by a surjective homomorphism $\mathrm{MT}_V \to \mathrm{MT}_W$ (apply Deligne and Milne 1982, 2.21a).

For every $s \in S$,

$$\Gamma(S,\mathsf{V}) = \Gamma(S,\mathsf{V}^f) = (\mathsf{V}^f)_s = \mathsf{V}_s^{\pi_1(S,s)} \overset{\text{(b)}}{=} \mathsf{V}_s^{M_s}.$$

The subspace $\mathsf{V}_s^{M_s}$ of V_s is stable under G_s when $s \in \mathring{S}$ because then M_s is normal in G_s, and it is stable under G_s when $s \notin \mathring{S}$ because then G_s is contained in some generic Mumford-Tate group. Because (V, F) satisfies the theorem of the fixed part, $\Gamma(S,\mathsf{V})$ has a Hodge structure (independent of s) for which the inclusion $\Gamma(S,\mathsf{V}) \to \mathsf{V}_s$ is a morphism of Hodge structures. From (*), the image of G_s in $\mathrm{GL}_{\Gamma(S,\mathsf{V})}$ is the Mumford-Tate group of $\Gamma(S,\mathsf{V})$, which does not depend on s. □

We now prove that $M_s = G_s^{\mathrm{der}}$ when some Mumford-Tate group $G_{s'}$ is commutative. We know that M_s is a normal subgroup of G_s^{der} for $s \in \mathring{S}$, and so it remains to show that G_s/M_s is commutative for $s \in \mathring{S}$ under the hypothesis.

We begin with a remark. Let N be a normal algebraic subgroup of an algebraic group G. The category of representations of G/N can be identified with the category of representations of G on which N acts trivially. Therefore, to show that G/N is commutative, it suffices to show that G acts through a commutative quotient on every V on which N acts trivially. If G is reductive and we are in characteristic zero, then it suffices to show that, for one faithful representation V of G, the group G acts through a commutative quotient on $(T^{m,n})^N$ for all $m,n \in \mathbb{N}$.

[26] Recall: let $G \to \mathrm{GL}_V$ be a representation of an algebraic group G, and let W be a subspace of V; then there exists an algebraic subgroup H of G such that

$$H(R) = \{g \in G(R) \mid g(W_R) = W_R\}$$

for all k-algebras R. The subgroup H is called the stabilizer of W. Clearly it commutes with extension of the base field.

Let $\mathsf{T} = \mathsf{T}^{m,n}$. According to the remark, it suffices to show that, for $s \in \mathring{S}$, G_s acts on $\mathsf{T}_s^{M_s}$ through a commutative quotient. This will follow from the hypothesis, once we check that T satisfies the hypotheses of Lemma 6.23. Certainly, M_s is a normal subgroup of G_s for $s \in \mathring{S}$, and $\pi_1(S,s)$ will be contained in M_s once we have passed to a finite cover. Finally, we are assuming that T satisfies the theorem of the fixed part.

NOTES. Theorem 6.19 is due to Deligne (see Deligne 1972, 7.5; Zarhin 1984, 7.3) except for the second statement of (b), which is taken from André 1992a, p. 12.

Variation of Mumford-Tate groups in algebraic families

When the underlying manifold is an algebraic variety, we have the following theorem.

THEOREM 6.24 (GRIFFITHS, SCHMID). *A variation of Hodge structures on a smooth algebraic variety over $\mathbb{C}$ satisfies the theorem of the fixed part if it is polarizable and admits an integral structure.*

PROOF. When the variation of Hodge structures arises from a projective smooth map $X \to S$ of algebraic varieties and S is complete, this is the original theorem of the fixed part (Griffiths 1970, §7). In the general case it is proved in Schmid 1973, 7.22. See also Deligne 1971a, 4.1.2 and the footnote on p. 45. □

THEOREM 6.25. *Let (V, F) variation of Hodge structures on a connected smooth complex algebraic variety S. If (V, F) is polarizable and admits an integral structure, then M_s is a normal subgroup of G_s^{der} for all $s \in \mathring{S}$, and the two groups are equal if G_s is commutative for some $s \in S$.*

PROOF. If (V, F) is polarizable variation of Hodge structures that admits an integral structure, then so also is $\mathsf{T}^{m,n}$, and so it satisfies the theorem of the fixed part (Theorem 6.24). Now Theorem 6.19 implies Theorem 6.25. □

7. Period subdomains

We define the notion of a period subdomain, and we show that the hermitian symmetric domains are exactly the period subdomains on which the universal family of Hodge structures is a polarizable variation of Hodge structures.

Flag manifolds

Let V be a complex vector space and let $\mathbf{d} = (d_1, \ldots, d_r)$ be a sequence of integers with $\dim V > d_1 > \cdots > d_r > 0$. The ***flag manifold*** $\mathrm{Gr}_{\mathbf{d}}(V)$ has as points the filtrations

$$V \supset F^1 V \supset \cdots \supset F^r V \supset 0, \qquad \dim F^i V = d_i.$$

It is a projective complex manifold, and the tangent space to $\mathrm{Gr}_{\mathbf{d}}(V)$ at the point corresponding to a filtration F is

$$T_F(\mathrm{Gr}_{\mathbf{d}}(V)) \simeq \mathrm{End}(V)/F^0\,\mathrm{End}(V)$$

where

$$F^j\,\mathrm{End}(V) = \{\alpha \in \mathrm{End}(V) \mid \alpha(F^i V) \subset F^{i+j} V \text{ for all } i\}.$$

THEOREM 7.1. *Let V_S be the constant sheaf on a connected complex manifold S defined by a real vector space V, and let (V_S, F) be a family of Hodge structures on S. Let $\mathbf{d}$ be the sequence of ranks of the subsheaves in F.*

(a) *The map $\varphi\colon S \to \mathrm{Gr}_{\mathbf{d}}(V_{\mathbb{C}})$ sending a point s of S to the point of $\mathrm{Gr}_{\mathbf{d}}(V_{\mathbb{C}})$ corresponding to the filtration F_s on V is holomorphic.*

(b) *The family (V_S, F) satisfies Griffiths transversality if and only if the image of the map*

$$(d\varphi)_s\colon T_s S \to T_{\varphi(s)} \mathrm{Gr}_{\mathbf{d}}(V_{\mathbb{C}})$$

lies in the subspace $F_s^{-1}\,\mathrm{End}(V_{\mathbb{C}})/F_s^0\,\mathrm{End}(V_{\mathbb{C}})$ of $\mathrm{End}(V_{\mathbb{C}})/F_s^0\,\mathrm{End}(V_{\mathbb{C}})$ for all $s \in S$.

PROOF. Statement (a) simply says that the filtration is holomorphic, and (b) restates the definition of Griffiths transversality. □

Period domains

Let V be a real vector space, and let F_0 be a Hodge filtration on V of weight m. Let $\psi\colon V \times V \to \mathbb{R}(m)$ be a polarization of the Hodge structure (V, F_0).

Let D be the set of Hodge filtrations of weight m on V for which ψ is a polarization and which have the same Hodge numbers as F_0. Thus D is the set of descending filtrations

$$V_{\mathbb{C}} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset 0$$

on $V_{\mathbb{C}}$ such that

(a) $\dim_{\mathbb{C}} F^p = \dim_{\mathbb{C}} F_0^p$ for all p,
(b) $V_{\mathbb{C}} = F^p \oplus \overline{F^q}$ whenever $p + q = m + 1$,
(c) $\psi(F^p, F^q) = 0$ whenever $p + q = m + 1$, and
(d) $(2\pi i)^m\, \psi_{\mathbb{C}}(v, C\overline{v}) > 0$ for all nonzero elements v of $V_{\mathbb{C}}$.

Condition (b) says that the filtration is a Hodge filtration of weight m, and the conditions (c) and (d) say that ψ is a polarization.

Let $D^{\vee}$ be the set of filtrations of $V_{\mathbb{C}}$ satisfying (a) and (c).

THEOREM 7.2. *The set $D^{\vee}$ is a compact complex submanifold of $\mathrm{Gr}_{\mathbf{d}}(V)$, and D is an open submanifold of $D^{\vee}$.*

PROOF. In the presence of (a), condition (c) says that F^{m+1-p} is the orthogonal complement of F^p for all p. In particular, each of F^p and F^{m+1-p} determines the other.

When m is odd, ψ is alternating, and the remark shows that $D^{\vee}$ can be identified with the set of filtrations

$$V_{\mathbb{C}} \supset F^{(m+1)/2} \supset F^{(m+3)/2} \supset \cdots \supset 0$$

satisfying (a) and such that $F^{(m+1)/2}$ is totally isotropic for ψ. Let S be the symplectic group for ψ. Then $S(\mathbb{C})$ acts transitively on these filtrations, and the stabilizer P of any one filtration is a parabolic subgroup of S. Therefore $S(\mathbb{C})/P(\mathbb{C})$ is a compact complex manifold, and the bijection $S(\mathbb{C})/P(\mathbb{C}) \simeq D^{\vee}$ is holomorphic. The proof when m is even is similar.

The submanifold D of $D^{\vee}$ is open because the conditions (b) and (d) are open. □

The complex manifold $D = D(V, F_0, \psi)$ is the (Griffiths) ***period domain*** defined by (V, F_0, ψ).

THEOREM 7.3. *Let (V, F, ψ) be a polarized family of Hodge structures on a complex manifold S. Let U be an open subset of S on which the local system V is trivial, and choose an isomorphism $\mathsf{V}|U \simeq V_U$. The map $\mathcal{P}: U \to D(V, F_0, \psi_0)$ sending a point $s \in U$ to the point (V_s, F_s, ψ_s) is holomorphic.*

PROOF. The map $s \mapsto F_s: U \to \mathrm{Gr}_{\mathbf{d}}(V)$ is holomorphic by (7.1) and it takes values in D. As D is a complex submanifold of $\mathrm{Gr}_{\mathbf{d}}(V)$ this implies that the map $U \to D$ is holomorphic (Grauert and Remmert 1984, 4.3.3). □

The map $\mathcal{P}$ is called the ***period map***.

THEOREM 7.4. *If the universal family of Hodge structures on $D = D(V, F_0, \psi)$ satisfies Griffiths transversality, then D is a hermitian symmetric domain.*

PROOF. Let G be the algebraic subgroup of GL_V of elements fixing ψ up to scalar. Then G is a real reductive algebraic group whose centre contains -1, and the Hodge structure on V defines a homomorphism $h: \mathbb{S} \to G$. As in (2.5), there exists a homomorphism $u: U^1 \to G^{\mathrm{ad}}(\mathbb{R})$ such that $u(z) = h(\sqrt{z}) \bmod Z(G)(\mathbb{Q})$.

Let $\mathfrak{g} = \mathrm{Lie}\, G$, with its Hodge structure provided by $\mathrm{Ad} \circ h$. Then

$$\mathfrak{g}_{\mathbb{C}}/\mathfrak{g}^{00} \simeq T_o(D) \subset T_o(\mathrm{Gr}_{\mathbf{d}}(V)) \simeq \mathrm{End}(V)/F^0\,\mathrm{End}(V).$$

If the universal family of Hodge structures satisfies Griffiths transversality, then $\mathfrak{g}_{\mathbb{C}} = F^{-1}\mathfrak{g}_{\mathbb{C}}$ (by 7.1b), and so $h(z)$ acts on $\mathfrak{g}_{\mathbb{C}}$ through the characters $z/\bar{z}$, 1, $\bar{z}/z$. This implies that $u(z)$ acts on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ through the characters $z, 1, z^{-1}$, and so u satisfies condition (SU1) of Theorem 2.6.

Let G^1 be the subgroup of G of elements fixing ψ. As ψ is a polarization of the Hodge structure, $(2\pi i)^m \psi$ is a C-polarization of V relative to G^1, and so $\mathrm{inn}\, C$ is a Cartan involution of G^1 (Theorem 2.3). Now $C = h(i) = u(-1)$, and so u satisfies condition (SU2) of Theorem 2.6. As G^1 obviously has no compact factors, and D can be identified with the set of conjugates of u by elements of $G^{\mathrm{ad}}(\mathbb{R})^+$, this shows that D is a hermitian symmetric domain. □

Period subdomains

7.5. Let G be a real algebraic group, and let X be a (topological) connected component of the space of homomorphisms $\mathbb{S} \to G$. Let G_1 be the smallest algebraic subgroup of G through which all the $h \in X$ factor. Then X is again a connected component of the space of homomorphisms of $\mathbb{S}$ into G_1. Since $\mathbb{S}$ is a torus, any two elements of X are conjugate, and so the space X is a $G_1(\mathbb{R})^+$-conjugacy class of morphisms from $\mathbb{S}$ into G. It is also a $G(\mathbb{R})^+$-conjugacy class, and G_1 is a normal subgroup of the identity component of G. (See Deligne 1979b, 1.1.12.)

Let V be a real vector space. By a ***tensor***[27] of V we mean a homomorphism $t: V^{\otimes r} \to \mathbb{R}(-m)$ for some r and m. When V is a Hodge structure and t is a morphism of Hodge structures, then we call it a ***Hodge tensor***. Concretely, this means that t is of type $(0,0)$ for the natural Hodge structure on $\mathrm{Hom}(V^{\otimes r}, \mathbb{R}(-m))$, or that it lies in $F^0\left(\mathrm{Hom}(V^{\otimes r}, \mathbb{R}(-m))\right)$.

Let $\mathfrak{t} = (t_i)_{i \in I}$ be a family of tensors of V. Assume that I contains an element 0 such that t_0 is a linear map $V \otimes V \to \mathbb{Q}(-m)$ for some m. Let h_0 be a Hodge structure on V such that each t_i

[27] Since we shall be considering only Hodge structures V together with a polarization, and hence a given nondegenerate pairing $V \times V \to \mathbb{Q}(-m)$, this definition of tensor is essentially the same as that in the last section. Explicitly, $V \simeq V^{\vee}(-m)$, and so $V^{\otimes n_1} \otimes V^{\vee \otimes n_2} \simeq \mathrm{Hom}(V^{\otimes n_1 + n_2}, \mathbb{Q}(-mn_1))$.

is a Hodge tensor and t_0 is a polarization, and let D be the connected component, containing h_0, of the Hodge structures on V such that each t_i is a Hodge tensor and t_0 is a polarization. In other words, D is a connected component of the subset of $D(V, h_0, t_0)$ consisting of the Hodge structures for which every t_i is a Hodge tensor.

Let G be the algebraic subgroup of $\mathrm{GL}_V \times \mathrm{GL}_{\mathbb{Q}(1)}$ fixing the t_i. Then $G(\mathbb{R})$ consists of the pairs (g, c) such that

$$t_i(gv_1, \ldots, gv_n) = c^m t_i(v_1, \ldots, v_n)$$

for $i \in I$. The t_i are Hodge tensors for $h: \mathbb{S} \to \mathrm{GL}_V$ on V if and only if $(h, t): \mathbb{S} \to \mathrm{GL}_V \times \mathbb{G}_m$ factors through G. Thus, to give a Hodge structure on V for which all the t_i are Hodge tensors is the same as giving a homomorphism $h: \mathbb{S} \to G$.

Let G_1 be the smallest algebraic subgroup of G through which all $h \in D$. According to (7.5), D is a $G_1(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to G_1$. The group $G_1(\mathbb{C})$ acts on $D^\vee(V, h_0, t_0)$, and we let $D^\vee$ denote the orbit of F_0.

THEOREM 7.6. *The set $D^\vee$ is a compact complex submanifold of $D^\vee(V, h_0, t_0)$, and D is an open complex submanifold of $D^\vee$.*

PROOF. In fact, $D^\vee$ is a closed algebraic subvariety of $D^\vee(V, h_0, t_0)$ (namely, the intersection of the zero-sets of certain sections of vector sheaves). It is also homogeneous, being isomorphic to $G_1(\mathbb{C})/P$ where P is the (parabolic) subgroup of G_1 stabilizing F_0, and hence it is smooth. Therefore it is a compact complex submanifold of $D^\vee(V, h_0, t_0)$. Finally, $D = D(V, h_0, t_0) \cap D^\vee(V, h_0, \mathfrak{t}_0)$. □

We call $D = D(V, h_0, \mathfrak{t})$ the ***period subdomain*** defined by $(V, h_0, \mathfrak{t})$.

THEOREM 7.7. *Let (V, F) be a family of Hodge structures on a complex manifold S, and let $\mathfrak{t} = (t_i)_{i \in I}$ be a family of Hodge tensors of V. Assume that I contains an element 0 such that t_0 is a polarization. Let U be an open subset of S on which the local system V is trivial, and fix a trivialization $\mathsf{V}|U \xrightarrow{\approx} V_U$. If $(\mathsf{V}_s, F_s, \mathfrak{t}_s) \in D$ for one s, then $(\mathsf{V}_s, F_s, \mathfrak{t}_s) \in D$ for all s, and the map $\mathcal{P}: U \to D(V, h_0, \mathfrak{t}_0)$ sending a point $s \in U$ to the point $(\mathsf{V}_s, F_s, \mathfrak{t}_s) \in D$ is holomorphic.*

PROOF. Same as that of Theorem 7.3. □

THEOREM 7.8. *If the universal family of Hodge structures on D satisfies Griffiths transversality, then D is a hermitian symmetric domain.*

PROOF. Essentially the same as that of Theorem 7.4. □

THEOREM 7.9. *Every hermitian symmetric domain arises as a period subdomain.*

PROOF. Let D be a hermitian symmetric domain, and let $o \in D$. Let H be the real adjoint algebraic group such that $H(\mathbb{R})^+ = \mathrm{Hol}(D)^+$, and let $u: \mathbb{S}^1 \to H$ be the homomorphism such that $u(z)$ fixes o and acts on $T_0(D)$ as multiplication by z (see §2). Let $h: \mathbb{S} \to H$ be the homomorphism such that $h(z) = u_o(z/\bar{z})$ for $z \in \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$. Choose a faithful representation $\rho: H \to \mathrm{GL}_V$ of G. Because u satisfies (2.6, SU2), the Hodge structure $(V, \rho \circ h)$ is polarizable. Choose a polarization and include it in a family $\mathfrak{t}$ of tensors for V such that H is the subgroup of $\mathrm{GL}_V \times \mathrm{GL}_{\mathbb{Q}(1)}$ fixing the elements of $\mathfrak{t}$. Then $D \simeq D(V, h, \mathfrak{t})$. □

NOTES. The interpretation of hermitian symmetric domains as moduli spaces for Hodge structures with tensors follows Deligne 1979b, 1.1.17.

Why moduli varieties are (sometimes) locally symmetric

Fix a base field k. A ***moduli problem*** over k is a contravariant functor $\mathcal{F}$ from the category of (some class of) schemes over k to the category of sets. A variety S over k together with a natural isomorphism $\phi\colon\mathcal{F}\to\mathrm{Hom}_k(-,S)$ is called a ***fine solution to the moduli problem***. A variety that arises in this way is called a ***moduli variety***.

Clearly, this definition is too general: every variety S represents the functor $h_S=\mathrm{Hom}_k(-,S)$. In practice, we only consider functors for which $\mathcal{F}(T)$ is the set of isomorphism classes of some algebro-geometric objects over T, for example, families of algebraic varieties with additional structure.

If S represents such a functor, then there is an object $\alpha\in\mathcal{F}(S)$ that is universal in the sense that, for any $\alpha'\in\mathcal{F}(T)$, there is a unique morphism $a\colon T\to S$ such that $\mathcal{F}(a)(\alpha)=\alpha'$. Suppose that α is, in fact, a smooth projective map $f\colon X\to S$ of smooth varieties over $\mathbb{C}$. Then $R^m f_*\mathbb{Q}$ is a polarizable variation of Hodge structures on S admitting an integral structure (Theorem 5.2). A polarization of X/S defines a polarization of $R^m f_*\mathbb{Q}$ and a family of algebraic classes on X/S of codimension m defines a family of global sections of $R^{2m} f_*\mathbb{Q}(m)$. Let D be the universal covering space of S^{an}. The pull-back of $R^m f_*\mathbb{Q}$ to D is a variation of Hodge structures whose underlying locally constant sheaf of $\mathbb{Q}$-vector spaces is constant, say, equal to V_S; thus we have a variation of Hodge structures (V_S,F) on D. We suppose that the additional structure on X/S defines a family $\mathfrak{t}=(t_i)_{i\in I}$ of Hodge tensors of V_S with t_0 a polarization. We also suppose that the family of Hodge structures on D is universal[28], i.e., that $D=D(V,F_0,\mathfrak{t})$. Because (V_S,F) is a variation of Hodge structures, D is a hermitian symmetric domain (by 7.8). The Margulis arithmeticity theorem (3.11) shows that Γ is an arithmetic subgroup of $G(D)$ except possibly when $G(D)$ has factors of small dimension. Thus, when looking at moduli varieties, we are naturally led to consider arithmetic locally symmetric varieties.

Application: Riemann's theorem in families

Let A be an abelian variety over $\mathbb{C}$. The exponential map defines an exact sequence

$$0\to H_1(A^{\mathrm{an}},\mathbb{Z})\to T_0(A^{\mathrm{an}})\xrightarrow{\exp} A^{\mathrm{an}}\to 0.$$

From the first map in this sequence, we get an exact sequence

$$0\to \mathrm{Ker}(\alpha)\to H_1(A^{\mathrm{an}},\mathbb{Z})_{\mathbb{C}}\xrightarrow{\alpha} T_0(A^{\mathrm{an}})\to 0.$$

The $\mathbb{Z}$-module $H_1(A^{\mathrm{an}},\mathbb{Z})$ is an integral Hodge structure with Hodge filtration

$$\begin{array}{ccccc} H_1(A^{\mathrm{an}},\mathbb{Z})_{\mathbb{C}} & \supset & \mathrm{Ker}(\alpha) & \supset & 0. \\ F^{-1} & & F^0 & & \end{array}$$

Let ψ be a Riemann form for A. Then $2\pi i\psi$ is a polarization for the Hodge structure $H_1(A^{\mathrm{an}},\mathbb{Z})$.

THEOREM 7.10. *The functor $A\rightsquigarrow H_1(A^{\mathrm{an}},\mathbb{Z})$ is an equivalence from the category of abelian varieties over $\mathbb{C}$ to the category of polarizable integral Hodge structures of type* $\{(-1,0),(0,-1)\}$.

PROOF. In view of the correspondence between complex structures and Hodge structures of type $\{(-1,0),(0,-1)\}$ (see 5.1), this is simply a restatement of Theorem 4.4. □

[28] This happens rarely!

THEOREM 7.11. *Let S be a smooth algebraic variety over $\mathbb{C}$. The functor*

$$(A \xrightarrow{f} S) \rightsquigarrow R_1 f_* \mathbb{Z}$$

is an equivalence from the category of families of abelian varieties[29] over S to the category of polarizable integral variations of Hodge structures of type $\{(-1,0),(0,-1)\}$.

PROOF. Let $f^A\colon A \to S$ be a family of abelian varieties over S. The exponential defines an exact sequence of sheaves on S^{an},

$$0 \to R_1 f_*^A \mathbb{Z} \to \mathcal{T}_0(A^{\mathrm{an}}) \to A^{\mathrm{an}} \to 0.$$

From this one sees that the map $\mathrm{Hom}(A^{\mathrm{an}}, B^{\mathrm{an}}) \to \mathrm{Hom}(R_1 f_*^A \mathbb{Z}, R_1 f_*^B \mathbb{Z})$ is an isomorphism. The S-scheme $\mathcal{H}om_S(A,B)$ is unramified over S, and so its algebraic sections coincide with its holomorphic sections (cf. Deligne 1971a, 4.4.3). Hence the functor is fully faithful. In particular, a family of abelian varieties is uniquely determined by its variation of Hodge structures up to a unique isomorphism. This allows us to construct the family of abelian varieties attached to a variation of Hodge structures locally. Thus, we may suppose that the underlying local system of $\mathbb{Z}$-modules is trivial. Assume initially that the variation of Hodge structures on S has a principal polarization, and endow it with a level-N structure. According Proposition 4.5, the variation of Hodge structures on S is the pull-back of the canonical variation of Hodge structures on D_N by a regular map $\alpha\colon S \to D_N$. Since the latter variation arises from a family of abelian varieties (Theorem 4.6), so does the former.

To remove the "principally polarized" condition in the above argument (a) rewrite it to allow polarizations of a fixed degree d, not necessarily 1, or (b) change the lattice (locally) so that the variation of Hodge structures is principally polarized, or (c) apply Zarhin's trick to show that (locally) the fourth multiple of the variation of Hodge structures is principally polarized. □

8. Variations of Hodge structures on locally symmetric varieties

In this section, we explain how to classify variations of Hodge structures on arithmetic locally symmetric varieties in terms of certain auxiliary reductive groups. Throughout, we write "family of integral Hodge structures" to mean "family of rational Hodge structures that admits an integral structure".

Generalities on u and h

8.1. Let H be a real algebraic group. The map $z \mapsto z/\bar{z}$ defines an isomorphism $\mathbb{S}/w(\mathbb{G}_m) \simeq \mathbb{S}^1$, and so the formula

$$h(z) = u(z/\bar{z}) \tag{11}$$

defines a one-to-one correspondence between homomorphisms $h\colon \mathbb{S} \to H$ trivial on $w(\mathbb{G}_m)$ and homomorphisms $u\colon \mathbb{S}^1 \to H$. Note that $h(z) = u(z)^2$ for $z \in \mathbb{S}^1(\mathbb{R}) = U^1$. We use $h\colon \mathbb{S}/\mathbb{G}_m \to H$ to denote a homomorphism $h\colon \mathbb{S} \to H$ trivial on $w(\mathbb{G}_m)$.

[29] By a family of abelian varieties over S we mean an abelian scheme over S, i.e., a smooth group scheme over S whose fibres are abelian varieties.

8.2. Let H be a real adjoint algebraic group. If $h\colon\mathbb{S}\to H$ satisfies (SV1), then it is trivial on $w(\mathbb{G}_m)$, and so (8.1) provides us with a one-to-one correspondence between homomorphisms $h\colon\mathbb{S}\to H$ satisfying (SV1) and homomorphisms $u\colon\mathbb{S}^1\to H$ satisfying the condition (SU1), p. 10. Since $h(i)=u(-1)$, h satisfies (SV2) if and only if u satisfies (SU2).

8.3. Let H be an adjoint algebraic group over $\mathbb{Q}$. As noted in §6, a homomorphism $h\colon\mathbb{S}/\mathbb{G}_m\to H$ satisfies (SV3) if and only if H is of noncompact type.

Existence of Hodge structures of CM-type in a family

PROPOSITION 8.4. *Let G be a reductive group over $\mathbb{Q}$, and let $h\colon\mathbb{S}\to G_{\mathbb{R}}$ be a homomorphism. There exists a $G(\mathbb{R})^+$-conjugate h_0 of h such that $h_0(\mathbb{S})\subset T_{0\mathbb{R}}$ for some maximal torus T_0 of G.*

PROOF (Cf. Mumford 1969, p. 348). Let K be the centralizer of h in $G_{\mathbb{R}}$. Let T be the centralizer in $G_{\mathbb{R}}$ of some regular element of $\operatorname{Lie}K$; it is a maximal torus in K. Because $h(\mathbb{S})$ is contained in the centre of K and T is maximal, $h(\mathbb{S})\subset T$. If T' is a torus in $G_{\mathbb{R}}$ containing T, then T' centralizes h, and so $T'\subset K$ and $T=T'$; therefore T is maximal in $G_{\mathbb{R}}$. For a regular element λ of $\operatorname{Lie}(T)$, T is the centralizer of λ. Choose a $\lambda_0\in\operatorname{Lie}(G)$ that is close to λ in $\operatorname{Lie}(G)_{\mathbb{R}}$, and let T_0 be its centralizer in G. Then T_0 is a maximal torus of G (over $\mathbb{Q}$). Because $T_{0\mathbb{R}}$ and $T_{\mathbb{R}}$ are close, they are conjugate: $T_{0\mathbb{R}}=gTg^{-1}$ for some $g\in G(\mathbb{R})^+$. Now $h_0\overset{\text{def}}{=}\operatorname{inn}(g)\circ h$ factors through $T_{0\mathbb{R}}$. □

A polarizable rational Hodge structure is of ***CM-type*** if its Mumford-Tate group is commutative (hence a torus). Let (G,h) be as in the proposition, and assume that h satisfies (SV2*). For any representation $\rho\colon G\to\mathrm{GL}_V$, the Hodge structure $(V,\rho_{\mathbb{R}}\circ h)$ is polarizable (by 6.4). The proposition shows that there exists a $G(\mathbb{R})^+$-conjugate h_0 of h such that $(V,\rho_{\mathbb{R}}\circ h_0)$ is of CM-type. For example, the universal family of Hodge structures on a period subdomain contains Hodge structures of CM-type.

Description of the variations of Hodge structures on $D(\Gamma)$

Consider an arithmetic locally symmetric variety $D(\Gamma)$. Recall that this means that $D(\Gamma)$ is an algebraic variety whose universal covering space is a hermitian symmetric domain D and that the group of covering transformations Γ is an arithmetic subgroup of the real Lie group $\operatorname{Hol}(D)^+$; moreover, $D(\Gamma)^{\mathrm{an}}=\Gamma\backslash D$.

According to Theorem 3.2, D decomposes into a product $D=D_1\times\cdots\times D_r$ of hermitian symmetric domains with the property that each group $\Gamma_i\overset{\text{def}}{=}\Gamma\cap\operatorname{Hol}(D_i)^+$ is an irreducible arithmetic subgroup of $\operatorname{Hol}(D_i)^+$ and the map

$$D_1(\Gamma_1)\times\cdots\times D_r(\Gamma_r)\to D(\Gamma)$$

is finite covering. In the remainder of this subsection, we assume:

$$\operatorname{rank}(\operatorname{Hol}(D_i))\geq 2 \text{ for each } i. \tag{12}$$

We also fix a point $o\in D$.

Recall (§2) that the group $\operatorname{Hol}(D)$ of holomorphic automorphisms of D is a semisimple Lie group without compact factors, and that there exists a unique homomorphism $u\colon U^1\to\operatorname{Hol}(D)$ such that $u(z)$ fixes o and acts as multiplication by z on $T_o(D)$. That Γ is arithmetic means that there exists a simply connected algebraic group H over $\mathbb{Q}$ and a surjective homomorphism

$\varphi\colon H(\mathbb{R}) \to \mathrm{Hol}(D)^+$ with compact kernel such that Γ is commensurable with $\varphi(H(\mathbb{Z}))$. The Margulis superrigidity theorem implies that the pair (H,φ) is unique up to a unique isomorphism (see 3.12).

Let

$$H_{\mathbb{R}}^{\mathrm{ad}} = H_{\mathrm{c}} \times H_{\mathrm{nc}}$$

where H_{c} (resp. H_{nc}) is the product of the compact (resp. noncompact) simple factors of $H_{\mathbb{R}}^{\mathrm{ad}}$. The homomorphism $\varphi(\mathbb{R})\colon H(\mathbb{R}) \to \mathrm{Hol}(D)^+$ factors through $H_{\mathrm{nc}}(\mathbb{R})^+$, and defines an isomorphism of Lie groups $H_{\mathrm{nc}}(\mathbb{R})^+ \to \mathrm{Hol}(D)^+$. Let $\bar{h}$ denote the homomorphism $\mathbb{S}/\mathbb{G}_m \to H_{\mathbb{R}}^{\mathrm{ad}}$ whose projection into H_{c} is trivial and whose projection into H_{nc} corresponds to u. In other words,

$$\bar{h}(z) = (h_{\mathrm{c}}(z), h_{\mathrm{nc}}(z)) \in H_{\mathrm{c}}(\mathbb{R}) \times H_{\mathrm{nc}}(\mathbb{R}) \tag{13}$$

where $h_{\mathrm{c}}(z) = 1$ and $h_{\mathrm{nc}}(z) = u(z/\bar{z})$ in $H_{\mathrm{nc}}(\mathbb{R})^+ \simeq \mathrm{Hol}(D)^+$. The map ${}^{g}h \mapsto go$ identifies D with the set of $H^{\mathrm{ad}}(\mathbb{R})^+$-conjugates of $\bar{h}$ (Theorem 2.6).

Let (V, F) be a polarizable variation of integral Hodge structures on $D(\Gamma)$, and let $V = \mathsf{V}_{\pi(o)}$. Then $\pi^*\mathsf{V} \simeq V_D$ where $\pi\colon D \to \Gamma\backslash D$ is the quotient map. Let $G \subset \mathrm{GL}_V$ be the generic Mumford-Tate group of (V, F) (see p. 6.14), and let $\mathfrak{t}$ be a family of tensors of V (in the sense of §7), including a polarization t_0, such that G is the subgroup of $\mathrm{GL}_V \times \mathrm{GL}_{\mathbb{Q}(1)}$ fixing the elements of $\mathfrak{t}$. As G contains the Mumford-Tate group at each point of D, $\mathfrak{t}$ is a family of Hodge tensors of (V_D, F). The period map $\mathcal{P}\colon D \to D(V, h_o, \mathfrak{t})$ is holomorphic (Theorem 7.7).

We now assume that the monodromy map $\varphi'\colon \Gamma \to \mathrm{GL}(V)$ has finite kernel, and we pass to a finite covering, so that $\Gamma \subset G(\mathbb{Q})$. Now the elements of $\mathfrak{t}$ are Hodge tensors of (V, F).

There exists an arithmetic subgroup Γ' of $H(\mathbb{Q})$ such that $\varphi(\Gamma') \subset \Gamma$. The Margulis superrigidity theorem 3.8, shows that there is a (unique) homomorphism $\varphi''\colon H \to G$ of algebraic groups that agrees with $\varphi' \circ \varphi$ on a subgroup of finite index in Γ',

$$\begin{array}{ccccc} H(\mathbb{Q})^+ & \xrightarrow{\varphi} & \mathrm{Hol}(D)^+ & & \\ \cup & & \cup & & \\ \Gamma' & \xrightarrow{\varphi|\Gamma'} & \Gamma & \xrightarrow{\varphi'} & G(\mathbb{Q}) \end{array} \qquad \begin{array}{ccc} H & & \\ & \searrow \varphi'' & \\ & & G. \end{array}$$

It follows from the Borel density theorem 3.9 that $\varphi''(H)$ is the connected monodromy group at each point of $D(\Gamma)$. Hence $H \subset G^{\mathrm{der}}$, and the two groups are equal if the Mumford-Tate group at some point of $D(\Gamma)$ is commutative (Theorem 6.19). When we assume that, the homomorphism $\varphi''\colon H \to G$ induces an isogeny $H \to G^{\mathrm{der}}$, and hence[30] an isomorphism $H^{\mathrm{ad}} \to G^{\mathrm{ad}}$. Let $(V, h_o) = (\mathsf{V}, F)_o$. Then

$$\mathrm{ad} \circ h_o\colon \mathbb{S} \to G_{\mathbb{R}}^{\mathrm{ad}} \simeq H^{\mathrm{ad}}$$

equals $\bar{h}$. Thus, we have a commutative diagram

$$\begin{tikzcd} H \arrow[d] \arrow[dr] & & \\ (H^{\mathrm{ad}}, \bar{h}) & (G, h) \arrow[l] \arrow[r, hook, "\rho"] & \mathrm{GL}_V \end{tikzcd} \tag{14}$$

[30]Let G be a reductive group. The algebraic subgroup $Z(G) \cdot G^{\mathrm{der}}$ is normal, and the quotient $G/\big(Z(G)^\circ \cdot G^{\mathrm{der}}\big)$ is both semisimple and commutative, and hence is trivial. Therefore $G = Z(G)^\circ \cdot G^{\mathrm{der}}$, from which it follows that $Z(G^{\mathrm{der}}) = Z(G) \cap G^{\mathrm{der}}$. For any isogeny $H \to G^{\mathrm{der}}$, the map $H^{\mathrm{ad}} \to (G^{\mathrm{der}})^{\mathrm{ad}}$ is certainly an isomorphism, and we have just shown that $(G^{\mathrm{der}})^{\mathrm{ad}} \to G^{\mathrm{ad}}$ is an isomorphism. Therefore $H^{\mathrm{ad}} \to G^{\mathrm{ad}}$ is an isomorphism.

in which G is a reductive group, the homomorphism $H \to G$ has image G^{der}, w_h is defined over $\mathbb{Q}$, and h satisfies (SV2*).

Conversely, suppose that we are given such a diagram (14). Choose a family $\mathfrak{t}$ of tensors for V, including a polarization, such that G is the subgroup of $\mathrm{GL}_V \times G_{\mathbb{Q}(1)}$ fixing the tensors. Then we get a period subdomain $D(V,h,\mathfrak{t})$ and a canonical variation of Hodge structures (V, F) on it. Pull this back to D using the period isomorphism, and descend it to a variation of Hodge structures on $D(\Gamma)$. The monodromy representation is injective, and some fibre is of CM-type by Proposition 8.4.

SUMMARY 8.5. *Let $D(\Gamma)$ be an arithmetic locally symmetric domain satisfying the condition (12) and fix a point $o \in D$. To give*

> *a polarizable variation of integral Hodge structures on $D(\Gamma)$ such that some fibre is of CM-type and the monodromy representation has finite kernel*

is the same as giving

> *a diagram (14) in which G is a reductive group, the homomorphism $H \to G$ has image G^{der}, w_h is defined over $\mathbb{Q}$, and h satisfies (SV2*).*

REMARK 8.6. When H is almost simple, it is not necessary to require the existence of a fibre of CM-type — it is automatic.

QUESTION 8.7. For which arithmetic locally symmetric varieties $D(\Gamma)$ is it possible to find a diagram (14) such that the corresponding variation of Hodge structures underlies a family of algebraic varieties? or, more generally, a family of motives?

In §§10,11, we shall answer Question 8.7 completely when "algebraic variety" and "motive" are replaced with "abelian variety" and "abelian motive".

Existence of variations of Hodge structures

In this subsection, we show that, for any arithmetic locally symmetric variety, there always exists a diagram (14), and hence a variation of polarizable integral Hodge structures on the variety.

PROPOSITION 8.8. *Let H be a semisimple algebraic group over $\mathbb{Q}$, and let $\bar{h}\colon \mathbb{S} \to H^{\mathrm{ad}}$ be a homomorphism satisfying (SV1,2,3). Then there exists a reductive algebraic group G over $\mathbb{Q}$ and a homomorphism $h\colon \mathbb{S} \to G_{\mathbb{R}}$ such that*

(a) *$G^{\mathrm{der}} = H$ and $\bar{h} = \mathrm{ad} \circ h$,*
(b) *the weight w_h is defined over $\mathbb{Q}$, and*
(c) *the centre of G is split by a CM field.*[31]

PROOF. We shall need the following statement:

> Let G be a reductive group over a field k (of characteristic zero), and let L be a finite Galois extension of k splitting G. Let $G' \to G^{\mathrm{der}}$ be a covering of the derived group of G. Then there exists a central extension
>
> $$1 \to N \to G_1 \to G \to 1$$
>
> such that G_1 is a reductive group, N is a product of copies of $(\mathbb{G}_m)_{L/k}$, and $(G_1^{\mathrm{der}} \to G^{\mathrm{der}}) = (G' \to G^{\mathrm{der}})$. (See Milne and Shih 1982, 3.1)

[31] A field E is CM if it is a totally imaginary quadratic extension of a totally real field.

A number field L is CM if it admits a nontrivial involution ι_L such that $\sigma \circ \iota_L = \iota \circ \sigma$ for every homomorphism $\sigma: L \to \mathbb{C}$. We may replace $\bar{h}$ with an $H^{\mathrm{ad}}(\mathbb{R})^+$-conjugate, and so assume (by Proposition 8.4) that there exists a maximal torus $\bar{T}$ of H^{ad} such that $\bar{h}$ factors through $\bar{T}_{\mathbb{R}}$. Then $\bar{T}_{\mathbb{R}}$ is anisotropic (by (SV2)), and so ι acts as -1 on $X^*(\bar{T})$. It follows that, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\sigma\iota$ and $\iota\sigma$ have the same action on $X^*(\bar{T})$, and so $\bar{T}$ splits over a CM-field L, which can be chosen to be Galois over $\mathbb{Q}$. From the statement, there exists a reductive group G and a central extension

$$1 \to N \to G \to H^{\mathrm{ad}} \to 1$$

such that $G^{\mathrm{der}} = H$ and N is a product of copies of $(\mathbb{G}_m)_{L/\mathbb{Q}}$. The inverse image T of $\bar{T}$ in G is a maximal torus, and the kernel of $T \twoheadrightarrow \bar{T}$ is N. Because N is connected, there exists a $\mu \in X_*(T)$ lifting $\mu_{\bar{h}} \in X_*(\bar{T})$.[32] The weight $w = -\mu - \iota\mu$ of μ lies in $X_*(Z)$, where $Z = Z(G) = N$. Clearly $\iota w = w$ and so, as the Tate cohomology group[33] $H_T^0(\mathbb{R}, X_*(Z)) = 0$, there exists a $\mu_0 \in X_*(Z)$ such that $(\iota + 1)\mu_0 = w$. When we replace μ with $\mu - \mu_0$, we find that $w = 0$; in particular, w is defined over $\mathbb{Q}$. Let $h: \mathbb{S} \to G_{\mathbb{R}}$ correspond to μ as in (5), p. 20. Then (G, h) fulfils the requirements. □

COROLLARY 8.9. *For any semisimple algebraic group H over $\mathbb{Q}$ and homomorphism $\bar{h}: \mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}_{\mathbb{R}}$ satisfying (SV1,2,3), there exists a reductive group G with $G^{\mathrm{der}} = H$ and a homomorphism $h: \mathbb{S} \to G_{\mathbb{R}}$ lifting $\bar{h}$ and satisfying (SV1,2*,3).*

PROOF. Let (G, h) be as in the proposition. Then G/G^{der} is a torus, and we let T be the smallest subtorus of it such that $T_{\mathbb{R}}$ contains the image of h. Then $T_{\mathbb{R}}$ is anisotropic, and when we replace G with the inverse image of T, we obtain a pair (G, h) satisfying (SV1,2*,3). □

Let G be a reductive group over $\mathbb{Q}$, and let $h: \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism satisfying (SV1,2,3). The homomorphism h is said to be ***special*** if $h(\mathbb{S}) \subset T_{\mathbb{R}}$ for some torus $T \subset G$.[34] In this case, there is a smallest such T, and when (T, h) is the Mumford-Tate group of a CM Hodge structure we say that h is ***CM***.

PROPOSITION 8.10. *Let $h: \mathbb{S} \to G_{\mathbb{R}}$ be special. Then h is CM if*

(a) *w_h is defined over $\mathbb{Q}$, and*
(b) *the connected centre of G is split by a CM-field.*

PROOF. It is known that a special h is CM if and only if it satisfies the Serre condition:

$$(\tau - 1)(\iota + 1)\mu_h = 0 = (\iota + 1)(\tau - 1)\mu_h \text{ for all } \tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}).$$

As $w_h = (\iota + 1)\mu_h$, the first condition says that

$$(\tau - 1)(\iota + 1)\mu_h = 0 \text{ for all } \tau \in \mathrm{Aut}(\mathbb{C}),$$

[32]The functor X^* is exact, and so

$$0 \to X^*(T') \to X^*(T) \to X^*(N) \to 0$$

is exact. In fact, it is split-exact because $X^*(N)$ is torsion-free. On applying $\mathrm{Hom}(-, \mathbb{Z})$ to it, we get the exact sequence

$$0 \to X_*(N) \to X_*(T) \to X_*(T') \to 0.$$

[33]Let $g = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The g-module $X_*(Z)$ is induced, and so the Tate cohomology group $H_T^0(g, X_*(Z)) = 0$. By definition, $H_T^0(g, X_*(Z)) = X_*(Z)^g/(\iota + 1)X_*(Z)$.

[34]Of course, $h(\mathbb{S})$ is always contained in a subtorus of $G_{\mathbb{R}}$, even a maximal subtorus; the point is that there should exist such a torus defined over $\mathbb{Q}$.

and the second condition implies that

$$\tau\iota\mu_h = \iota\tau\mu_h \text{ for all } \tau \in \mathrm{Aut}(\mathbb{C}).$$

Let $T \subset G$ be a maximal torus such that $h(\mathbb{S}) \subset T_{\mathbb{R}}$. The argument in the proof of (8.8) shows that $\tau\iota\mu = \iota\tau\mu$ for $\mu \in X_*(T)$, and since

$$X_*(T)_{\mathbb{Q}} = X_*(Z)_{\mathbb{Q}} \oplus X_*(T/Z)_{\mathbb{Q}}$$

we see that the same equation holds for $\mu \in X_*(T)$. Therefore $(\iota+1)(\tau-1)\mu = (\tau-1)(\iota+1)\mu$, and we have already observed that this is zero. □

9. Absolute Hodge classes and motives

In order to be able to realize all but a handful of Shimura varieties as moduli varieties, we shall need to replace algebraic varieties and algebraic classes by more general objects, namely, by motives and absolute Hodge classes.

The standard cohomology theories

Let X be a smooth complete[35] algebraic variety over an algebraically closed field k (of characteristic zero as always).

For each prime number ℓ, the étale cohomology groups[36] $H^r_\ell(X)(m) \stackrel{\text{def}}{=} H^r_\ell(X_{\mathrm{et}}, \mathbb{Q}_\ell(m))$ are finite dimensional $\mathbb{Q}_\ell$-vector spaces. For any homomorphism $\sigma\colon k \to k'$ of algebraically closed fields, there is a canonical base change isomorphism

$$H^r_\ell(X)(m) \xrightarrow{\sigma} H^r_\ell(\sigma X)(m), \quad \sigma X \stackrel{\text{def}}{=} X \otimes_{k,\sigma} k'. \tag{15}$$

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

$$\mathbb{Q}_\ell \otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_\ell(X)(m). \tag{16}$$

Here $H^r_B(X)$ denotes the Betti cohomology group $H^r(X^{\mathrm{an}}, \mathbb{Q})$.

The de Rham cohomology groups $H^r_{\mathrm{dR}}(X)(m) \stackrel{\text{def}}{=} \mathbb{H}^r(X_{\mathrm{Zar}}, \Omega^\bullet_{X/k})(m)$ are finite dimensional k-vector spaces. For any homomorphism $\sigma\colon k \to k'$ of fields, there is a canonical base change isomorphism

$$k' \otimes_k H^r_{\mathrm{dR}}(X)(m) \xrightarrow{\sigma} H^r_{\mathrm{dR}}(\sigma X)(m). \tag{17}$$

When $k = \mathbb{C}$, there is a canonical comparison isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_{\mathrm{dR}}(X)(m). \tag{18}$$

We let $H^r_{k\times\mathbb{A}_f}(X)(m)$ denote the product of $H^r_{\mathrm{dR}}(X)(m)$ with the restricted product of the topological spaces $H^r_\ell(X)(m)$ relative to their subspaces $H^r(X_{\mathrm{et}}, \mathbb{Z}_\ell)(m)$. This is a finitely generated

[35]Many statements hold without this hypothesis, but we shall need to consider only this case.

[36]The “(m)” denotes a Tate twist. Specifically, for Betti cohomology it denotes the tensor product with the Tate Hodge structure $\mathbb{Q}(m)$, for de Rham cohomology it denotes a shift in the numbering of the filtration, and for étale cohomology it denotes a change in Galois action by a multiple of the cyclotomic character.

free module over the ring $k \times \mathbb{A}_f$. For any homomorphism $\sigma\colon k \to k'$ of algebraically closed fields, the maps (15) and (17) give a base change homomorphism

$$H^r_{k\times\mathbb{A}_f}(X)(m) \xrightarrow{\sigma} H^r_{k'\times\mathbb{A}_f}(\sigma X)(m). \tag{19}$$

When $k = \mathbb{C}$, the maps (16) and (18) give a comparison isomorphism

$$(\mathbb{C}\times\mathbb{A}_f)\otimes_{\mathbb{Q}} H^r_B(X)(m) \to H^r_{\mathbb{C}\times\mathbb{A}_f}(X)(m). \tag{20}$$

NOTES. For more details and references, see Deligne 1982, §1.

Absolute Hodge classes

Let X be a smooth complete algebraic variety over $\mathbb{C}$. The cohomology group $H^{2r}_B(X)(r)$ has a Hodge structure of weight 0, and an element of type $(0,0)$ in it is called a ***Hodge class of codimension*** r on X.[37] We wish to extend this notion to all base fields of characteristic zero. Of course, given a variety X over a field k, we can choose a homomorphism $\sigma\colon k \to \mathbb{C}$ and define a Hodge class on X to be a Hodge class on σX, but this notion depends on the choice of the embedding. Deligne's idea for avoiding this problem is to use all embeddings (Deligne 1979a, 0.7).

$$\begin{tikzcd}
H^{2r}_B(\sigma X)(r)\cap H^{0,0} \arrow[r, hook, "(20)"] & H^{2r}_{\mathbb{C}\times\mathbb{A}_f}(\sigma X)(r) \\
AH^r(X) \arrow[u, dashed] \arrow[r, hook] & H^{2r}_{k\times\mathbb{A}_f}(X)(r) \arrow[u, "(19)"', "\sigma"]
\end{tikzcd}$$

Let X be a smooth complete algebraic variety over an algebraically closed field k of characteristic zero, and let σ be a homomorphism $k \to \mathbb{C}$. An element γ of $H^{2r}_{k\times\mathbb{A}_f}(X)(r)$ is a σ***-Hodge class of codimension*** r if $\sigma\gamma$ lies in the subspace $H^{2r}_B(\sigma X)(r)\cap H^{0,0}$ of $H^{2r}_{\mathbb{C}\times\mathbb{A}_f}(\sigma X)(r)$. When k has finite transcendence degree over $\mathbb{Q}$, an element γ of $H^{2r}_{k\times\mathbb{A}}(X)(r)$ is an ***absolute Hodge class*** if it is σ-Hodge for all homomorphisms $\sigma\colon k \to \mathbb{C}$. The absolute Hodge classes of codimension r on X form a $\mathbb{Q}$-subspace $AH^r(X)$ of $H^{2r}_{k\times\mathbb{A}_f}(X)(r)$.

We list the basic properties of absolute Hodge classes.

9.1. The inclusion $AH^r(X) \subset H^{2r}_{k\times\mathbb{A}_f}(X)(r)$ induces an injective map

$$\left(k\times\mathbb{A}_f\right)\otimes_{\mathbb{Q}} AH^r(X) \to H^{2r}_{k\times\mathbb{A}_f}(X)(r);$$

in particular $AH^r(X)$ is a finite dimensional $\mathbb{Q}$-vector space.

This follows from (20) because $AH^r(X)$ is isomorphic to a $\mathbb{Q}$-subspace of $H^{2r}_B(\sigma X)(r)$ (each σ).

9.2. For any homomorphism $\sigma\colon k \to k'$ of algebraically closed fields of finite transcendence degree over $\mathbb{Q}$, the map (19) induces an isomorphism $AH^r(X) \to AH^r(\sigma X)$ (Deligne 1982, 2.9a).

This allows us to define $AH^r(X)$ for a smooth complete variety over an arbitrary algebraically closed field k of characteristic zero: choose a model X_0 of X over an algebraically closed subfield k_0 of k of finite transcendence degree over $\mathbb{Q}$, and define $AH^r(X)$ to be the image of $AH^r(X_0)$ under the map $H^{2r}_{k_0\times\mathbb{A}_f}(X_0)(r) \to H^{2r}_{k\times\mathbb{A}_f}(X)(r)$. With this definition, (9.2) holds for all homomorphisms of algebraically closed fields k of characteristic zero. Moreover, if k admits an embedding in $\mathbb{C}$, then a cohomology class is absolutely Hodge if and only if it is σ-Hodge for every such embedding.

[37] As $H^{2r}_B(X)(r) \simeq H^{2r}_B(X)\otimes\mathbb{Q}(r)$, this is essentially the same as an element of $H^{2r}_B(X)$ of type (r,r).

9.3. The cohomology class of an algebraic cycle on X is absolutely Hodge; thus, the algebraic cohomology classes of codimension r on X form a $\mathbb{Q}$-subspace $A^r(X)$ of $AH^r(X)$ (Deligne 1982, 2.1a).

9.4. The Künneth components of the diagonal are absolute Hodge classes (ibid., 2.1b).

9.5. Let X_0 be a model of X over a subfield k_0 of k such that k is algebraic over k_0; then $\mathrm{Gal}(k/k_0)$ acts on $AH^r(X)$ through a finite discrete quotient (ibid. 2.9b).

9.6. Let

$$AH^*(X) = \bigoplus\nolimits_{r\geq 0} AH^r(X);$$

then $AH^*(X)$ is a $\mathbb{Q}$-subalgebra of $\bigoplus H^{2r}_{k\times\mathbb{A}_f}(X)(r)$. For any regular map $\alpha\colon Y\to X$ of complete nonsingular varieties, the maps α_* and α^* send absolute Hodge classes to absolute Hodge classes. (This follows easily from the definitions.)

THEOREM 9.7 (DELIGNE 1982, 2.12, 2.14). *Let S be a smooth connected algebraic variety over $\mathbb{C}$, and let $\pi\colon X\to S$ be a smooth proper morphism. Let $\gamma\in\Gamma(S,R^{2r}\pi_*\mathbb{Q}(r))$, and let γ_s be the image of γ in $H^{2r}_B(X_s)(r)$ ($s\in S(\mathbb{C})$).*

(a) *If γ_s is a Hodge class for one $s\in S(\mathbb{C})$, then it is a Hodge class for every $s\in S(\mathbb{C})$.*
(b) *If γ_s is an absolute Hodge class for one $s\in S(\mathbb{C})$, then it is an absolute Hodge class for every $s\in S(\mathbb{C})$.*

PROOF. Let $\bar{X}$ be a smooth compactification of X whose boundary $\bar{X}\smallsetminus X$ is a union of smooth divisors with normal crossings, and let $s\in S(\mathbb{C})$. According to Deligne 1971b, 4.1.1, 4.1.2, there are maps

$$H^{2r}_B(\bar{X})(r)\xrightarrow{\text{onto}}\Gamma(S,R^{2r}\pi_*\mathbb{Q}(r))\xrightarrow{\text{injective}}H^{2r}_B(X_s)(r)$$

whose composite $H^{2r}_B(\bar{X})(r)\to H^{2r}_B(X_s)(r)$ is defined by the inclusion $X_s\hookrightarrow\bar{X}$; moreover $\Gamma(S,R^{2r}\pi_*\mathbb{Q}(r))$ has a Hodge structure (independent of s) for which the injective maps are morphisms of Hodge structures (theorem of the fixed part).

Let $\gamma\in\Gamma(S,R^{2r}\pi_*\mathbb{Q}(r))$. If γ_s is of type $(0,0)$ for one s, then so also is γ; then γ_s is of type $(0,0)$ for all s. This proves (a).

Let σ be an automorphism of $\mathbb{C}$ (as an abstract field). It suffices to prove (b) with "absolute Hodge" replaced with "σ-Hodge". We shall use the commutative diagram ($\mathbb{A}=\mathbb{C}\times\mathbb{A}_f$):

$$\begin{tikzcd}
H^{2r}_B(\bar{X})(r) \arrow[r, "\text{onto}"] \arrow[d, "?\mapsto ?_{\mathbb{A}}"] & \Gamma(S,R^{2r}\pi_*\mathbb{Q}(r)) \arrow[r, "\text{injective}"] \arrow[d] & H^{2r}_B(X_s)(r) \arrow[d] \\
H^{2r}_{\mathbb{A}}(\bar{X})(r) \arrow[r, "\text{onto}"] \arrow[d, "\sigma"] & \Gamma(S,R^{2r}\pi_*\mathbb{A}(r)) \arrow[r, "\text{injective}"] \arrow[d, "\sigma"] & H^{2r}_{\mathbb{A}}(X_s)(r) \arrow[d, "\sigma"] \\
H^{2r}_{\mathbb{A}}(\sigma\bar{X})(r) \arrow[r, "\text{onto}"] & \Gamma(\sigma S,R^{2r}(\sigma\pi)_*\mathbb{A}(r)) \arrow[r, "\text{injective}"] & H^{2r}_{\mathbb{A}}(\sigma X_s)(r) \\
H^{2r}_B(\sigma\bar{X})(r) \arrow[r, "\text{onto}"] \arrow[u] & \Gamma(\sigma S,R^{2r}(\sigma\pi)_*\mathbb{Q}(r)) \arrow[r, "\text{injective}"] \arrow[u] & H^{2r}_B(\sigma X_s)(r). \arrow[u]
\end{tikzcd}$$

The middle map σ uses a relative version of the base change map (19). The other maps σ are the base change isomorphisms and the remaining vertical maps are essential tensoring with $\mathbb{A}$ (and are denoted $? \mapsto ?_{\mathbb{A}}$).

Let γ be an element of $\Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$ such that γ_s is σ-Hodge for one s. Recall that this means that there is a $\gamma_s^\sigma \in H_B^{2r}(\sigma X_s)(r)$ of type (0,0) such that $\left(\gamma_s^\sigma\right)_{\mathbb{A}} = \sigma(\gamma_s)_{\mathbb{A}}$ in $H_{\mathbb{A}}^{2r}(\sigma X_s)(r)$. As γ_s is in the image of

$$H_B^{2r}(\bar{X})(r) \to H_B^{2r}(X_s)(r),$$

$\sigma(\gamma_s)_{\mathbb{A}}$ is in the image of

$$H_{\mathbb{A}}^{2r}(\sigma\bar{X})(r) \to H_{\mathbb{A}}^{2r}(\sigma X_s)(r).$$

Therefore $\left(\gamma_s^\sigma\right)_{\mathbb{A}}$ is also, which implies (by linear algebra[38]) that γ_s^σ is in the image of

$$H_B^{2r}(\sigma\bar{X})(r) \to H_B^{2r}(\sigma X_s)(r).$$

Let $\widetilde{\gamma}^\sigma$ be a pre-image of γ_s^σ in $H_B^{2r}(\sigma\bar{X})(r)$.

Let s' be a second point of S, and let $\widetilde{\gamma}_{s'}^\sigma$ be the image of $\widetilde{\gamma}^\sigma$ in $H_B^{2r}(\sigma X_{s'})(r)$. By construction, $(\widetilde{\gamma}^\sigma)_{\mathbb{A}}$ maps to $\sigma\gamma_{\mathbb{A}}$ in $\Gamma(\sigma S, R^{2r}(\sigma\pi)_*\mathbb{A}(r))$, and so $\left(\widetilde{\gamma}_{s'}^\sigma\right)_{\mathbb{A}} = \sigma(\gamma_{s'})_{\mathbb{A}}$ in $H_{\mathbb{A}}^{2r}(\sigma X_{s'})(r)$, which demonstrates that $\gamma_{s'}$ is σ-Hodge. □

CONJECTURE 9.8 (DELIGNE 1979a, 0.10). *Every σ-Hodge class on a smooth complete variety over an algebraically closed field of characteristic zero is absolutely Hodge, i.e.,*

$$\sigma\text{-Hodge (for one } \sigma) \implies \text{absolutely Hodge.}$$

THEOREM 9.9 (DELIGNE 1982, 2.11). *Conjecture 9.8 is true for abelian varieties.*

To prove the theorem, it suffices to show that every Hodge class on an abelian variety over $\mathbb{C}$ is absolutely Hodge.[39] We defer the proof of the theorem to the next subsection.

ASIDE 9.10. Let $X_{\mathbb{C}}$ be a smooth complete algebraic variety over $\mathbb{C}$. Then $X_{\mathbb{C}}$ has a model X_0 over a subfield k_0 of $\mathbb{C}$ finitely generated over $\mathbb{Q}$. Let k be the algebraic closure of k_0 in $\mathbb{C}$, and let $X = X_{0k}$. For a prime number ℓ, let

$$\mathcal{T}_\ell^r(X) = \bigcup\nolimits_U H_\ell^{2r}(X)(r)^U \qquad \text{(space of Tate classes)}$$

where U runs over the open subgroups of $\mathrm{Gal}(k/k_0)$ — as the notation suggests, $\mathcal{T}_\ell^r(X)$ depends only on X/k. The Tate conjecture (Tate 1964, Conjecture 1) says that the $\mathbb{Q}_\ell$-vector space $\mathcal{T}_\ell^r(X)$ is spanned by algebraic classes. Statement 9.5 implies that $AH^r(X)$ projects into $\mathcal{T}_\ell^r(X)$, and (9.1) implies that the map $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} AH^r(X) \to \mathcal{T}_\ell^r(X)$ is injective. Therefore the Tate conjecture implies that $A^r(X) = AH^r(X)$, and so the Tate conjecture for X and one ℓ implies that all absolute Hodge classes on $X_{\mathbb{C}}$ are algebraic. Thus, in the presence of Conjecture 9.8, the Tate conjecture implies the Hodge conjecture. In particular, Theorem 9.9 shows that, for an abelian variety, the Tate conjecture implies the Hodge conjecture.

[38] Apply the following elementary statement:

> Let E, W, and V be vector spaces, and let $\alpha: W \to V$ be a linear map; let $v \in V$; if $e \otimes v$ is in the image of $1 \otimes \alpha: E \otimes W \to E \otimes V$ for some nonzero $e \in E$, then v is in the image of α.

To prove the statement, choose a basis $(e_i)_{i \in I}$ for E with $e_0 = e$ for some $0 \in I$. Then $(1 \otimes \alpha)(\sum e_i \otimes w_i) = \sum e_i \otimes \alpha(w_i)$, which equals $e_0 \otimes v$ if and only if $\alpha(w_0) = v$ and $\alpha(w_i) = 0$ for $i \neq 0$.

[39] Let A be an abelian variety over k, and suppose that γ is σ_0-Hodge for some homomorphism $\sigma_0: k \to \mathbb{C}$. We have to show that it is σ-Hodge for every $\sigma: k \to \mathbb{C}$. But, using the axiom of choice (!), one can show that there exists a homomorphism $\sigma': \mathbb{C} \to \mathbb{C}$ such that $\sigma = \sigma' \circ \sigma_0$. Now γ is σ-Hodge if and only if $\sigma_0\gamma$ is σ'-Hodge.

Proof of Deligne's theorem

It is convenient to prove Theorem 9.9 in the following more abstract form.

THEOREM 9.11. *Suppose that for each abelian variety A over $\mathbb{C}$ we have a $\mathbb{Q}$-subspace $C^r(A)$ of the Hodge classes of codimension r on A. Assume:*

(a) *$C^r(A)$ contains all algebraic classes of codimension r on A;*
(b) *pull-back by a homomorphism $\alpha: A \to B$ of abelian varieties maps $C^r(B)$ into $C^r(A)$;*
(c) *let $\pi: \mathcal{A} \to S$ be an abelian scheme over a connected smooth complex algebraic variety S, and let $t \in \Gamma(S, R^{2r}\pi_*\mathbb{Q}(r))$; if t_s lies in $C^r(A_s)$ for one $s \in S(\mathbb{C})$, then it lies in $C^r(A_s)$ for all s.*

Then $C^r(A)$ contains all the Hodge classes of codimension r on A.

COROLLARY 9.12. *If hypothesis (c) of the theorem holds for algebraic classes on abelian varieties, then the Hodge conjecture holds for abelian varieties. (In other words, for abelian varieties, the variational Hodge conjecture implies the Hodge conjecture.)*

PROOF. Immediate consequence of the theorem, because the algebraic classes satisfy (a) and (b).□

The proof of Theorem 9.11 requires four steps.

STEP 1: THE HODGE CONJECTURE HOLDS FOR POWERS OF AN ELLIPTIC CURVE

As Tate observed (1964, p. 19), the $\mathbb{Q}$-algebra of Hodge classes on a power of an elliptic curve is generated by those of type $(1,1)$.[40] These are algebraic by a theorem of Lefschetz.

STEP 2: SPLIT WEIL CLASSES LIE IN C

Let A be a complex abelian variety, and let ν be a homomorphism from a CM-field E into $\mathrm{End}(A)_{\mathbb{Q}}$. The pair (A,ν) is said to be of ***Weil type*** if the tangent space $T_0(A)$ is a free $E \otimes_{\mathbb{Q}} \mathbb{C}$-module. In this case, $d \stackrel{\text{def}}{=} \dim_E H^1_B(A)$ is even and the subspace $\bigwedge_E^d H^1_B(A)(\frac{d}{2})$ of $H^d_B(A)(\frac{d}{2})$ consists of Hodge classes (Deligne 1982, 4.4). When E is quadratic over $\mathbb{Q}$, these Hodge classes were studied by Weil (1977), and for this reason are called ***Weil classes***. A ***polarization*** of (A,ν) is a polarization λ of A whose whose Rosati involution acts on $\nu(E)$ as complex conjugation. The Riemann form of such a polarization can be written

$$(x,y) \mapsto \mathrm{Tr}_{E/\mathbb{Q}}(f\phi(x,y))$$

for some totally imaginary element f of E and E-hermitian form ϕ on $H_1(A,\mathbb{Q})$. If λ can be chosen so that ϕ is split (i.e., admits a totally isotropic subspace of dimension $d/2$), then the Weil classes are said to be ***split***.

LEMMA 9.13. *All split Weil classes of codimension r on an abelian variety A lie in $C^r(A)$.*

PROOF. Let (A,ν,λ) be a polarized abelian variety of split Weil type. Let $V = H_1(A,\mathbb{Q})$, and let ψ be the Riemann form of λ. The Hodge structures on V for which the elements of E act as morphisms and ψ is a polarization are parametrized by a period subdomain, which is hermitian symmetric domain (cf. 7.8). On dividing by a suitable arithmetic subgroup, we get a smooth proper map $\pi: \mathcal{A} \to S$ of smooth algebraic varieties whose fibres are abelian varieties with an action of E (Theorem 7.11). There is a $\mathbb{Q}$-subspace W of $\Gamma(S, R^d\pi_*\mathbb{Q}(\frac{d}{2}))$ whose fibre at every point s is the

[40] This is most conveniently proved by applying the criterion Milne 1999, 4.8.

space of Weil classes on A_s. One fibre of π is (A,ν) and another is a power of an elliptic curve. Therefore the lemma follows from Step 1 and hypotheses (a,c). (See Deligne 1982, 4.8, for more details.) □

STEP 3: THEOREM 9.11 FOR ABELIAN VARIETIES OF CM-TYPE

A simple abelian variety A is of ***CM-type*** if $\mathrm{End}(A)_{\mathbb{Q}}$ is a field of degree $2\dim A$ over $\mathbb{Q}$, and a general abelian variety is of ***CM-type*** if every simple isogeny factor of it is of CM-type. Equivalently, it is of CM-type if the Hodge structure $H_1(A^{\mathrm{an}},\mathbb{Q})$ is of CM-type. According to André 1992b:

> For any complex abelian variety A of CM-type, there exist complex abelian varieties B_J of CM-type and homomorphisms $A \to B_J$ such that every Hodge class on A is a linear combination of the pull-backs of split Weil classes on the B_J.

Thus Theorem 9.11 for abelian varieties of CM-type follows from Step 2 and hypothesis (b). (See Deligne 1982, §5, for the original proof of this step.)

STEP 4: COMPLETION OF THE PROOF OF THEOREM 9.11

Let t be a Hodge class on a complex abelian variety A. Choose a polarization λ for A. Let $V = H_1(A,\mathbb{Q})$ and let h_A be the homomorphism defining the Hodge structure on $H_1(A,\mathbb{Q})$. Both t and the Riemann form t_0 of λ can be regarded as Hodge tensors for V. The period subdomain $D = D(V,h_A,\{t,t_0\})$ is a hermitian symmetric domain (see 7.8). On dividing by a suitable arithmetic subgroup, we get a smooth proper map $\pi\colon \mathcal{A} \to S$ of smooth algebraic varieties whose fibres are abelian varieties (Theorem 7.11) and a section t of $R^{2r}\pi_*\mathbb{Q}(r)$. For one $s \in S$, the fibre $(\mathcal{A},t)_s = (A,t)$, and another fibre is an abelian variety of CM-type (apply 8.4), and so the theorem follows from Step 3 and hypothesis (c). (See Deligne 1982, §6, for more details.)

Motives for absolute Hodge classes

We fix a base field k of characteristic zero; "variety" will mean "smooth projective variety over k".

For varieties X and Y with X connected, we let

$$C^r(X,Y) = AH^{\dim X + r}(X \times Y)$$

(correspondences of degree r from X to Y). When X has connected components X_i, $i \in I$, we let

$$C^r(X,Y) = \bigoplus\nolimits_{i\in I} C^r(X_i,Y).$$

For varieties X,Y,Z, there is a bilinear pairing

$$f,g \mapsto g\circ f\colon C^r(X,Y)\times C^s(Y,Z) \to C^{r+s}(X,Z)$$

with

$$g\circ f \stackrel{\mathrm{def}}{=} (p_{XZ})_*(p_{XY}^* f \cdot p_{YZ}^* g).$$

Here the p's are projection maps from $X\times Y\times Z$. These pairings are associative and so we get a "category of correspondences", which has one object hX for every variety over k, and whose Homs are defined by

$$\mathrm{Hom}(hX,hY) = C^0(X,Y).$$

Let $f: Y \to X$ be a regular map of varieties. The transpose of the graph of f is an element of $C^0(X,Y)$, and so way $X \rightsquigarrow hX$ becomes a contravariant functor.

The category of correspondences is additive, but not abelian, and so we enlarge it by adding the images of idempotents. More precisely, we define a "category of effective motives", which has one object $h(X,e)$ for each variety X and idempotent e in the ring $\mathrm{End}(hX) = AH^{\dim X}(X \times X)$, and whose Homs are defined by

$$\mathrm{Hom}(h(X,e),h(Y,f)) = f \circ C^0(X,Y) \circ e.$$

This contains the old category by $hX \leftrightarrow h(X,\mathrm{id})$, and $h(X,e)$ is the image of $hX \xrightarrow{e} hX$.

The category of effective motives is abelian, but objects need not have duals. In the enlarged category, the motive $h\mathbb{P}^1$ decomposes into $h\mathbb{P}^1 = h^0\mathbb{P}^1 \oplus h^2\mathbb{P}^1$, and it turns out that, to obtain duals for all objects, we only have to "invert" the motive $h^2\mathbb{P}^1$. This is most conveniently done by defining a "category of motives" which has one object $h(X,e,m)$ for each pair (X,e) as before and integer m, and whose Homs are defined by

$$\mathrm{Hom}(h(X,e,m),h(Y,f,n)) = f \circ C^{n-m}(X,Y) \circ e.$$

This contains the old category by $h(X,e) \leftrightarrow h(X,e,0)$.

We now list some properties of the category $\mathsf{Mot}(k)$ of motives.

9.14. The Hom's in $\mathsf{Mot}(k)$ are finite dimensional $\mathbb{Q}$-vector spaces, and $\mathsf{Mot}(k)$ is a semisimple abelian category.

9.15. Define a tensor product on $\mathsf{Mot}(k)$ by

$$h(X,e,m) \otimes h(X,f,n) = h(X \times Y, e \times f, m+n).$$

With the obvious associativity constraint and a suitable[41] commutativity constraint, $\mathsf{Mot}(k)$ becomes a tannakian category.

9.16. The standard cohomology functors factor through $\mathsf{Mot}(k)$. For example, define

$$\omega_\ell(h(X,e,m)) = e\left(\bigoplus\nolimits_i H^i_\ell(X)(m)\right)$$

(image of e acting on $\bigoplus_i H^i_\ell(X)(m)$). Then ω_ℓ is an exact faithful functor $\mathsf{Mot}(k) \to \mathsf{Vec}_{\mathbb{Q}_\ell}$ commuting with tensor products. Similarly, de Rham cohomology defines an exact tensor functor $\omega_{\mathrm{dR}}: \mathsf{Mot}(k) \to \mathsf{Vec}_k$, and, when $k = \mathbb{C}$, Betti cohomology defines an exact tensor functor $\mathsf{Mot}(k) \to \mathsf{Vec}_{\mathbb{Q}}$. The functors ω_ℓ, ω_{dR}, and ω_B are called the ℓ-adic, de Rham, and Betti fibre functors, and they send a motive to its ℓ-adic, de Rham, or Betti ***realization***.

The Betti fibre functor on $\mathsf{Mot}(\mathbb{C})$ takes values in $\mathsf{Hdg}_{\mathbb{Q}}$, and is faithful (almost by definition). Deligne's conjecture 9.8 is equivalent to saying that it is full.

Abelian motives

Definition 9.17. A motive is ***abelian*** if it lies in the tannakian subcategory $\mathsf{Mot}^{\mathrm{ab}}(k)$ of $\mathsf{Mot}(k)$ generated by the motives of abelian varieties.

The Tate motive, being isomorphic to $\bigwedge^2 h_1 E$ for any elliptic curve E, is an abelian motive. It is known that $h(X)$ is an abelian motive if X is a curve, a unirational variety of dimension ≤ 3, a Fermat hypersurface, or a $K3$ surface.

Deligne's theorem 9.9 implies that $\omega_B: \mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) \to \mathsf{Hdg}_{\mathbb{Q}}$ is fully faithful.

[41] Not the obvious one! It is necessary to change some signs.

CM MOTIVES

DEFINITION 9.18. A motive over $\mathbb{C}$ is of ***CM-type*** if its Hodge realization is of CM-type.

LEMMA 9.19. *Every Hodge structure of CM-type is the Betti realization of an abelian motive.*

PROOF. Elementary (see, for example, Milne 1994a, 4.6). □

Therefore ω_B defines an equivalence from the category of abelian motives of CM-type to the category of Hodge structures of CM-type.

PROPOSITION 9.20. *Let G_{Hdg} (resp. G_{Mab}) be the affine group scheme attached to $\mathsf{Hdg}_{\mathbb{Q}}$ and its forgetful fibre functor (resp. $\mathsf{Mot}^{ab}(\mathbb{C})$ and its Betti fibre functor). The kernel of the homomorphism $G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}}$ defined by the tensor functor $\omega_B\colon \mathsf{Mot}^{ab}(\mathbb{C}) \to \mathsf{Hdg}_{\mathbb{Q}}$ is contained in $(G_{\mathrm{Hdg}})^{\mathrm{der}}$.*

PROOF. Let S be the affine group scheme attached to the category $\mathsf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}}$ of Hodge structures of CM-type and its forgetful fibre functor. The lemma shows that the functor $\mathsf{Hdg}_{\mathbb{Q}}^{\mathrm{cm}} \hookrightarrow \mathsf{Hdg}_{\mathbb{Q}}$ factors through $\mathsf{Mot}^{\mathrm{ab}}(\mathbb{C}) \hookrightarrow \mathsf{Hdg}_{\mathbb{Q}}$, and so $G_{\mathrm{Hdg}} \to S$ factors through $G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}}$:

$$G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}} \twoheadrightarrow S.$$

Hence

$$\mathrm{Ker}(G_{\mathrm{Hdg}} \to G_{\mathrm{Mab}}) \subset \mathrm{Ker}(G_{\mathrm{Hdg}} \twoheadrightarrow S) = \left(G_{\mathrm{Hdg}}\right)^{\mathrm{der}}.$$ □

SPECIAL MOTIVES

DEFINITION 9.21. A motive over $\mathbb{C}$ is ***special*** if its Hodge realization is special (see p. 24).

It follows from (6.5) that the special motives form a tannakian subcategory of $\mathsf{Mot}(k)$, which includes the abelian motives (see 6.8).

QUESTION 9.22. Is every special Hodge structure the Betti realization of a motive?

More explicitly: for each simple special Hodge structure (V,h), does there exist an algebraic variety X over $\mathbb{C}$ and an integer m such that (V,h) is a direct factor of $\bigoplus_{r\geq 0} H_B^r(X)(m)$ and the projection $\bigoplus_{r\geq 0} H_B^r(X)(m) \to V \subset \bigoplus_{r\geq 0} H_B^r(X)(m)$ is an absolute Hodge class on X.

A positive answer to (9.22) would imply that all connected Shimura varieties are moduli varieties for motives (see §11). Apparently, no special motive is known that is not abelian.

FAMILIES OF ABELIAN MOTIVES

For an abelian variety A over k, let

$$\omega_f(A) = \varprojlim A_N(k^{\mathrm{al}}), \quad A_N(k^{\mathrm{al}}) = \mathrm{Ker}(N\colon A(k^{\mathrm{al}}) \to A(k^{\mathrm{al}})).$$

This is a free $\mathbb{A}_f$-module of rank $2\dim A$ with a continuous action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$.

Let S be a smooth connected variety over k, and let $k(S)$ be its function field. Fix an algebraic closure $k(S)^{\mathrm{al}}$ of $k(S)$, and let $k(S)^{\mathrm{un}}$ be the union of the subfields L of $k(S)^{\mathrm{al}}$ such that the normalization of S in L is unramified over S. We say that an action of $\mathrm{Gal}(k(S)^{\mathrm{al}}/k(S))$ on a module is ***unramified*** if it factors throught $\mathrm{Gal}(k(S)^{\mathrm{un}}/k(S))$.

THEOREM 9.23. *Let S be a smooth connected variety over k. The functor $A \rightsquigarrow A_\eta \stackrel{\text{def}}{=} A_{k(S)}$ is a fully faithful functor from the category of families of abelian varieties over S to the category of abelian varieties over $k(S)$, with essential image the abelian varieties B over $k(S)$ such that $\omega_f(B)$ is unramified.*

PROOF. When S has dimension 1, this follows from the theory of Néron models. In general, this theory shows that an abelian variety (or a morphism of abelian varieties) extends to an open subvariety U of S such that $S \smallsetminus U$ has codimension at least 2. Now we can apply[42] Chai and Faltings 1990, I 2.7, V 6.8. □

The functor ω_f extends to a functor on abelian motives such that $\omega_f(h_1 A) = \omega_f(A)$ if A is an abelian variety.

DEFINITION 9.24. Let S be a smooth connected variety over k. A ***family M of abelian motives*** over S is an abelian motive M_η over $k(S)$ such that $\omega_f(M_\eta)$ is unramified.

Let M be a family of motives over a smooth connected variety S, and let $\overline{\eta} = \mathrm{Spec}(k(S)^{\mathrm{al}})$. The fundamental group $\pi_1(S,\overline{\eta}) = \mathrm{Gal}(k(S)^{\mathrm{un}}/k(S))$, and so the representation of $\pi_1(S,\overline{\eta})$ on $\omega_f(M_\eta)$ defines a local system of $\mathbb{A}_f$-modules $\omega_f(M)$. Less obvious is that, when the ground field is $\mathbb{C}$, M defines a polarizable variation of Hodge structures on S, $\mathcal{H}_B(M/S)$. When M can be represented in the form (A,p,m) on S, this is obvious. However, M can always be represented in this fashion on an open subset of S, and the underlying local system of $\mathbb{Q}$-vector spaces extends to the whole of S because the monodromy representation is unramified. Now it is possible to show that the variation of Hodge structures itself extends (uniquely) to the whole of S, by using results from Schmid 1973, Cattani et al. 1986, and Griffiths and Schmid 1969. See Milne 1994b, 2.40, for the details.

THEOREM 9.25. *Let S be a smooth connected variety over $\mathbb{C}$. The functor sending a family M of abelian motives over S to its associated polarizable Hodge structure is fully faithful, with essential image the variations of Hodge structures (V,F) such that there exists a dense open subset U of S, an integer m, and a family of abelian varieties $f\colon A \to S$ such that (V,F) is a direct summand of $Rf_*\mathbb{Q}$.*

PROOF. This follows from the similar statement (7.11) for families of abelian varieties (see Milne 1994b, 2.42). □

10. Symplectic Representations

In this subsection, we classify the symplectic representations of groups. These were studied by Satake in a series of papers (see especially Satake 1965, 1967, 1980). Our exposition follows that of Deligne 1979b.

In §8 we proved that there exists a correspondence between variations of Hodge structures on locally symmetric varieties and certain commutative diagrams

$$\begin{tikzcd} H \arrow[d] \arrow[dr] & & \\ (H^{\mathrm{ad}},\overline{h}) & (G,h) \arrow[l] \arrow[r, hook, "\rho"] & \mathrm{GL}_V \end{tikzcd} \tag{21}$$

[42] Recall that we are assuming that the base field has characteristic zero — the theorem is false without that condition.

In this section, we study whether there exists such a diagram and a nondegenerate alternating form ψ on V such that $\rho(G) \subset G(\psi)$ and $\rho_{\mathbb{R}} \circ h \in D(\psi)$. Here $G(\psi)$ is the group of ***symplectic similitudes*** (algebraic subgroup of GL_V whose elements fix ψ up to a scalar) and $D(\psi)$ is the Siegel upper half space (set of Hodge structures h on V of type $\{(-1,0),(0,-1)\}$ for which $2\pi i \psi$ is a polarization[43]). Note that $G(\psi)$ is a reductive group whose derived group is the symplectic group $S(\psi)$.

Preliminaries

10.1. The ***universal covering torus*** $\widetilde{T}$ of a torus T is the projective system $(T_n, T_{nm} \xrightarrow{m} T_n)$ in which $T_n = T$ for all n and the indexing set is $\mathbb{N} \smallsetminus \{0\}$ ordered by divisibility. For any algebraic group G,

$$\mathrm{Hom}(\widetilde{T}, G) = \varinjlim_{n \geq 1} \mathrm{Hom}(T_n, G).$$

Concretely, a homomorphism $\widetilde{T} \to G$ is represented by a pair (f,n) with f a homomorphism $T \to G$ and $n \in \mathbb{N} \smallsetminus \{0\}$; two pairs (f,n) and (g,m) represent the same homomorphism $\widetilde{T} \to G$ if and only if $f \circ m = g \circ n$. A homomorphism $f\colon \widetilde{T} \to G$ factors through T if and only if it is represented by a pair $(f,1)$. A homomorphism $\widetilde{\mathbb{G}}_m \to \mathrm{GL}_V$ represented by (μ, n) defines a gradation $V = \bigoplus V_r$, $r \in \frac{1}{n}\mathbb{Z}$, and a homomorphism $\widetilde{\mathbb{S}} \to \mathrm{GL}_V$ represented by (h,n) defines a fractional Hodge decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$ with $p,q \in \frac{1}{n}\mathbb{Z}$.

The real case

Let H be a simply connected real algebraic group without compact factors, and let $\bar{h}$ be a homomorphism $\mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}$ satisfying the conditions (SV1,2), p. 6, and whose projection on each simple factor of H^{ad} is nontrivial.

DEFINITION 10.2. A homomorphism $H \to \mathrm{GL}_V$ with finite kernel is a ***symplectic representation*** of $(H, \bar{h})$ if there exists a commutative diagram

$$\begin{tikzcd} H \arrow[d] \arrow[dr] & & \\ (H^{\mathrm{ad}}, \bar{h}) & (G,h) \arrow[l] \arrow[r] & (G(\psi), D(\psi)), \end{tikzcd}$$

in which G is a reductive group, the homomorphism $H \to G$ has image G^{der}, and ψ is a nondegenerate alternating form on V.

In more detail, this means there exists a real reductive group G, a nondegenerate alternating form ψ on V, and a factorization

$$H \xrightarrow{a} G \xrightarrow{b} \mathrm{GL}_V$$

of $H \to \mathrm{GL}_V$ such that $a(H) = G^{\mathrm{der}}$, $b(G) \subset G(\psi)$, and $b \circ h \in D(\psi)$; the isogeny $H \to G^{\mathrm{der}}$ induces an isomorphism $H^{\mathrm{ad}} \xrightarrow{c} G^{\mathrm{ad}}$ (see footnote 30), and it is required that $\bar{h} = c^{-1} \circ \mathrm{ad} \circ h$.

We shall determine the complex representations of H that occur in the complexification of a symplectic representation (and we shall omit "the complexification of").

[43] This description agrees with that in §2 because of the correspondence in (5.1).

PROPOSITION 10.3. *A homomorphism $H \to \mathrm{GL}_V$ with finite kernel is a symplectic representation of $(H,\bar{h})$ if there exists a commutative diagram*

$$\begin{tikzcd} H \arrow[d] \arrow[dr] & & \\ (H^{\mathrm{ad}},\bar{h}) & (G,h) \arrow[l] \arrow[r, "\rho"] & \mathrm{GL}_V, \end{tikzcd}$$

in which G is a reductive group, the homomorphism $H \to G$ has image G^{der}, and $(V,\rho\circ h)$ has type $\{(-1,0),(0,-1)\}$.

PROOF. Let G' be the algebraic subgroup of G generated by G^{der} and $h(\mathbb{S})$. After replacing G with G', we may suppose that G itself is generated by G^{der} and $h(\mathbb{S})$. Then (G,h) satisfies (SV2*), and it follows from Theorem 2.3 that there exists a polarization ψ of $(V,\rho\circ h)$ such that G maps into $G(\psi)$ (cf. the proof of 6.2). □

Let $(H,\bar{h})$ be as before. The cocharacter $\mu_{\bar{h}}$ of $H^{\mathrm{ad}}_{\mathbb{C}}$ lifts to a fractional cocharacter $\tilde{\mu}$ of $H_{\mathbb{C}}$:

$$\begin{tikzcd} \widetilde{\mathbb{G}}_m \arrow[r, "\tilde{\mu}"] \arrow[d] & H_{\mathbb{C}} \arrow[d, "\mathrm{ad}"] \\ \mathbb{G}_m \arrow[r, "\mu_{\bar{h}}"] & H^{\mathrm{ad}}_{\mathbb{C}}. \end{tikzcd}$$

LEMMA 10.4. *If an irreducible complex representation W of H occurs in a symplectic representation, then $\tilde{\mu}$ has at most two weights a and $a+1$ on W.*

PROOF. Let $H \xrightarrow{\varphi} (G,h) \longrightarrow \mathrm{GL}_V$ be a symplectic representation of $(H,\bar{h})$, and let W be an irreducible direct summand of $V_{\mathbb{C}}$. The homomorphisms $\varphi_{\mathbb{C}}\circ\tilde{\mu}\colon\widetilde{\mathbb{G}}_m \to G_{\mathbb{C}}$ coincides with μ_h when composed with $G_{\mathbb{C}} \to G^{\mathrm{ad}}_{\mathbb{C}}$, and so $\varphi_{\mathbb{C}}\circ\tilde{\mu} = \mu_h\cdot\nu$ with ν central. On V, μ_h has weights $0,1$. If a is the unique weight of ν on W, then the only weights of $\tilde{\mu}$ on W are a and $a+1$. □

LEMMA 10.5. *Assume that H is almost simple. A nontrivial irreducible complex representation W of H occurs in a symplectic representation if and only if $\tilde{\mu}$ has exactly two weights a and $a+1$ on W.*

PROOF. $\Rightarrow$: Let (μ,n) represent $\tilde{\mu}$. As $H_{\mathbb{C}}$ is almost simple and W nontrivial, the homomorphism $\mathbb{G}_m \to \mathrm{GL}_W$ defined by μ is nontrivial, therefore noncentral, and the two weights a and $a+1$ occur.

$\Leftarrow$: Let (W,r) be an irreducible complex representation of H with weights $a, a+1$, and let V be the real vector space underlying W. Define G to be the subgroup of GL_V generated by the image of H and the homotheties: $G = r(H)\cdot\mathbb{G}_m$. Let $\tilde{h}$ be a fractional lifting of $\bar{h}$ to $\widetilde{H}$:

$$\begin{tikzcd} \widetilde{\mathbb{S}} \arrow[r, "\tilde{h}"] \arrow[d] & H_{\mathbb{C}} \arrow[d, "\mathrm{ad}"] \\ \mathbb{S} \arrow[r, "\bar{h}"] & H^{\mathrm{ad}}_{\mathbb{C}}. \end{tikzcd}$$

Let W_a and W_{a+1} be the subspaces of weight a and $a+1$ of W. Then $\tilde{h}(z)$ acts on W_a as $(z/\bar{z})^a$ and on W_{a+1} as $(z/\bar{z})^{a+1}$, and so $h(z) \stackrel{\mathrm{def}}{=} \tilde{h}(z)z^{-a}\bar{z}^{1+a}$ acts on these spaces as $\bar{z}$ and z respectively. Therefore h is a true homomorphism $\mathbb{S} \to G$, projecting to $\bar{h}$ on H^{ad}, and V is of type $\{(-1,0),(0,-1)\}$ relative to h. We may now apply Lemma 10.3. □

We interprete the condition in Lemma 10.5 in terms of roots and weights. Let $\overline{\mu} = \mu_{\overline{h}}$. Fix a maximal torus T in $H_{\mathbb{C}}$, and let $R = R(H,T) \subset X^*(T)_{\mathbb{Q}}$ be the corresponding root system. Choose a base S for R such that $\langle \alpha, \overline{\mu} \rangle \geq 0$ for all $\alpha \in S$ (cf. §2).

Recall that, for each $\alpha \in R$, there exists a unique $\alpha^\vee \in X_*(T)_{\mathbb{Q}}$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and the symmetry $s_\alpha : x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ preserves R; moreover, for all $\alpha \in R$, $\langle R, \alpha^\vee \rangle \subset \mathbb{Z}$. The lattice of weights is

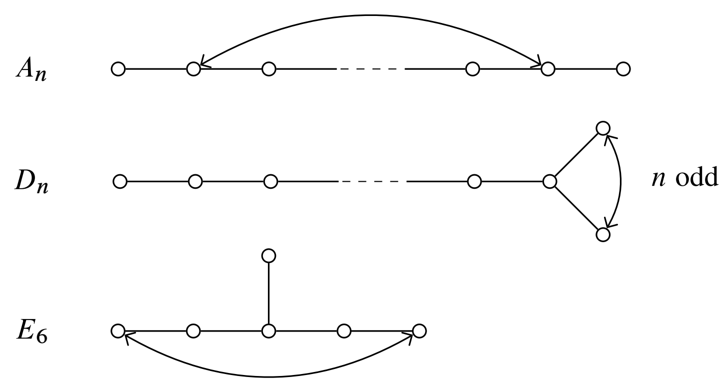
$$P(R) = \{\varpi \in X^*(T)_{\mathbb{Q}} \mid \langle \varpi, \alpha^\vee \rangle \in \mathbb{Z} \text{ all } \alpha \in R\},$$

the fundamental weights are the elements of the dual basis $\{\varpi_1, \ldots, \varpi_n\}$ to $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$, and that the dominant weights are the elements $\sum n_i \varpi_i$, $n_i \in \mathbb{N}$. The quotient $P(R)/Q(R)$ of $P(R)$ by the lattice $Q(R)$ generated by R is the character group of $Z(H)$:

$$P(R)/Q(R) \simeq X^*(Z(H)).$$

The irreducible complex representations of H are classified by the dominant weights. We shall determine the dominant weights of the irreducible complex representations such that $\widetilde{\mu}$ has exactly two weights a and $a+1$.

There is a unique permutation τ of the simple roots, called the ***opposition involution***, such that the $\tau^2 = 1$ and the map $\alpha \mapsto -\tau(\alpha)$ extends to an action of the Weyl group. Its action on the Dynkin diagram is determined by the following rules: it preserves each connected component; on a connected component of type A_n, D_n (n odd), or E_6, it acts as the unique nontrivial involution, and on all other connected components, it acts trivially (Tits 1966, 1.5.1). Thus:



PROPOSITION 10.6. *Let W be an irreducible complex representation of H, and let ϖ be its highest weight. The representation W occurs in a symplectic representation if and only if*

$$\langle \varpi + \tau\varpi, \overline{\mu} \rangle = 1. \tag{22}$$

PROOF. The lowest weight of W is $-\tau(\varpi)$. The weights β of W are of the form

$$\beta = \varpi + \sum_{\alpha \in R} m_\alpha \alpha, \quad m_\alpha \in \mathbb{Z},$$

and

$$\langle \beta, \overline{\mu} \rangle \in \mathbb{Z}.$$

Thus, $\langle \beta, \overline{\mu} \rangle$ takes only two values $a, a+1$ if and only if

$$\langle -\tau(\varpi), \overline{\mu} \rangle = \langle \varpi, \overline{\mu} \rangle - 1,$$

i.e., if and only if (22) holds. □

COROLLARY 10.7. *If ξ is symplectic, then ϖ is a fundamental weight. Therefore the representation factors through an almost simple quotient of H.*

PROOF. For every dominant weight ϖ, $\langle \varpi + \tau\varpi, \bar{\mu}\rangle \in \mathbb{Z}$ because $\varpi + \tau\varpi \in Q(R)$. If $\varpi \neq 0$, then $\langle \varpi + \tau\varpi, \bar{\mu}\rangle > 0$ unless $\bar{\mu}$ kills all the weights of the representation corresponding to ϖ. Hence a dominant weight satisfying (22) can not be a sum of two dominant weights. □

The corollary allows us to assume that H is almost simple. Recall from §2 that there is a unique special simple root α_s such that, for $\alpha \in S$,

$$\langle \alpha, \bar{\mu}\rangle = \begin{cases} 1 & \text{if } \alpha = \alpha_s \\ 0 & \text{otherwise.} \end{cases}$$

When a weight ϖ is expressed as a $\mathbb{Q}$-linear combination of the simple roots, $\langle \varpi, \bar{\mu}\rangle$ is the coefficient of α_s. For the fundamental weights, these coefficients can be found in the tables in Bourbaki Lie, VI. A fundamental weight ϖ satisfies (22) if and only if

$$(\text{coefficient of } \alpha_s \text{ in } \varpi + \tau\varpi) = 1. \tag{23}$$

In the following, we write $\alpha_1, \ldots, \alpha_n$ for the simple roots and $\varpi_1, \ldots, \varpi_n$ for the fundamental weights with the usual numbering. In the diagrams, the solid node is the special node corresponding to α_s, and the nodes ✩ correspond to symplectic representations (and we call them ***symplectic nodes***).

TYPE A_n.

The opposition involution τ switches the nodes i and $n+1-i$. According to the tables in Bourbaki, for $1 \leq i \leq (n+1)/2$,

$$\varpi_i = \tfrac{n+1-i}{n+1}\alpha_1 + \tfrac{2(n+1-i)}{n+1}\alpha_2 + \cdots + \tfrac{i(n+1-i)}{n+1}\alpha_i + \cdots + \tfrac{2i}{n+1}\alpha_{n-1} + \tfrac{i}{n+1}\alpha_n.$$

Replacing i with $n+1-i$ reflects the coefficients, and so

$$\tau\varpi_i = \varpi_{n+1-i} = \tfrac{i}{n+1}\alpha_1 + \tfrac{2i}{n+1}\alpha_2 + \cdots + \tfrac{2(n+1-i)}{n+1}\alpha_{n-1} + \tfrac{n+1-i}{n+1}\alpha_n.$$

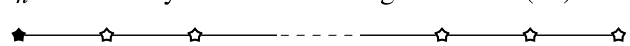
Therefore,

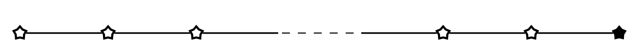
$$\varpi_i + \tau\varpi_i = \alpha_1 + 2\alpha_2 + \cdots + i\alpha_i + i\alpha_{i+1} + \cdots + i\alpha_{n+1-(i+1)} + i\alpha_{n+1-i} + \cdots + 2\alpha_{n-1} + \alpha_n,$$

i.e., the sequence of coefficients is

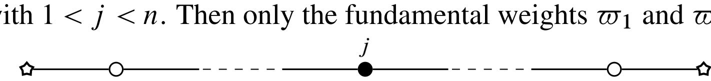
$$(1, 2, \ldots, i, i, \ldots, i, i, \ldots, 2, 1).$$

Let $\alpha_s = \alpha_1$ or α_n. Then every fundamental weight satisfies (23):[44]

$A_n(1)$ 

$A_n(n)$ 

Let $\alpha_s = \alpha_j$, with $1 < j < n$. Then only the fundamental weights ϖ_1 and ϖ_n satisfy (23):

$A_n(j)$ 

As P/Q is generated by ϖ_1, the symplectic representations form a faithful family.

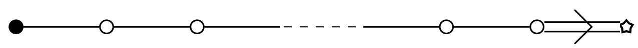
[44] Deligne 1979b, Table 1.3.9, overlooks this possibility.

TYPE B_n.

In this case, $\alpha_s = \alpha_1$ and the opposition involution acts trivially on the Dynkin diagram, and so we seek a fundamental weight ϖ_i such that $\varpi_i = \frac{1}{2}\alpha_1 + \cdots$. According to the tables in Bourbaki,

$$\begin{aligned}\varpi_i &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_n) \quad (1 \le i < n)\\ \varpi_n &= \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n),\end{aligned}$$

and so only ϖ_n satisfies (23):

$B_n(1)$ 

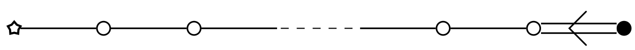
As P/Q is generated by ϖ_n, the symplectic representations form a faithful family.

TYPE C_n.

In this case $\alpha_s = \alpha_n$ and the opposition involution acts trivially on the Dynkin diagram, and so we seek a fundamental weight ϖ_i such that $\varpi_i = \cdots + \frac{1}{2}\alpha_n$. According to the tables in Bourbaki,

$$\varpi_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1} + \tfrac{1}{2}\alpha_n),$$

and so only ϖ_1 satisfies (23):

C_n 

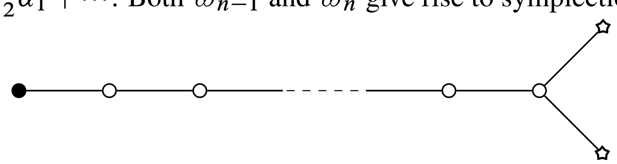
As P/Q is generated by ϖ_1, the symplectic representations form a faithful family.

TYPE D_n.

The opposition involution acts trivially if n is even, and switches α_{n-1} and α_n if n is odd. According to the tables in Bourbaki,

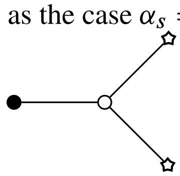
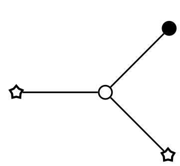
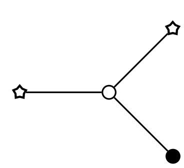
$$\begin{aligned}\varpi_i &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{n-2}) + \tfrac{i}{2}(\alpha_{n-1} + \alpha_n), \quad 1 \le i \le n-2\\ \varpi_{n-1} &= \tfrac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \tfrac{1}{2}n\alpha_{n-1} + \tfrac{1}{2}(n-2)\alpha_n\right)\\ \varpi_n &= \tfrac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \tfrac{1}{2}(n-2)\alpha_{n-1} + \tfrac{1}{2}n\alpha_n\right)\end{aligned}$$

Let $\alpha_s = \alpha_1$. As α_1 is fixed by the opposition involution, we seek a fundamental weight ϖ_i such that $\varpi_i = \frac{1}{2}\alpha_1 + \cdots$. Both ϖ_{n-1} and ϖ_n give rise to symplectic representations:

$D_n(1)$ 

When n is odd, ϖ_{n-1} and ϖ_n each generates P/Q, and when n is even ϖ_{n-1} and ϖ_n together generate P/Q. Therefore, in both cases, the symplectic representations form a faithful family.

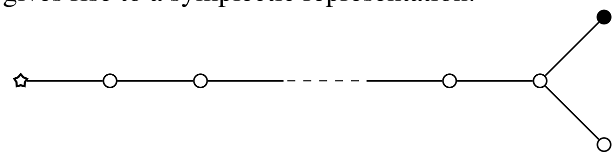
Let $\alpha_s = \alpha_{n-1}$ or α_n and let $n = 4$. The nodes α_1, α_3, and α_4 are permuted by automorphisms of the Dynkin diagram (hence by outer automorphisms of the corresponding group), and so this case is the same as the case $\alpha_s = \alpha_1$:

$D_4(1)$  $D_4(3)$  $D_4(4)$ 

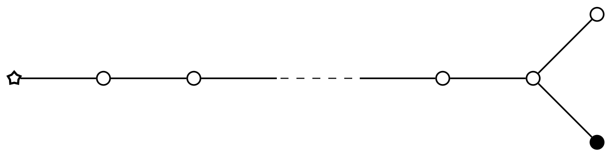
The symplectic representations form a faithful family.

Let $\alpha_s = \alpha_{n-1}$ or α_n and let $n \geq 5$. When n is odd, τ interchanges α_{n-1} and α_n, and so we seek a fundamental weight ϖ_i such that $\varpi_i = \cdots + a\alpha_{n-1} + b\alpha_n$ with $a+b=1$; when n is even, τ is trivial, and we seek a fundamental weight ϖ_i such that $\varpi_i = \cdots + \frac{1}{2}\alpha_{n-1} + \cdots$ or $\cdots + \frac{1}{2}\alpha_n$. In each case, only ϖ_1 gives rise to a symplectic representation:

$D_n(n-1)$



$D_n(n)$

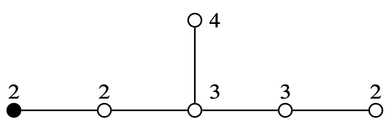


The weight ϖ_1 generates a subgroup of order 2 (and index 2) in P/Q. Let $C \subset Z(H)$ be the kernel of ϖ_1 regarded as a character of $Z(H)$. Then every symplectic representation factors through H/C, and the symplectic representations form a faithful family of representations of H/C.

TYPE E_6.

In this case, $\alpha_s = \alpha_1$ or α_6, and the opposition involution interchanges α_1 and α_6. Therefore, we seek a fundamental weight ϖ_i such that $\varpi_i = a\alpha_1 + \cdots + b\alpha_6$ with $a+b=1$. In the following diagram, we list the value $a+b$ for each fundamental weight ϖ_i:

$E_6(1)$

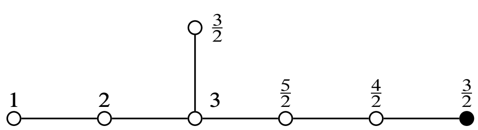


As no value equals 1, there are no symplectic representations.

TYPE E_7.

In this case, $\alpha_s = \alpha_7$, and the opposition involution is trivial. Therefore, we seek a fundamental weight ϖ_i such that $\varpi_i = \cdots + \frac{1}{2}\alpha_7$. In the following diagram, we list the coefficient of α_7 for each fundamental weight ϖ_i:

$E_7(7)$



As no value is $\frac{1}{2}$, there are no symplectic representations.

Following Deligne 1979b, 1.3.9, we write $D^{\mathbb{R}}$ for the case $D_n(1)$ and $D^{\mathbb{H}}$ for the cases $D_n(n-1)$ and $D_n(n)$.

SUMMARY 10.8. *Let H be a simply connected almost simple group over $\mathbb{R}$, and let $\overline{h}: \mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}$ be a nontrivial homomorphism satisfying (SV1,2). There exists a symplectic representation of $(H, \overline{h})$ if and only if it is of type A, B, C, or D. Except when $(H, \overline{h})$ is of type $D_n^{\mathbb{H}}$, $n \geq 5$, the symplectic representations form a faithful family of representations of H; when $(H, \overline{h})$ is of type $D_n^{\mathbb{H}}$, $n \geq 5$, they form a faithful family of representations of the quotient of the simply connected group by the kernel of ϖ_1.*

The rational case

Let H be a semisimple algebraic group over $\mathbb{Q}$, and let $\bar{h}$ be a homomorphism $\mathbb{S}/\mathbb{G}_m \to H^{\mathrm{ad}}_{\mathbb{R}}$ satisfying (SV1,2) and generating H^{ad}.

DEFINITION 10.9. A homomorphism $H \to \mathrm{GL}_V$ with finite kernel is a ***symplectic representation*** of $(H, \bar{h})$ if there exists a commutative diagram

$$\begin{tikzcd} & H \arrow[d] \arrow[dr] & \\ & (H^{\mathrm{ad}}, \bar{h}) & (G, h) \arrow[l] \arrow[r, "\rho"] & (G(\psi), D(\psi)), \end{tikzcd} \tag{24}$$

in which G is a reductive group (over $\mathbb{Q}$), the homomorphism $H \to G$ has image G^{der}, and ψ is a nondegenerate alternating form on V.

Given a diagram (24), we may replace G with its image in GL_V and so assume that the representation ρ is faithful.

We now assume that H is simply connected and almost simple. Then $H = (H^s)_{F/\mathbb{Q}}$ for some geometrically almost simple algebraic group H^s over a number field F. Because $H_{\mathbb{R}}$ is an inner form of its compact form, the field F is totally real (see the proof of 3.12). Let $I = \mathrm{Hom}(F, \mathbb{R})$. Then,

$$H_{\mathbb{R}} = \prod_{v \in I} H_v, \quad H_v = H^s \otimes_{F,v} \mathbb{R}.$$

The Dynkin diagram D of $H_{\mathbb{C}}$ is a disjoint union of the Dynkin diagrams D_v of the group $H_{v\mathbb{C}}$. The Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts on it in a manner consistent with its projection to I. In particular, it acts transitively on D and so all the factors H_v of $H_{\mathbb{R}}$ are of the same type. We let I_{c} (resp. I_{nc}) denote the subset of I of v for which H_v is compact (resp. not compact), and we let $H_{\mathrm{c}} = \prod_{v \in I_{\mathrm{c}}} H_v$ and $H_{\mathrm{nc}} = \prod_{v \in I_{\mathrm{nc}}} H_v$. Because $\bar{h}$ generates H^{ad}, I_{nc} is nonempty.

PROPOSITION 10.10. *Let F be a totally real number field. Suppose that for each real prime v of F, we are given a pair $(H_v, \bar{h}_v)$ in which H_v is a simply connected algebraic group over $\mathbb{R}$ of a fixed type, and $\bar{h}_v$ is a homomorphism $\mathbb{S}/\mathbb{G}_m \to H_v^{\mathrm{ad}}$ satisfying (SV1,2) (possibly trivial). Then there exists an algebraic group H over $\mathbb{Q}$ such that $H \otimes_{F,v} \mathbb{R} \approx H_v$ for all v.*

PROOF. There exists an algebraic group H over F such that $H \otimes_{F,v} \mathbb{R}$ is an inner form of its compact form for all real primes v of F. For each such v, H_v is an inner form of $H \otimes_{F,v} \mathbb{R}$, and so defines a cohomology class in $H^1(F_v, H^{\mathrm{ad}})$. The proposition now follows from the surjectivity of the map

$$H^1(F, H^{\mathrm{ad}}) \to \prod\nolimits_{v \text{ real}} H^1(F_v, H^{\mathrm{ad}})$$

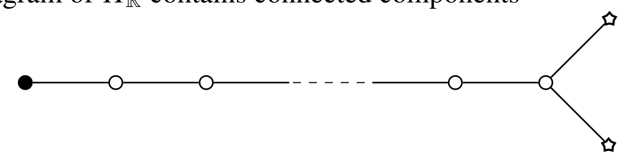
(Prasad and Rapinchuk 2006, Proposition 1). □

PAIRS $(H, \bar{h})$ FOR WHICH THERE DO NOT EXIST SYMPLECTIC REPRESENTATIONS

H is of exceptional type Assume that H is of exceptional type. If there exists an $\bar{h}$ satisfying (SV1,2), then H is of type E_6 or E_7 (see §2). A symplectic representation of $(H, \bar{h})$ over $\mathbb{Q}$ gives rise to a symplectic representation of $(H_{\mathbb{R}}, \bar{h})$ over $\mathbb{R}$, but we have seen (10.8) that no such representations exist.

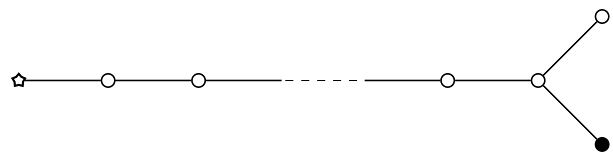
$(H,\overline{h})$ is of mixed type D. By this we mean that H is of type D_n with $n \geq 5$ and that at least one factor $(H_v,\overline{h}_v)$ is of type $D_n^{\mathbb{R}}$ and one of type $D_n^{\mathbb{H}}$. Such pairs $(H,\overline{h})$ exist by Proposition 10.10. The Dynkin diagram of $H_{\mathbb{R}}$ contains connected components

$D_n(1)$



and

$D_n(n)$



or $D_n(n-1)$. To give a symplectic representation for $H_{\mathbb{R}}$, we have to choose a symplectic node for each real prime v such that H_v is noncompact. In order for the representation to be rational, the collection of symplectic nodes must be stable under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, but this is impossible, because there is no automorphism of the Dynkin diagram of type D_n, $n \geq 5$, carrying the node 1 into either the node $n-1$ or the node n.

PAIRS $(H,\overline{h})$ FOR WHICH THERE EXIST SYMPLECTIC REPRESENTATIONS

LEMMA 10.11. *Let G be a reductive group over $\mathbb{Q}$ and let h be a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$ satisfying (SV1,2*) and generating G. For any representation (V,ρ) of G such that $(V,\rho\circ h)$ is of type $\{(-1,0),(0,-1)\}$, there exists an alternating form ψ on V such that ρ induces a homomorphism $(G,h) \to (G(\psi),D(\psi))$.*

PROOF. The pair $(\rho G,\rho\circ h)$ is the Mumford-Tate group of $(V,\rho\circ h)$ and satisfies (SV2*). The proof of Proposition 6.2 constructs a polarization ψ for $(V,\rho\circ h)$ such that $\rho G \subset G(\psi)$. □

PROPOSITION 10.12. *A homomorphism $H \to \mathrm{GL}_V$ is a symplectic representation of $(H,\overline{h})$ if there exists a commutative diagram*

$$\begin{tikzcd} H \arrow[d] \arrow[dr] & & \\ (H^{\mathrm{ad}},\overline{h}) & (G,h) \arrow[l] \arrow[r, hook, "\rho"] & \mathrm{GL}_V, \end{tikzcd}$$

in which G is a reductive group whose connected centre splits over a CM-field, the homomorphism $H \to G$ has image G^{der}, the weight w_h is defined over $\mathbb{Q}$, and the Hodge structure $(V,\rho\circ h)$ is of type $\{(-1,0),(0,-1)\}$.

PROOF. The hypothesis on the connected centre Z° says that the largest compact subtorus of $Z^{\circ}_{\mathbb{R}}$ is defined over $\mathbb{Q}$. Take G' to be the subgroup of G generated by this torus, G^{der}, and the image of w_h. Now (G',h) satisfies (SV2*), and we can apply 10.11. □

We classify the symplectic representations of $(H,\overline{h})$ with ρ faithful. Note that the quotient of H acting faithfully on V is isomorphic to G^{der}.

Let (V,r) be a symplectic representation of $(H,\overline{h})$. The restriction of the representation to H_{nc} is a real symplectic representation of H_{nc}, and so, according to Corollary 10.7, every nontrivial irreducible direct summand of $r_{\mathbb{C}}|H_{\mathrm{nc}}$ factors through H_v for some $v \in I_{\mathrm{nc}}$ and corresponds to a symplectic node of the Dynkin diagram D_v of H_v.

Let W be an irreducible direct summand of $V_{\mathbb{C}}$. Then

$$W \approx \bigotimes\nolimits_{v \in T} W_v$$

$$\begin{tikzcd} D \arrow[d] & \\ I & T \arrow[l, hook'] \arrow[ul, "s"'] \end{tikzcd}$$

for some irreducible symplectic representations W_v of $H_{v\mathbb{C}}$ indexed by a subset T of I. The irreducible representation W_v corresponds to a symplectic node $s(v)$ of D_v. Because r is defined over $\mathbb{Q}$, the set $s(T)$ is stable under the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$. For $v \in I_{\mathrm{nc}}$, the set $s(T) \cap D_v$ consists of a single symplectic node.

Given a diagram (24), we let $\mathcal{S}(V)$ denote the set of subsets $s(T)$ of the nodes of D as W runs over the irreducible direct summands of V. The set $\mathcal{S}(V)$ satisfies the following conditions:

(10.13a) for $S \in \mathcal{S}(V)$, $S \cap D_{\mathrm{nc}}$ is either empty or consists of a single symplectic node of D_v for some $v \in I_{\mathrm{nc}}$;

(10.13b) $\mathcal{S}$ is stable under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ and contains a nonempty subset.

Given such a set $\mathcal{S}$, let $H(\mathcal{S})_{\mathbb{C}}$ be the quotient of $H_{\mathbb{C}}$ that acts faithfully on the representation defined by $\mathcal{S}$. The condition (10.13b) ensures that $H(\mathcal{S})$ is defined over $\mathbb{Q}$. According to Galois theory (in the sense of Grothendieck), there exists an étale $\mathbb{Q}$-algebra $K_{\mathcal{S}}$ such that

$$\mathrm{Hom}(K_{\mathcal{S}}, \mathbb{Q}^{\mathrm{al}}) \simeq \mathcal{S} \qquad \text{(as sets with an action of } \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})).$$

THEOREM 10.14. *For any set $\mathcal{S}$ satisfying the conditions (10.13), there exists a diagram (24) such that the quotient of H acting faithfully on V is $H(\mathcal{S})$.*

PROOF. We prove this only in the case that $\mathcal{S}$ consists of one-point sets. For an $\mathcal{S}$ as in the theorem, the set $\mathcal{S}'$ of $\{s\}$ for $s \in S \in \mathcal{S}$ satisfies (10.13) and $H(\mathcal{S})$ is a quotient of $H(\mathcal{S}')$.

Recall that $H = (H^s)_{F/\mathbb{Q}}$ for some totally real field F. We choose a totally imaginary quadratic extension E of F and, for each real embedding v of F in I_{c}, we choose an extension σ of v to a complex embedding of E. Let T denote the set of σ's. Thus

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & \mathbb{C} \\ \cup & & \cup \\ F & \xrightarrow{v} & \mathbb{R} \end{array} \qquad T = \{\sigma \mid v \in I_c\}.$$

We regard E as a $\mathbb{Q}$-vector space, and define a Hodge structure h_T on it as follows: $E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$ and the factor with index σ is of type $(-1,0)$ if $\sigma \in T$, type $(0,-1)$ if $\bar{\sigma} \in T$, and of type $(0,0)$ if σ lies above I_{nc}. Thus ($\mathbb{C}_\sigma = \mathbb{C}$):

$$\begin{array}{ccccccc} E \otimes_{\mathbb{Q}} \mathbb{C} & = & \bigoplus_{\sigma \in T} \mathbb{C}_\sigma & \oplus & \bigoplus_{\bar{\sigma} \in T} \mathbb{C}_\sigma & \oplus & \bigoplus_{\sigma \notin T \cup \bar{T}} \mathbb{C}_\sigma . \\ h_T(z) & & z & & \bar{z} & & 1 \end{array}$$

Because the elements of $\mathcal{S}$ are one-point subsets of D, we can identify them with elements of D, and so regard $\mathcal{S}$ as a subset of D. It has the properties:

(a) if $s \in \mathcal{S} \cap D_{\mathrm{nc}}$, then s is a symplectic node;
(b) $\mathcal{S}$ is stable under $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ and is nonempty.

Let K_D be the smallest subfield of $\mathbb{Q}^{\mathrm{al}}$ such that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/K_D)$ acts trivially on D. Then K_D is a Galois extension of $\mathbb{Q}$ in $\mathbb{Q}^{\mathrm{al}}$ such that $\mathrm{Gal}(K_D/K)$ acts faithfully on D. Complex conjugation acts as the opposition involution on D, which lies in the centre of $\mathrm{Aut}(D)$; therefore K_D is either totally real or CM.

The $\mathbb{Q}$-algebra $K_{\mathcal{S}}$ can be taken to be a product of subfields of K_D. In particular, $K_{\mathcal{S}}$ is a product of totally real fields and CM fields. The projection $\mathcal{S} \to I$ corresponds to a homomorphism $F \to K_{\mathcal{S}}$.

For $s \in \mathcal{S}$, let $V(s)$ be a complex representation of $H_{\mathbb{C}}$ with dominant weight the fundamental weight corresponding to s. The isomorphism class of the representation $\bigoplus_{s\in\mathcal{S}} V(s)$ is defined over $\mathbb{Q}$. The obstruction to the representation itself being defined over $\mathbb{Q}$ lies in the Brauer group of $\mathbb{Q}$, which is torsion, and so some multiple of the representation is defined over $\mathbb{Q}$. Let V be a representation of H over $\mathbb{Q}$ such that $V_{\mathbb{C}} \approx \bigoplus_{s\in\mathcal{S}} nV(s)$ for some integer n, and let V_s denote the direct summand of $V_{\mathbb{C}}$ isomorphic to $nV(s)$. These summands are permuted by $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ in a fashion compatible with the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on $\mathcal{S}$, and the decomposition $V_{\mathbb{C}} = \bigoplus_{s\in\mathcal{S}} V_s$ corresponds therefore to a structure of a $K_{\mathcal{S}}$-module on V: let $s'\colon K_{\mathcal{S}} \to \mathbb{Q}^{\mathrm{al}}$ be the homomorphism corresponding to $s \in \mathcal{S}$; then $a \in K_{\mathcal{S}}$ acts on V_s as multiplication by $s'(a)$.

Let H' denote the quotient of H that acts faithfully on V. Then $H'_{\mathbb{R}}$ is the quotient of $H_{\mathbb{R}}$ described in (10.8).

A lifting of $\bar{h}$ to a fractional morphism of $\mathbb{S}$ into $H'_{\mathbb{R}}$ defines a fractional Hodge structure on V of weight 0, which can be described as follows. Let $s \in \mathcal{S}$, and let v be its image in I; if $v \in I_{\mathrm{c}}$, then V_s is of type $(0,0)$; if $v \in I_{\mathrm{nc}}$, then V_s is of type $\{(r,-r),(r-1,1-r)\}$ where $r = \langle \varpi_s, \bar{\mu}\rangle$ (notations as in 10.6). We renumber this Hodge structure to obtain a new Hodge structure on V:

	old	new
V_s, $v \in I_{\mathrm{c}}$	$(0,0)$	$(0,0)$
V_s, $v \in I_{\mathrm{nc}}$	$(r,-r)$	$(0,-1)$
V_s, $v \in I_{\mathrm{nc}}$	$(r-1,1-r)$	$(-1,0)$.

We endow the $\mathbb{Q}$-vector space $E \otimes_F V$ with the tensor product Hodge structure. The decomposition

$$(E \otimes_F V) \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus\nolimits_{v\in I} (E \otimes_{F,v} \mathbb{R}) \otimes_{\mathbb{R}} (V \otimes_{F,v} \mathbb{R}),$$

is compatible with the Hodge structures. The type of the Hodge structure on each direct summand is given by the following table:

	$E \otimes_{F,v} \mathbb{R}$	$V \otimes_{F,v} \mathbb{R}$
$v \in I_{\mathrm{c}}$	$\{(-1,0),(0,-1)\}$	$\{(0,0)\}$
$v \in I_{\mathrm{nc}}$	$\{(0,0)\}$	$\{(-1,0),(0,-1)\}$.

Therefore, $E \otimes_F V$ has type $\{(-1,0),(0,-1)\}$. Let G be the algebraic subgroup of $\mathrm{GL}_{E\otimes_F V}$ generated by $E^{\times}$ and H'. The homomorphism $h\colon \mathbb{S} \to (\mathrm{GL}_{E\otimes_F V})_{\mathbb{R}}$ corresponding to the Hodge structure factors through $G_{\mathbb{R}}$, and the derived group of G is H'. Now apply (10.12). □

ASIDE 10.15. The trick of using a quadratic imaginary extension E of F in order to obtain a Hodge structure of type $\{(-1,0),(0,-1)\}$ from one of type $\{(-1,0),(0,0),(0,-1)\}$ in essence goes back to Shimura (cf. Deligne 1971b, §6).

CONCLUSION

Now let H be a semisimple algebraic group over $\mathbb{Q}$, and let $\bar{h}$ be a homomorphism $\mathbb{S} \to H^{\mathrm{ad}}_{\mathbb{R}}$ satisfying (SV1,2) and generating H.

DEFINITION 10.16. The pair $(H,\overline{h})$ is of ***Hodge type*** if it admits a faithful family of symplectic representations.

THEOREM 10.17. *A pair $(H,\overline{h})$ is of Hodge type if it is a product of pairs $(H_i,\overline{h}_i)$ such that either*

(a) *$(H_i,\overline{h}_i)$ is of type A, B, C, or $D^{\mathbb{R}}$, and H is simply connected, or*

(b) *$(H_i,\overline{h}_i)$ is of type $D_n^{\mathbb{H}}$ $(n \geq 5)$ and equals $(H^s)_{F/\mathbb{Q}}$ for the quotient H^s of the simply connected group of type $D_n^{\mathbb{H}}$ by the kernel of ϖ_1 (cf. 10.8).*

Conversely, if $(H,\overline{h})$ is a Hodge type, then it is a quotient of a product of pairs satisfying (a) or (b).

PROOF. Suppose that $(H,\overline{h})$ is a product of pairs satisfying (a) and (b), and let $(H',\overline{h}')$ be one of these factors with H' almost simple. Let $\widetilde{H}'$ be the simply connected covering group of H. Then (10.8) allows us to choose a set $\mathcal{S}$ satisfying (10.13) and such that $H' = H(\mathcal{S})$. Now Theorem 10.14 shows that $(H',\overline{h}')$ admits a faithful symplectic representation. A product of pairs of Hodge type is clearly of Hodge type.

Conversely, suppose that $(H,\overline{h})$ is of Hodge type, let $\widetilde{H}$ be the simply connected covering group of H, and let $(H',\overline{h}')$ be an almost simple factor of $(\widetilde{H},\overline{h})$. Then $(H',\overline{h}')$ admits a symplectic representation with finite kernel, and so $(H',\overline{h}')$ is not of type E_6, E_7, or mixed type D (see p. 55). Moreover, if $(H',\overline{h}')$ is of type $D_n^{\mathbb{H}}$, $n \geq 5$, then (10.8) shows that it factors through the quotient described in (b). □

Notice that we haven't completely classified the pairs $(H,\overline{h})$ of Hodge type because we haven't determined exactly which quotients of products of pairs satisfying (a) or (b) occur as $H(\mathcal{S})$ for some set $\mathcal{S}$ satisfying (10.13).

11. Moduli

In this section, we determine (a) the pairs (G,h) that arise as the Mumford-Tate group of an abelian variety (or an abelian motive); (b) the arithmetic locally symmetric varieties that carry a faithful family of abelian varieties (or abelian motives); (c) the Shimura varieties that arise as moduli varieties for polarized abelian varieties (or motives) with Hodge class and level structure.

Mumford-Tate groups

THEOREM 11.1. *Let G be an algebraic group over $\mathbb{Q}$, and let $h\colon \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism that generates G and whose weight is rational. The pair (G,h) is the Mumford-Tate group of an abelian variety if and only if h satisfies (SV2*) and there exists a faithful representation $\rho\colon G \to \mathrm{GL}_V$ such that $(V,\rho\circ h)$ is of type $\{(-1,0),(0,-1)\}$*

PROOF. The necessity is obvious (apply (6.2) to see that (G,h) satisfies (SV2*)). For the sufficiency, note that (G,h) is the Mumford-Tate group of $(V,\rho\circ h)$ because h generates G. The Hodge structure is polarizable because (G,h) satisfies (SV2*) (apply 6.2), and so it is the Hodge structure $H_1(A^{\mathrm{an}},\mathbb{Q})$ of an abelian variety A by Riemann's theorem 4.4. □

THEOREM 11.2. *Let (G,h) be an algebraic group over $\mathbb{Q}$, and let $h\colon \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism satisfying (SV1,2*) and generating G. Assume that w_h is defined over $\mathbb{Q}$. The pair (G,h) is the Mumford-Tate group of an abelian motive if and only if $(G^{\mathrm{der}},\overline{h})$ is a quotient of a product of pairs satisfying (a) and (b) of (10.17).*

The proof will occupy the rest of this subsection. Recall that G_{Hdg} is the affine group scheme attached to the tannakian category $\mathsf{Hdg}_{\mathbb{Q}}$ of polarizable rational Hodge structures and the forgetful fibre functor (see 9.20). It is equipped with a homomorphism $h_{\mathrm{Hdg}}\colon\mathbb{S}\to(G_{\mathrm{Hdg}})_{\mathbb{R}}$. If (G,h) is the Mumford-Tate group of a polarizable Hodge structure, then there is a unique homomorphism $\rho(h)\colon G_{\mathrm{Hdg}}\to G$ such that $h=\rho(h)_{\mathbb{R}}\circ h_{\mathrm{Hdg}}$. Moreover, $(G_{\mathrm{Hdg}},h_{\mathrm{Hdg}})=\varprojlim(G,h)$.

LEMMA 11.3. *Let H be a semisimple algebraic group over $\mathbb{Q}$, and let $\bar{h}\colon\mathbb{S}/\mathbb{G}_m\to H_{\mathbb{R}}^{\mathrm{ad}}$ be a homomorphism satisfying (SV1,2,3). There exists a unique homomorphism*

$$\rho(H,\bar{h})\colon\left(G_{\mathrm{Hdg}}\right)^{\mathrm{der}}\to H$$

such that the following diagram commutes:

$$\begin{tikzcd}
(G_{\mathrm{Hdg}})^{\mathrm{der}} \arrow[r, "\rho(H{,}\bar{h})"] \arrow[d] & H \arrow[d] \\
G_{\mathrm{Hdg}} \arrow[r, "\rho(\bar{h})"] & H^{\mathrm{ad}}.
\end{tikzcd}$$

PROOF. Two such homomorphisms $\rho(H,\bar{h})$ would differ by a map into $Z(H)$. Because $(G_{\mathrm{Hdg}})^{\mathrm{der}}$ is connected, any such map is constant, and so the homomorphisms are equal.

For the existence, choose a pair (G,h) as in (8.9). Then (G,h) is the Mumford-Tate group of a polarizable Hodge structure, and we can take $\rho(H,\bar{h})=\rho(h)|(G_{\mathrm{Hdg}})^{\mathrm{der}}$. □

LEMMA 11.4. *The assignment $(H,\bar{h})\mapsto\rho(H,\bar{h})$ is functorial: if $\alpha\colon H\to H'$ is a homomorphism mapping $Z(H)$ into $Z(H')$ and carrying $\bar{h}$ to $\bar{h}'$, then $\rho(H',\bar{h}')=\alpha\circ\rho(H,\bar{h})$.*

PROOF. The homomorphism $\bar{h}'$ generates H'^{ad} (by SV3), and so the homomorphism α is surjective. Choose a pair (G,h) for $(H,\bar{h})$ as in (8.9), and let $G'=G/\operatorname{Ker}(\alpha)$. Write α again for the projection $G\to G'$ and let $h'=\alpha_{\mathbb{R}}\circ h$. This equality implies that

$$\rho(h')=\alpha\circ\rho(h).$$

On restricting this to $(G_{\mathrm{Hdg}})^{\mathrm{der}}$, we obtain the equality

$$\rho(H',\bar{h}')=\alpha\circ\rho(H,\bar{h}).$$

□

Recall that G_{Mab} is the affine group scheme attached to the category of abelian motives over $\mathbb{C}$ and the Betti fibre functor. The functor $\mathsf{Mot}^{\mathrm{ab}}(\mathbb{C})\to\mathsf{Hdg}_{\mathbb{Q}}$ is fully faithful by Deligne's theorem (9.9), and so it induces a surjective map $G_{\mathrm{Hdg}}\to G_{\mathrm{Mab}}$.

LEMMA 11.5. *If (H,h) is of Hodge type, then $\rho(H,\bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$.*

PROOF. Let (G,h) be as in the definition (10.9), and replace G with the algebraic subgroup generated by h. Then (G,h) is the Mumford-Tate group of an abelian variety (Riemann's theorem 4.4), and so $\rho(h)\colon G_{\mathrm{Hdg}}\to G$ factors through $G_{\mathrm{Hdg}}\to G_{\mathrm{Mab}}$. Therefore $\rho(H,\bar{h})$ maps the kernel of $\left(G_{\mathrm{Hdg}}\right)^{\mathrm{der}}\to(G_{\mathrm{Mab}})^{\mathrm{der}}$ into the kernel of $H\to G$. By assumption, the intersection of these kernels is trivial. □

LEMMA 11.6. *The homomorphism $\rho(H,\bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$ if and only if $(H,\bar{h})$ has a finite covering by a pair of Hodge type.*

PROOF. Suppose that there is a finite covering $\alpha: H' \to H$ such that $(H', \bar{h})$ is of Hodge type. By Lemma 11.5, $\rho(H', \bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$, and therefore so also does $\rho(H, \bar{h}) = \alpha \circ \rho(H', \bar{h})$.

Conversely, suppose that $\rho(H, \bar{h})$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$. There will be an algebraic quotient (G, h) of $(G_{\mathrm{Mab}}, h_{\mathrm{Mab}})$ such that $(H, \bar{h})$ is a quotient of $(G^{\mathrm{der}}, \mathrm{ad} \circ h)$. Consider the category of abelian motives M such that the action of G_{Mab} on $\omega_B(M)$ factors through G. By definition, this category is contained in the tensor category generated by $h_1(A)$ for some abelian variety A. We can replace G with the Mumford-Tate group of A. Then $(G^{\mathrm{der}}, \mathrm{ad} \circ h)$ has a faithful symplectic embedding, and so it is of Hodge type. □

We can now complete the proof of the Theorem 11.2. From (9.20), we know that $\rho(h)$ factors through G_{Mab} if and only if $\rho(G^{\mathrm{der}}, \mathrm{ad} \circ h)$ factors through $(G_{\mathrm{Mab}})^{\mathrm{der}}$, and from (11.6) we know that this is true if and only if $(G^{\mathrm{der}}, \mathrm{ad} \circ h)$ has a finite covering by a pair of Hodge type.

NOTES. This subsection follows §1 of Milne 1994b.

Families of abelian varieties and motives

Let S be a connected smooth algebraic variety over $\mathbb{C}$, and let $o \in S(\mathbb{C})$. A family $f: A \to S$ of abelian varieties over S defines a local system $\mathsf{V} = R_1 f_* \mathbb{Z}$ of $\mathbb{Z}$-modules on S^{an}. We say that the family is ***faithful*** if the monodromy representation $\pi_1(S^{\mathrm{an}}, o) \to \mathrm{GL}(\mathsf{V}_o)$ is injective.

Let $D(\Gamma) = \Gamma \backslash D$ be an arithmetic locally symmetric variety, and let $o \in D$. By definition, there exists a simply connected algebraic group H over $\mathbb{Q}$ and a surjective homomorphism $\varphi: H(\mathbb{R}) \to \mathrm{Hol}(D)^+$ with compact kernel such that $\varphi(H(\mathbb{Z}))$ is commensurable with Γ. Moreover, the pair (H, φ) is uniquely determined up to a unique isomorphism (see 3.12). Let $\bar{h}: \mathbb{S} \to H^{\mathrm{ad}}$ be the homomorphism whose projection into a compact factor of H^{ad} is trivial and is such that $\varphi(\bar{h}(z/\bar{z}))$ fixes o and acts on $T_o(D)$ as multiplication by $z/\bar{z}$ (cf. (13), p. 37).

THEOREM 11.7. *There exists a faithful family of abelian varieties on $D(\Gamma)$ having a fibre of CM-type if and only if $(H, \bar{h})$ admits a symplectic representation (10.9).*

PROOF. Let $f: A \to D(\Gamma)$ be a faithful family of abelian varieties on $D(\Gamma)$, and let (V, F) be the variation of Hodge structures $R_1 f_* \mathbb{Q}$. Choose a trivialization $\pi^* \mathsf{V} \approx V_D$, and let $G \subset \mathrm{GL}_V$ be the generic Mumford-Tate group (see 6.14). As in (§8), we get a commutative diagram

$$\begin{tikzcd} H \arrow[d] \arrow[dr] & & \\ (H^{\mathrm{ad}}, \bar{h}) & (G, h) \arrow[l] \arrow[r, hook, "\rho"] & \mathrm{GL}_V \end{tikzcd} \tag{25}$$

in which the image of $H \to G$ is G^{der}. Because the family is faithful, the map $H \to G^{\mathrm{der}}$ is an isogeny, and so $(H, \bar{h})$ admits a symplectic representation.

Conversely, a symplectic representation of $(H, \bar{h})$ defines a variation of Hodge structures (8.5), which arises from a family of abelian varieties by Theorem 7.11 (Riemann's theorem in families). □

THEOREM 11.8. *There exists a faithful family of abelian motives on $D(\Gamma)$ having a fibre of CM-type if and only if $(H, \bar{h})$ has finite covering by a pair of Hodge type.*

PROOF. Similar to that of 11.7. The key point is the determination of the Mumford-Tate groups of abelian motives in (11.2). □

Shimura varieties

In the above, we have always considered connected varieties. As Deligne (1971) observed, it is often more convenient to consider nonconnected varieties.

DEFINITION 11.9. A ***Shimura datum*** is a pair (G,X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S}\to G_{\mathbb{R}}$ satisfying (SV1,2,3).[45]

EXAMPLE 11.10. Let (V,ψ) be a symplectic space over $\mathbb{Q}$. The group $G(\psi)$ of symplectic similitudes together with the space $X(\psi)$ of all complex structures J on $V_{\mathbb{R}}$ such that $(x,y)\mapsto\psi(x,Jy)$ is positive definite is a Shimura datum.

Let (G,X) be a Shimura datum. The map $h\mapsto\overline{h}\overset{\text{def}}{=}\operatorname{ad}\circ h$ identifies X with a $G^{\text{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $\overline{h}\colon\mathbb{S}/\mathbb{G}_m\to G^{\text{ad}}_{\mathbb{R}}$ (satisfying SV1,2,3). Thus X is a hermitian symmetric domain. More canonically, the set X has a unique structure of a complex manifold such that, for every representation $\rho_{\mathbb{R}}\colon G_{\mathbb{R}}\to\mathrm{GL}_V$, $(V_X,\rho\circ h)_{h\in X}$ is a holomorphic family of Hodge structures. For this complex structure, $(V_X,\rho\circ h)_{h\in X}$ is a variation of Hodge structures, and so X is a hermitian symmetric domain.

The Shimura variety attached to (G,X) and the choice of an compact open subgroup K of $G(\mathbb{A}_f)$ is[46]

$$\mathrm{Sh}_K(G,X)=G(\mathbb{Q})_+\backslash X\times G(\mathbb{A}_f)/K$$

where $G(\mathbb{Q})_+=G(\mathbb{Q})\cap G(\mathbb{R})^+$. In this quotient, $G(\mathbb{Q})_+$ acts on both X (by conjugation) and $G(\mathbb{A}_f)$, and K acts on $G(\mathbb{A}_f)$. Let $\mathcal{C}$ be a set of representatives for the (finite) double coset space $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$; then

$$G(\mathbb{Q})_+\backslash X\times G(\mathbb{A}_f)/K\simeq\bigsqcup\nolimits_{g\in\mathcal{C}}\Gamma_g\backslash X,\quad \Gamma_g=gKg^{-1}\cap G(\mathbb{Q})_+.$$

Because Γ_g is a congruence subgroup of $G(\mathbb{Q})$, its image in $G^{\text{ad}}(\mathbb{Q})$ is arithmetic (3.3), and so $\mathrm{Sh}_K(G,X)$ is a finite disjoint union of connected Shimura varieties. It therefore has a unique structure of an algebraic variety. As K varies, these varieties form a projective system.

We make this more explicit in the case that G^{der} is simply connected. Let $\nu\colon G\to T$ be the quotient of G by G^{der}, and let Z be the centre of G. Then ν defines an isogeny $Z\to T$, and we let

$$\begin{aligned}T(\mathbb{R})^\dagger&=\mathrm{Im}(Z(\mathbb{R})\to T(\mathbb{R})),\\ T(\mathbb{Q})^\dagger&=T(\mathbb{Q})\cap T(\mathbb{R})^\dagger.\end{aligned}$$

The set $T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/\nu(K)$ is finite and discrete. For K sufficiently small, the map

$$[x,a]\mapsto[\nu(a)]\colon G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K\to T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/\nu(K) \tag{26}$$

is surjective, and each fibre is isomorphic to $\Gamma\backslash X$ for some congruence subgroup Γ of $G^{\text{der}}(\mathbb{Q})$. For the fibre over [1], the congruence subgroup Γ is contained in $K\cap G^{\text{der}}(\mathbb{Q})$, and equals it if $Z(G^{\text{der}})$ satisfies the Hasse principal for H^1, for example, if G^{der} has no factors of type A.

[45] In the usual definition, X is taken to be a $G(\mathbb{R})$-conjugacy class. For our purposes, it is convenient to choose a connected component of X.

[46] This agrees with the usual definition because of Milne 2005, 5.11.

EXAMPLE 11.11. Let $G = \mathrm{GL}_2$. Then $(G \xrightarrow{\nu} T) = (\mathrm{GL}_2 \xrightarrow{\det} \mathbb{G}_m)$ and $(Z \xrightarrow{\nu} T) = (\mathbb{G}_m \xrightarrow{2} \mathbb{G}_m)$; therefore
$$T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K) = \mathbb{Q}^{>0} \backslash \mathbb{A}_f^\times / \det(K).$$
Note that $\mathbb{A}_f^\times = \mathbb{Q}^{>0} \cdot \widehat{\mathbb{Z}}^\times$ (direct product) where $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \simeq \prod_\ell \mathbb{Z}_\ell$. For $K = K(N) \stackrel{\text{def}}{=} \{a \in \widehat{\mathbb{Z}}^\times \mid a \equiv 1 \bmod N\}$, we find that
$$T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K) \simeq (\mathbb{Z}/N\mathbb{Z})^\times.$$

DEFINITION 11.12. A Shimura datum (G, X) is of ***Hodge type*** if there exists an injective homomorphism $G \to G(\psi)$ sending X into $X(\psi)$ for some symplectic pair (V, ψ) over $\mathbb{Q}$.

DEFINITION 11.13. A Shimura datum (G, X) is of ***abelian type*** if, for one (hence all) $h \in X$, the pair $(G^{\mathrm{der}}, \mathrm{ad} \circ h)$ is a quotient of a product of pairs satisfying (a) or (b) of (10.17).

A Shimura variety $\mathrm{Sh}(G, X)$ is said to be of Hodge or abelian type if (G, X) is.

NOTES. See Milne 2005, §5, for proofs of the statements in this subsection. For the structure of the Shimura variety when G^{der} is not simply connected, see Deligne 1979b, 2.1.16.

Shimura varieties as moduli varieties

Throughout this subsection, (G, X) is a Shimura datum such that

(a) w_X is defined over $\mathbb{Q}$ and the connected centre of G is split by a CM-field, and
(b) there exists a homomorphism $\nu: G \to \mathbb{G}_m \simeq \mathrm{GL}_{\mathbb{Q}(1)}$ such that $\nu \circ w_X = -2$.

Fix a faithful representation $\rho: G \to \mathrm{GL}_V$. Assume that there exists a pairing $t_0: V \times V \to \mathbb{Q}(m)$ such that (i) $gt_0 = \nu(g)^m t_0$ for all $g \in G$ and (ii) t_0 is a polarization of $(V, \rho_\mathbb{R} \circ h)$ for all $h \in X$. Then there exist homomorphisms $t_i: V^{\otimes r_i} \to \mathbb{Q}(\frac{mr_i}{2})$, $1 \leq i \leq n$, such that G is the subgroup of GL_V whose elements fix $t_0, t_1, \ldots, t_n$. When (G, X) is of Hodge type, we choose ρ to be a symplectic representation.

Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Define $\mathcal{H}_K(\mathbb{C})$ to be the set of triples
$$(W, (s_i)_{0 \leq i \leq n}, \eta K)$$
in which

- $W = (W, h_W)$ is a rational Hodge structure,
- each s_i is a morphism of Hodge structures $W^{\otimes r_i} \to \mathbb{Q}(\frac{mr_i}{2})$ and s_0 is a polarization of W,
- ηK is a K-orbit of $\mathbb{A}_f$-linear isomorphisms $V_{\mathbb{A}_f} \to W_{\mathbb{A}_f}$ sending each t_i to s_i,

satisfying the following condition:

> (*) there exists an isomorphism $\gamma: W \to V$ sending each s_i to t_i and h_W onto an element of X.

LEMMA 11.14. *For $(W, \ldots)$ in $\mathcal{H}_K(\mathbb{C})$, choose an isomorphism γ as in (*), let h be the image of h_W in X, and let $a \in G(\mathbb{A}_f)$ be the composite $V_{\mathbb{A}_f} \xrightarrow{\eta} W_{\mathbb{A}_f} \xrightarrow{\gamma} V_{\mathbb{A}_f}$. The class $[h, a]$ of the pair (h, a) in $G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f)/K$ is independent of all choices, and the map*
$$(W, \ldots) \mapsto [h, a]: \mathcal{H}_K(\mathbb{C}) \to \mathrm{Sh}_K(G, X)(\mathbb{C})$$
is surjective with fibres equal to the isomorphism classes.

PROOF. The proof involves only routine checking. □

For a smooth algebraic variety S over $\mathbb{C}$, let $\mathcal{F}_K(S)$ be the set of isomorphism classes of triples $(A,(s_i)_{0\le i\le n},\eta K)$ in which

- ◇ A is a family of abelian motives over S,
- ◇ each s_i is a morphism of abelian motives $A^{\otimes r_i}\to\mathbb{Q}(\frac{mr_i}{2})$, and
- ◇ ηK is a K-orbit of $\mathbb{A}_f$-linear isomorphisms $V_S\to\omega_f(A/S)$ sending each t_i to s_i,[47]

satisfying the following condition:

(**) for each $s\in S(\mathbb{C})$, the Betti realization of $(A,(s_i),\eta K)_s$ lies in $\mathcal{H}_K(\mathbb{C})$.

With the obvious notion of pullback, $\mathcal{F}_K$ becomes a functor from smooth complex algebraic varieties to sets. There is a well-defined injective map $\mathcal{F}_K(\mathbb{C})\to\mathcal{H}_K(\mathbb{C})/\approx$, which is surjective when (G,X) is of abelian type. Hence, in this case, we get an isomorphism $\alpha\colon\mathcal{F}_K(\mathbb{C})\to\mathrm{Sh}_K(\mathbb{C})$.

THEOREM 11.15. *Assume that (G,X) is of abelian type. The map α realizes Sh_K as a coarse moduli variety for $\mathcal{F}_K$, and even a fine moduli variety when $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{R})$ ($Z=Z(G)$).*

PROOF. To say that (Sh_K,α) is coarse moduli variety means the following:

(a) for any smooth algebraic variety S over $\mathbb{C}$, and $\xi\in\mathcal{F}(S)$, the map $s\mapsto\alpha(\xi_s)\colon S(\mathbb{C})\to\mathrm{Sh}_K(\mathbb{C})$ is regular;

(b) (Sh_K,α) is universal among pairs satisfying (a) .

To prove (a), we use that ξ defines a variation of Hodge structures on S (see p. 48). Now the universal property of hermitian symmetric domains (7.7) shows that the map $s\mapsto\alpha(\xi_s)$ is holomorphic (on the universal covering space, and hence on the variety), and Borel's theorem 4.3 shows that it is regular.

Next assume that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{R})$. Then the representation ρ defines a variation of Hodge structures on Sh_K itself (not just its universal covering space), which arises from a family of abelian motives. This family is universal, and so Sh_K is a fine moduli variety.

We now prove (b). Let S' be a smooth algebraic variety over $\mathbb{C}$ and let $\alpha'\colon\mathcal{F}_K(\mathbb{C})\to S'(\mathbb{C})$ be a map with the following property: for any smooth algebraic variety S over $\mathbb{C}$ and $\xi\in\mathcal{F}(S)$, the map $s\mapsto\alpha'(\xi_s)\colon S(\mathbb{C})\to S'(\mathbb{C})$ is regular. We have to show that the map $s\mapsto\alpha'\alpha^{-1}(s)\colon\mathrm{Sh}_K(\mathbb{C})\to S'(\mathbb{C})$ is regular. When $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{R})$, the map is that defined by α' and the universal family of abelian motives on Sh_K, and so it is regular by definition. In the general case, we let G' be the smallest algebraic subgroup of G such that $h(\mathbb{S})\subset G'_{\mathbb{R}}$ for all $h\in X$. Then (G',X) is a Shimura datum (cf. 7.5), which now is such that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{R})$; moreover, $\mathrm{Sh}_{K\cap G'(\mathbb{A}_f)}(G',X)$ consists of a certain number of connected components of $\mathrm{Sh}_K(G,X)$. As the map is regular on $\mathrm{Sh}_{K\cap G'(\mathbb{A}_f)}(G',X)$, and $\mathrm{Sh}_K(G,X)$ is a union of translates of $\mathrm{Sh}_{K\cap G'(\mathbb{A}_f)}(G',X)$, this shows that the map is regular on $\mathrm{Sh}_K(G,X)$. □

REMARKS

11.16. When (G,X) is of Hodge type in Theorem 11.15, the Shimura variety is a moduli variety for abelian *varieties* with additional structure. In this case, the moduli problem can be defined for all schemes algebraic over $\mathbb{C}$ (not necessarily smooth), and Mumford's theorem can be used to prove that the Shimura variety is moduli variety for the expanded functor.

[47] The isomorphism η is defined only on the universal covering space of S^{an}, but the family ηK is stable under $\pi_1(S,o)$, and so is "defined" on S.

11.17. It is possible to describe the structure ηK by passing only to a finite covering, rather than the full universal covering. This means that it can be described purely algebraically.

11.18. For certain compact open groups K, the structure ηK can be interpreted as a level-N structure in the usual sense.

11.19. Consider a pair $(H, \overline{h})$ having a finite covering of Hodge type. Then there exists a Shimura datum (G, X) of abelian type such that $(G^{\mathrm{der}}, \mathrm{ad} \circ h) = (H, \overline{h})$ for some $h \in X$. The choice of a faithful representation ρ for G gives a realization of the connected Shimura variety defined by any (sufficiently small) congruence subgroup of $H(\mathbb{Q})$ as a fine moduli variety for abelian motives with additional structure. For example, when H is simply connected, there is a map $\mathcal{H}_K(\mathbb{C}) \to T(\mathbb{Q})^{\dagger} \backslash T(\mathbb{A}_f)/\nu(K)$ (see (26)), and the moduli problem is obtained from $\mathcal{F}_K$ by replacing $\mathcal{H}_K(\mathbb{C})$ with its fibre over [1]. Note that the realization involves many choices.

11.20. For each Shimura variety, there is a well-defined number field $E(G, X)$, called the reflex field. When the Shimura variety is a moduli variety, it is possible choose the moduli problem so that it is defined over $E(G, X)$. Then an elementary descent argument shows that the Shimura variety itself has a model over $E(G, X)$. A priori, it may appear that this model depends on the choice of the moduli problem. However, the theory of complex multiplication shows that the model satisfies a certain reciprocity law at the special points, which characterize it.

11.21. The (unique) model of a Shimura variety over the reflex field $E(G, X)$ satisfying (Shimura's) reciprocity law at the special points is called the ***canonical model***. As we have just noted, when a Shimura variety can be realized as a moduli variety, it has a canonical model. More generally, when the associated connected Shimura variety is a moduli variety, then $\mathrm{Sh}(G, X)$ has a canonical model (Shimura 1970, Deligne 1979b). Otherwise, the Shimura variety can be embedded in a larger Shimura variety that contains many Shimura subvarieties of type A_1, and this can be used to prove that the Shimura variety has a canonical model (Milne 1983).

NOTES. For more details on this subsection, see Milne 1994b.

References

ANDRÉ, Y. 1992a. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compositio Math.* 82:1–24.

ANDRÉ, Y. 1992b. Une remarque à propos des cycles de Hodge de type CM, pp. 1–7. *In* Séminaire de Théorie des Nombres, Paris, 1989–90, volume 102 of *Progr. Math.* Birkhäuser Boston, Boston, MA.

BAILY, JR., W. L. AND BOREL, A. 1966. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)* 84:442–528.

BOREL, A. 1969. Introduction aux groupes arithmétiques. Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341. Hermann, Paris.

BOREL, A. 1972. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem. *J. Differential Geometry* 6:543–560.

BOREL, A. AND HARISH-CHANDRA 1962. Arithmetic subgroups of algebraic groups. *Ann. of Math. (2)* 75:485–535.

BOURBAKI, N. Lie. Groupes et Algèbres de Lie. Éléments de mathématique. Hermann; Masson, Paris. Chap. I, Hermann 1960; Chap. II,III, Hermann 1972; Chap. IV,V,VI, Masson 1981;Chap. VII,VIII, Masson 1975; Chap. IX, Masson 1982 (English translation available from Springer).

CATTANI, E., DELIGNE, P., AND KAPLAN, A. 1995. On the locus of Hodge classes. *J. Amer. Math. Soc.* 8:483–506.

CATTANI, E., KAPLAN, A., AND SCHMID, W. 1986. Degeneration of Hodge structures. *Ann. of Math. (2)* 123:457–535.

CHAI, C.-L. AND FALTINGS, G. 1990. Degeneration of abelian varieties, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin.

CHOW, W.-L. 1949. On compact complex analytic varieties. *Amer. J. Math.* 71:893–914.

COMESSATTI, A. 1938. Sugl'indici di singolarita a più dimensioni della varieta abeliane. *Rendiconti del Seminario Matematico della Università di Padova* pp. 50–79.

DELIGNE, P. 1971a. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.* pp. 5–57.

DELIGNE, P. 1971b. Travaux de Shimura, pp. 123–165. Lecture Notes in Math., Vol. 244. *In* Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389. Springer, Berlin.

DELIGNE, P. 1972. La conjecture de Weil pour les surfaces $K3$. *Invent. Math.* 15:206–226.

DELIGNE, P. 1973. Les constantes des équations fonctionnelles des fonctions L, pp. 501–597. Lecture Notes in Math., Vol. 349. *In* Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972). Springer, Berlin.

DELIGNE, P. 1974. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.* pp. 273–307.

DELIGNE, P. 1979a. Valeurs de fonctions L et périodes d'intégrales, pp. 313–346. *In* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I.

DELIGNE, P. 1979b. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, pp. 247–289. *In* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I.

DELIGNE, P. 1982. Hodge cycles on abelian varieties (notes by J.S. Milne), pp. 9–100. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics. Springer-Verlag, Berlin. A corrected version is available at `jmilne.org`.

DELIGNE, P. AND MILNE, J. S. 1982. Tannakian categories, pp. 101–228. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics 900. Springer-Verlag, Berlin.

GRAUERT, H. AND REMMERT, R. 1984. Coherent analytic sheaves, volume 265 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin.

GRIFFITHS, P. AND SCHMID, W. 1969. Locally homogeneous complex manifolds. *Acta Math.* 123:253–302.

GRIFFITHS, P. A. 1968. Periods of integrals on algebraic manifolds. I, II. *Amer. J. Math.* 90:568–626, 805–865.

GRIFFITHS, P. A. 1970. Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems. *Bull. Amer. Math. Soc.* 76:228–296.

HELGASON, S. 1978. Differential geometry, Lie groups, and symmetric spaces, volume 80 of *Pure and Applied Mathematics*. Academic Press Inc., New York.

HIRONAKA, H. 1964. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math.* 79:109–326.

KATZ, N. M. AND MAZUR, B. 1985. Arithmetic moduli of elliptic curves, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ.

KOBAYASHI, S. 1959. Geometry of bounded domains. *Trans. Amer. Math. Soc.* 92:267–290.

MARGULIS, G. A. 1991. Discrete subgroups of semisimple Lie groups, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin.

MATTUCK, A. 1958. Cycles on abelian varieties. *Proc. Amer. Math. Soc.* 9:88–98.

MILNE, J. S. 1983. The action of an automorphism of **C** on a Shimura variety and its special points, pp. 239–265. *In* Arithmetic and geometry, Vol. I, Progr. Math. Birkhäuser Boston, Boston, MA.

MILNE, J. S. 1994a. Motives over finite fields, pp. 401–459. *In* Motives (Seattle, WA, 1991), volume 55 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI.

MILNE, J. S. 1994b. Shimura varieties and motives, pp. 447–523. *In* Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI.

MILNE, J. S. 1999. Lefschetz classes on abelian varieties. *Duke Math. J.* 96:639–675.

MILNE, J. S. 2005. Introduction to Shimura varieties, pp. 265–378. *In* J. Arthur and R. Kottwitz (eds.), Harmonic analysis, the trace formula, and Shimura varieties, volume 4 of *Clay Math. Proc.* Amer. Math. Soc., Providence, RI. Available at: `http://www.claymath.org/library/`.

MILNE, J. S. AND SHIH, K.-Y. 1982. Conjugates of Shimura varieties, pp. 280–356. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics. Springer-Verlag, Berlin.

MORRIS, D. 2008. Introduction to arithmetic groups. arXive:math/0106063v3.

MOSTOW, G. D. AND TAMAGAWA, T. 1962. On the compactness of arithmetically defined homogeneous spaces. *Ann. of Math. (2)* 76:446–463.

MUMFORD, D. 1965. Geometric invariant theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin.

MUMFORD, D. 1966. Families of abelian varieties, pp. 347–351. *In* Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965). Amer. Math. Soc., Providence, R.I.

MUMFORD, D. 1969. A note of Shimura's paper "Discontinuous groups and abelian varieties". *Math. Ann.* 181:345–351.

MUMFORD, D. 1970. Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay.

PRASAD, G. AND RAPINCHUK, A. S. 2006. On the existence of isotropic forms of semi-simple algebraic groups over number fields with prescribed local behavior. *Adv. Math.* 207:646–660.

PRASAD, G. AND RAPINCHUK, A. S. 2008. Developments on the congruence subgroup problem after the work of Bass, Milnor and Serre. arXiv:0809.1622.

RAPOPORT, M. 1972. Complément à l'article de P. Deligne "La conjecture de Weil pour les surfaces $K3$". *Invent. Math.* 15:227–236.

SATAKE, I. 1965. Holomorphic imbeddings of symmetric domains into a Siegel space. *Amer. J. Math.* 87:425–461.

SATAKE, I. 1967. Symplectic representations of algebraic groups satisfying a certain analyticity condition. *Acta Math.* 117:215–279.

SATAKE, I. 1980. Algebraic structures of symmetric domains, volume 4 of *Kanô Memorial Lectures*. Iwanami Shoten, Tokyo.

SCHMID, W. 1973. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.* 22:211–319.

SERRE, J.-P. 1990. Lectures on the Mordell-Weil theorem, 2nd edition. Aspects of Mathematics, E15. Friedr. Vieweg & Sohn, Braunschweig. Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt.

SHIMURA, G. 1964. On the field of definition for a field of automorphic functions. *Ann. of Math. (2)* 80:160–189.

SHIMURA, G. 1966. Moduli and fibre systems of abelian varieties. *Ann. of Math. (2)* 83:294–338.

SHIMURA, G. 1967a. Algebraic number fields and symplectic discontinuous groups. *Ann. of Math. (2)* 86:503–592.

SHIMURA, G. 1967b. Construction of class fields and zeta functions of algebraic curves. *Ann. of Math. (2)* 85:58–159.

SHIMURA, G. 1970. On canonical models of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)* 91:144–222; II ibid. 92:528–549.

TATE, J. T. 1964. Algebraic cohomology classes. *In* Lecture Notes Prepared in Connection with Seminars held at the Summer Institute on Algebraic Geometry, Woods Hole, MA, July 6 – July 31, 1964. American Mathematical Society. Reprinted as: Algebraic cycles and poles of zeta functions, Arithmetical Algebraic Geometry, Harper & Row, (1965), 93–110.

TAYLOR, J. L. 2002. Several complex variables with connections to algebraic geometry and Lie groups, volume 46 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.

TITS, J. 1966. Classification of algebraic semisimple groups, pp. 33–62. *In* Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965). Amer. Math. Soc., Providence, R.I., 1966.

WEIL, A. 1977. Abelian varieties and the Hodge ring. Talk at a conference held at Harvard in honor of Lars Ahlfors (Oevres Scientifiques 1977c, 421–429).

ZARHIN, Y. G. 1984. Weights of simple Lie algebras in the cohomology of algebraic varieties. *Izv. Akad. Nauk SSSR Ser. Mat.* 48:264–304.

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