The Tate conjecture over finite fields (AIM talk)

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Abstract

These are my notes for a talk at the The Tate Conjecture workshop at the American Institute of Mathematics in Palo Alto, CA, July 23–July 27, 2007, somewhat revised and expanded. The intent of the talk was to review what is known and to suggest directions for research.
v1 (September 19, 2007): first version on the web.
v2 (October 10, 2007): revised and expanded.

Contents

CONVENTIONS All varieties are smooth and projective. Complex conjugation on $\mathbb{C}$ is denoted by ι. The symbol $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_p$, and ℓ always denotes a prime $\neq p$. On the other hand, l is allowed to equal p. For a variety X, $H^*(X) = \bigoplus_i H^i(X)$ and $H^{2*}(X)(*) = \bigoplus_i H^{2i}(X)(i)$; both are graded algebras. I denote a canonical (or a specifically given) isomorphism by $\simeq$. I assume that the reader is familiar with the basic theory of abelian varieties as, for example, in Milne 1986.

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1 The conjecture, and some folklore

Let X be a variety over $\mathbb{F}$. A model X_1 of X over a finite subfield k_1 of $\mathbb{F}$ gives rise to a commutative diagram:

$$\begin{array}{ccc} Z^r(X) & \xrightarrow{c^r} & H^{2r}(X, \mathbb{Q}_\ell(r)) \\ \uparrow & & \uparrow \\ Z^r(X_1) & \xrightarrow{c^r} & H^{2r}(X_1, \mathbb{Q}_\ell(r)). \end{array}$$

Here $Z^r(*)$ denotes the group of algebraic cycles of codimension r on a variety (free $\mathbb{Z}$-module generated by the closed irreducible subvarieties of codimension r) and c^r is the cycle map. The image of the vertical arrow at right is contained in $H^{2r}(X, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(\mathbb{F}/k_1)}$ and $Z^r(X) = \varinjlim_{X_1/k_1} Z^r(X_1)$, and so the image of the top cycle map is contained in

$$H^{2r}(X, \mathbb{Q}_\ell(r))' \overset{\text{def}}{=} \bigcup\nolimits_{X_1/k_1} H^{2r}(X, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(\mathbb{F}/k_1)}.$$

In his talk at the AMS Summer Institute at Woods Hole in July, 1964, Tate conjectured the following:[1]

CONJECTURE $T^r(X, \ell)$ (TATE). *The $\mathbb{Q}_\ell$-vector space $H^{2r}(X, \mathbb{Q}_\ell(r))'$ is spanned by algebraic classes.*

The conjecture implies that, for any model X_1/k_1, the $\mathbb{Q}_\ell$-subspace $H^{2r}(X, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(\mathbb{F}/k_1)}$ is spanned by the classes of algebraic cycles on X_1; conversely, if this is true for all models X_1/k_1 over (sufficiently large) finite fields k_1, then $T^r(X, \ell)$ is true.

I write $T^r(X)$ (resp. $T(X, \ell)$, resp. $T(X)$) for the conjecture that $T^r(X, \ell)$ is true for all ℓ (resp. all r, resp. all r and ℓ).

In the same talk, Tate mentioned the following "conjectural statement":

CONJECTURE $E^r(X, \ell)$ (EQUALITY OF EQUIVALENCE RELATIONS). *The kernel of the cycle class map $c^r : Z^r(X) \to H^{2r}(X_{\mathbb{F}}, \mathbb{Q}_\ell(r))$ consists exactly of the cycles numerically equivalent to zero.*

Both conjectures are existence statements for algebraic classes. It is well known that Conjecture $E^1(X)$ holds for all X (see Tate 1994, §5).

Let X be a variety over $\mathbb{F}$. The choice of a model of X over a finite subfield of $\mathbb{F}$ defines a Frobenius map $\pi : X \to X$. For example, for a model $X_1 \subset \mathbb{P}^n$ over $\mathbb{F}_q$, π acts as

$$(a_1 : a_2 : \ldots) \mapsto (a_1^q : a_2^q : \ldots) : X_1(\mathbb{F}) \to X_1(\mathbb{F}), \quad X_1(\mathbb{F}) \simeq X(\mathbb{F}).$$

Any such map will be called a ***Frobenius map*** of X (or a q-***Frobenius map*** if it is defined by a model over $\mathbb{F}_q$). If π_1 and π_2 are p^{n_1}- and p^{n_2}-Frobenius maps of X, then $\pi_1^{n_2 N} = \pi_2^{n_1 N}$

[1] Tate's talk is included in the mimeographed proceedings of the conference, which were distributed by the AMS to only a select few. Despite their great historical importance — for example, they contain the only written account by Artin and Verdier of their duality theorem, and the only written account by Serre and Tate of their lifting theorem — the AMS has ignored requests to make the proceedings more widely available. Fortunately, Tate's talk was reprinted in the proceedings of an earlier conference (Arithmetical algebraic geometry. Proceedings of a Conference held at Purdue University, December 5–7, 1963. Edited by O. F. G. Schilling, Harper & Row, Publishers, New York 1965).

for some $N > 1$.[2] For a Frobenius map π of X, we use a subscript a to denote the generalized eigenspace with eigenvalue a, i.e., $\bigcup_N \operatorname{Ker}\left((\pi - a)^N\right)$.

CONJECTURE $S^r(X,\ell)$ (PARTIAL SEMISIMPLICITY). *Every Frobenius map π of X acts semisimply on $H^{2r}(X,\mathbb{Q}_\ell(r))_1$ (i.e., it acts as 1).*

Weil proved that, for an abelian variety A over $\mathbb{F}$, the Frobenius maps act semisimply on $H^1(A,\mathbb{Q}_\ell)$. It follows that they act semisimply on all the cohomology groups $H^i(A,\mathbb{Q}_\ell) \simeq \bigwedge^i H^1(A,\mathbb{Q}_\ell)$. In particular, Conjecture $S(X)$ holds when X is an abelian variety over $\mathbb{F}$.

From now on, I'll write $\mathcal{T}^r_\ell(X)$ for $H^{2r}(X,\mathbb{Q}_\ell(r))'$ and call its elements the ***Tate classes*** of degree r on X. Note that $\mathcal{T}^*_\ell(X) \overset{\text{def}}{=} \bigoplus_r \mathcal{T}^r_\ell(X)$ is a graded $\mathbb{Q}_\ell$-subalgebra of $H^{2*}(X,\mathbb{Q}_\ell(*))$.

ASIDE 1.4. One can ask whether the Tate conjecture holds integrally, i.e., whether the map

$$c^r\colon Z^r(X)\otimes\mathbb{Z}_\ell \to H^{2r}(X_{\mathbb{F}},\mathbb{Z}_\ell(r))' \tag{1}$$

is surjective for all varieties X over $\mathbb{F}$. Clearly $H^{2r}(X_{\mathbb{F}},\mathbb{Z}_\ell(r))'$ contains all torsion classes, and essentially the same argument that shows that not all torsion classes in Betti cohomology are algebraic, shows that not all torsion classes in étale cohomology are algebraic.[3] However, I don't know of any varieties over $\mathbb{F}$ for which the map (1) is not surjective modulo torsion. It is known that if $T^r(X,\ell)$ and $E^r(X,\ell)$ hold for a single ℓ, then the map (1) is surjective for all but possibly finitely many ℓ. See Milne and Ramachandran 2004, §3.

Folklore

The next three theorems are folklore.

THEOREM 1.5. *Let X be a variety over $\mathbb{F}$ of dimension d, and let $r\in\mathbb{N}$. The following statements are equivalent:*

(a) *$T^r(X,\ell)$ and $E^r(X,\ell)$ are true for a single ℓ.*
(b) *$T^r(X,\ell)$, $S^r(X,\ell)$, and $T^{d-r}(X,\ell)$ are true for a single ℓ.*
(c) *$T^r(X,\ell)$, $E^r(X,\ell)$, $S^r(X,\ell)$, $T^{d-r}(X,\ell)$, and $E^{d-r}(X,\ell)$ are true for all ℓ, and the $\mathbb{Q}$-subspace $\mathcal{A}^r_\ell(X)$ of $\mathcal{T}^r_\ell(X)$ generated by the algebraic classes is a $\mathbb{Q}$-structure on $\mathcal{T}^r_\ell(X)$, i.e., $\mathcal{A}^r_\ell(X)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \simeq \mathcal{T}^r_\ell(X)$.*
(d) *the order of the pole of the zeta function $Z(X,t)$ at $t=q^{-r}$ is equal to the rank of the group of numerical equivalence classes of algebraic cycles of codimension r.*

The proof is explained in Tate 1994, §2.

THEOREM 1.6. *Let X be a variety over $\mathbb{F}$ of dimension d. If $S^{2d}(X\times X,\ell)$ is true, then every Frobenius map π acts semisimply on $H^*(X,\mathbb{Q}_\ell)$.*

PROOF. If a occurs as an eigenvalue of π on $H^r(X,\mathbb{Q}_\ell)$, then $1/a$ occurs as an eigenvalue of π on $H^{2d-r}(X,\mathbb{Q}_\ell(d))$ (by Poincaré duality), and

$$H^r(X,\mathbb{Q}_\ell)_a \otimes H^{2d-r}(X,\mathbb{Q}_\ell(d))_{1/a} \subset H^{2d}(X\times X,\mathbb{Q}_\ell(d))_1$$

(Künneth formula), from which the claim follows. □

[2] Because any two models of X become isomorphic over a finite subfield of $\mathbb{F}$; when the model X_1/k_1 is replaced by X_{1K}/K, then its Frobenius map π is replaced by $\pi^{[K:k_1]}$.

[3] The proof shows that the odd dimensional Steenrod operations are zero on the torsion algebraic classes but not on all torsion cohomology classes.

An ℓ***-adic Tate*** q***-structure*** is a finite dimensional $\mathbb{Q}_\ell$-vector space together with a linear (Frobenius) map π whose characteristic polynomial has rational coefficients and whose eigenvalues are ***Weil*** q***-numbers***, i.e., algebraic numbers α such that, for some integer m called the ***weight*** of α, $|\rho(\alpha)| = q^{m/2}$ for every homomorphism $\rho\colon \mathbb{Q}[\alpha] \to \mathbb{C}$ and, for some integer n, $q^n\alpha$ is an algebraic integer. When the eigenvalues of π are all algebraic integers, the Tate structure is said to be ***effective***. For example, for a variety X over $\mathbb{F}_q$, $H^r(X_{\mathbb{F}}, \mathbb{Q}_\ell(s))$ is a Tate q-structure of weight $r - 2s$, which is effective if $s = 0$.

A Tate p^{n_1}-structure π_1 and a Tate p^{n_2}-structure π_2 on a $\mathbb{Q}_\ell$-vector space V are said to be equivalent if $\pi_1^{n_1 N} = \pi_2^{n_2 N}$ for some N. This is an equivalence relation, and an ℓ***-adic Tate structure*** is a finite dimensional $\mathbb{Q}_\ell$-vector space together with an equivalence class of Tate q-structures. For example, for a variety X over $\mathbb{F}$, $H^r(X, \mathbb{Q}_\ell(s))$ is a Tate structure of weight $r - 2s$, which is effective if $s = 0$.

Let X be a smooth projective variety over $\mathbb{F}$. For each r, let $F_a^r H^i(X, \mathbb{Q}_\ell)$ denote the subspace of $H^i(X, \mathbb{Q}_\ell)$ of classes with support in codimension r, i.e.,

$$F_a^r H^i(X, \mathbb{Q}_\ell) = \bigcup\nolimits_U \operatorname{Ker}(H^i(X, \mathbb{Q}_\ell) \to H^i(U, \mathbb{Q}_\ell))$$

where U runs over the open subvarieties of X such that $X \smallsetminus U$ has codimension $\geq r$. If $Z = X \smallsetminus U$ has codimension r and $\widetilde{Z} \to Z$ is a desingularization of Z, then

$$H^{i-2r}(\widetilde{Z}, \mathbb{Q}_\ell)(-r) \to H^i(X, \mathbb{Q}_\ell) \to H^i(U, \mathbb{Q}_\ell)$$

is exact (see Deligne 1974, 8.2.8; a similar proof applies to étale cohomology). This shows that $F_a^r H^i(X, \mathbb{Q}_\ell)$ is an effective Tate substructure of $H^i(X, \mathbb{Q}_\ell)$ such that $F_a^r H^i(X, \mathbb{Q}_\ell)(r)$ is still effective.

CONJECTURE (GENERALIZED TATE CONJECTURE). *For a smooth projective variety X over $\mathbb{F}$, every Tate substructure $V \subset H^i(X, \mathbb{Q}_\ell)$ such that $V(r)$ is effective is contained in $F_a^r H^i(X, \mathbb{Q}_\ell)$ (cf. Grothendieck 1968, 10.3).*

THEOREM 1.8. *Let X be a variety over $\mathbb{F}$. If the Tate conjecture holds for all varieties of the form $A \times X$ with A an abelian variety (and some ℓ), then the generalized Tate conjecture holds for X (and the same ℓ).*

The proof is explained in Milne and Ramachandran 2006, 1.10.

REMARK 1.9. There are p-analogues of all of the above conjectures and statements. Let $W(k)$ be the ring of Witt vectors with coefficients in a perfect field k, and let $B(k)$ be its field of fractions. Let σ be the automorphism of $W(k)$ (or $B(k)$) that acts as $x \mapsto x^p$ on k. For a variety X over k, let $H_p^r(X)$ denote the crystalline cohomology group with coefficients in $B(k)$. It is a finite dimensional $B(k)$-vector space with a σ-linear Frobenius map F. Define

$$\mathcal{T}_p^r(X) = \bigcup\nolimits_{X_1/k_1} H_p^{2r}(X_1)(r)^{F^{n_1}=1} \quad (p^{n_1} = |k_1|).$$

This is a finite dimensional $\mathbb{Q}_p$-vector space, which the Tate conjecture $T^r(X, p)$ says is spanned by algebraic classes.

Motivic interpretation

Let $\mathsf{Mot}(\mathbb{F})$ be the category of motives over $\mathbb{F}$ defined using algebraic cycles modulo numerical equivalence. It is known that $\mathsf{Mot}(\mathbb{F})$ is a semisimple Tannakian category (Jannsen 1992). Etale cohomology defines a functor ω_ℓ on $\mathsf{Mot}(\mathbb{F})$ if and only if Conjecture $E(X,\ell)$ holds for all varieties X. Assuming this, conjecture $T(X,\ell)$ holds for all X if and only if, for all X and Y, the image of the map

$$\mathrm{Hom}(X,Y)\otimes\mathbb{Q}_\ell \to \mathrm{Hom}_{\mathbb{Q}_\ell}(\omega_\ell(X),\omega_\ell(Y)) \tag{2}$$

consists of the homomorphisms $\alpha\colon\omega_\ell(X)\to\omega_\ell(Y)$ such that $\alpha\circ\pi_X=\pi_Y\circ\alpha$ for some Frobenius maps π_X and π_Y of X and Y (necessarily q-Frobenius maps for the same q). In other words, conjectures $E(X,\ell)$ and $T(X,\ell)$ hold for all X if and only if ℓ-adic étale cohomology defines an equivalence from $\mathsf{Mot}(\mathbb{F})\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ to the category of ℓ-adic Tate structures.

2 Divisors on abelian varieties

Tate (1966) proved the Tate conjecture for divisors on abelian varieties over $\mathbb{F}$, in the form:

THEOREM 2.1. *For all abelian varieties A and B over $\mathbb{F}_q$, the map*

$$\mathrm{Hom}(A,B)\otimes\mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Q}_\ell}(T_\ell A, T_\ell B)^{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)} \tag{3}$$

is an isomorphism.

SKETCH OF PROOF. It suffices to prove this with $A=B$ (because $\mathrm{Hom}(A,B)$ is a direct summand of $\mathrm{End}(A\times B)$). Choose a polarization on A, of degree d^2 say. It defines a nondegenerate skew-symmetric form on $V_\ell A$, and a maximal isotropic subspace W of $V_\ell A$ stable under $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ will have dimension $g=\frac{1}{2}\dim V_\ell A$. For $n\in\mathbb{N}$, let

$$X(n)=T_\ell A\cap W+\ell^n T_\ell A\subset T_\ell A.$$

For each n, there exists an abelian variety $A(n)$ and an isogeny $A(n)\to A$ mapping $T_\ell A(n)$ isomorphically onto $X(n)$. There are only finitely many isomorphism classes of abelian varieties in the set $\{A(n)\}$ because each $A(n)$ has a polarization of degree d^2, and hence can be realized as a closed subvariety of $\mathbb{P}^{3^g d-1}$ of degree $3^g d(g!)$. Thus two of the $A(n)$'s are isomorphic, and we have constructed a (nonobvious) isogeny. From this beginning, Tate was able to deduce the theorem by exploiting the semisimplicity of the Frobenius map. □

COROLLARY 2.2. *For varieties X and Y over $\mathbb{F}$,*

$$T^1(X\times Y,\ell)\iff T^1(X,\ell)+T^1(Y,\ell).$$

PROOF. Compare the decomposition

$$\mathrm{NS}(X\times Y)\simeq\mathrm{NS}(X)\oplus\mathrm{NS}(Y)\oplus\mathrm{Hom}(\mathrm{Alb}(X),\mathrm{Pic}^0(Y))$$

with the similar decomposition of $H^2(X\times Y,\mathbb{Q}_\ell(1))$ given by the Künneth formula. □

COROLLARY 2.3. *The Tate conjecture $T^1(X)$ is true when X is a product of curves and abelian varieties over $\mathbb{F}$.*

PROOF. Let A be an abelian variety over $\mathbb{F}$. Choose a polarization $\lambda\colon A \to A^\vee$ of A, and let $\dagger$ be the Rosati involution on $\mathrm{End}^0(A)$ defined by λ. The map $D \mapsto \lambda^{-1} \circ \lambda_D$ defines an isomorphism

$$NS(A) \otimes \mathbb{Q} \simeq \{\alpha \in \mathrm{End}^0(A) \mid \alpha^\dagger = \alpha\}.$$

Similarly,

$$\mathcal{T}^1_\ell(A) \simeq \{\alpha \in \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell A) \mid \alpha^\dagger = \alpha \text{ and } \alpha\pi = \pi\alpha \text{ for some Frobenius map } \pi\},$$

and so T^1 for abelian varieties follows from Theorem 2.1. Since T^1 is obvious for curves, the general statement follows from Corollary 2.2. □

As $E^1(X,\ell)$ is true for all varieties X, the equivalent statements in Theorem 1.5 hold for products of curves and abelian varieties in the case $r = 1$. According to some more folklore (Tate 1994, 5.2), $T^1(X)$ and $E^1(X)$ hold for any variety X for which there exists a dominant rational map $Y \to X$ with Y a product of curves and abelian varieties.

ASIDE 2.4. The reader will have noted the similarity of (2) and (3). Tate (1994) describes how he was led to his conjecture partly by his belief that (3) was true. Today, one would say that if (3) is true, then so must (2) because "everything that's true for abelian varieties is true for motives".[4] However, when Tate was thinking about these things, motives didn't exist. Apparently, the first text in which the notion of a *motif* appears is Grothendieck's letter to Serre of August 16, 1964 (Grothendieck and Serre 2001, p173, p276).

ASIDE 2.5. Theorem 2.1 has been extended to abelian varieties over fields finitely generated over the prime field by Zarhin and Faltings (Zarhin 1974a,b; Faltings 1983; Faltings and Wüstholz 1984).

Abelian varieties with no exotic Tate classes

THEOREM 2.6. *The Tate conjecture $T(A)$ holds for any abelian variety A such that, for some ℓ, the $\mathbb{Q}_\ell$-algebra $\mathcal{T}^*_\ell(A)$ is generated by $\mathcal{T}^1_\ell(A)$; in fact, the equivalent statements of (1.5) hold for such an A and all $r \geq 0$.*

PROOF. If the $\mathbb{Q}_\ell$-algebra $\mathcal{T}^*_\ell(A)$ is generated by $\mathcal{T}^1_\ell(A)$, then, because the latter is spanned by algebraic classes (2.3), so is the former. Thus $T^r(A,\ell)$ holds for all r, and as $S(A)$ is known, this implies that the equivalent statements of (1.5) hold for A. □

For an abelian variety A, let $\mathcal{L}^*_\ell(A)$ be the $\mathbb{Q}$-subalgebra of $H^{2*}(A, \mathbb{Q}_\ell(*))$ generated by the divisor classes. Then $\mathcal{L}^*_\ell(A) \cdot \mathbb{Q}_\ell \subset \mathcal{T}^*_\ell(A)$. The elements of $\mathcal{L}^*_\ell(A)$ are called the ***Lefschetz classes*** on A, and the Tate classes not in $\mathcal{L}^*_\ell(A)$ are said to be ***exotic***.

An abelian variety A is said to have ***sufficiently many endomorphisms*** if $\mathrm{End}^0(A)$ contains an étale $\mathbb{Q}$-subalgebra of degree $2\dim A$ over $\mathbb{Q}$. Tate's theorem (2.1) implies that every abelian variety over $\mathbb{F}$ has sufficiently many endomorphisms.

Let A be an abelian variety with sufficiently many endomorphisms over an algebraically closed field k, and let $C(A)$ be the centre of $\mathrm{End}^0(A)$. The Rosati involution $\dagger$ defined by any polarization of A stabilizes $C(A)$, and its restriction to $C(A)$ is independent of the choice of the polarization. Define $L(A)$ to be the algebraic group over $\mathbb{Q}$ such that, for any commutative $\mathbb{Q}$-algebra R,

$$L(A)(R) = \{a \in C(A) \otimes R \mid aa^\dagger \in R^\times\}.$$

[4]This reasoning is circular: what we hope to be true for motives is partly based on our hope that the Tate and Hodge conjectures are true.

It is a group of multiplicative type (not necessarily connected), which acts in a natural way on the cohomology groups $H^{2*}(A^n, \mathbb{Q}_\ell(*))$ for all n. It is known that the subspace fixed by $L(A)$ consists of the Lefschetz classes,

$$H^{2*}(A^n, \mathbb{Q}_\ell(*))^{L(A)} = \mathcal{L}_\ell^*(A^n) \cdot \mathbb{Q}_\ell, \text{ all } n \text{ and } \ell, \tag{4}$$

(see Milne 1999a). Let π be a Frobenius endomorphism of A. Some power π^N of π lies in $C(A)$, hence in $L(A)(\mathbb{Q})$, and (4) shows that no power of A has an exotic Tate class if and only if π^N is Zariski dense in $L(A)$. This gives the following explicit criterion:

> 2.7 Let A be a simple abelian variety over $\mathbb{F}$, let π_A be a q-Frobenius endomorphism of A lying in $C(A)$, and let $(\pi_i)_{1 \le i \le 2s}$ be the roots in $\mathbb{C}$ of the minimum polynomial of π_A, numbered so that $\pi_i \pi_{i+s} = q$. Then no power of A has an exotic Tate class (and so the Tate conjecture holds for all powers of A) if and only if $\{\pi_1, \ldots, \pi_s, q\}$ is a $\mathbb{Z}$-linearly independent in $\mathbb{C}^\times$ (i.e., $\pi_1^{m_1} \cdots \pi_s^{m_s} = q^m$, $m_i, m \in \mathbb{Z}$, implies $m_1 = \cdots = m_s = 0 = m$).

Kowalski (2005, 2.2(2), 2.7) verifies the criterion for any simple ordinary abelian variety A such that the Galois group of $\mathbb{Q}[\pi_1, \ldots, \pi_{2g}]$ over $\mathbb{Q}$ is the full group of permutations of $\{\pi_1, \ldots, \pi_{2g}\}$ preserving the relations $\pi_i \pi_{i+g} = q$.

More generally, let $P(A)$ be the smallest algebraic subgroup of $L(A)$ containing a Frobenius element. Then no power of A has an exotic Tate class if and only if $P(A) = L(A)$. Spiess (1999) proves this for products of elliptic curves, and Zarhin (1991) and Lenstra and Zarhin (1993) prove it for certain abelian varieties. See also Milne 2001, A7.

REMARK 2.8. Let K be a CM subfield of $\mathbb{C}$, finite and Galois over $\mathbb{Q}$, and let $G = \mathrm{Gal}(K/\mathbb{Q})$. We say that an abelian variety A is ***split*** by K if $\mathrm{End}^0(A)$ is ***split*** by K, i.e., $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} K$ is isomorphic to a product of matrix algebras over $\mathbb{Q}$.

Let A be a simple abelian variety over $\mathbb{F}$, and let π be a q-Frobenius endomorphism for A. If A is split by K, then, for any p-adic prime w of K

$$f_A(w) \stackrel{\text{def}}{=} \frac{\mathrm{ord}_w(\pi)}{\mathrm{ord}_w(q)} [K_w : \mathbb{Q}_p]$$

lies in $\mathbb{Z}$ (apply Tate 1966, p142). Clearly, $f_A(w)$ is independent of the choice of π, and the equality $\pi \cdot \iota\pi = q$ implies that $f_A(w) + f_A(\iota w) = [K_w : \mathbb{Q}_p]$. Let W be the set of p-adic primes of K, and let d be the local degree $[K_w : \mathbb{Q}_p]$. The map $A \mapsto f_A$ defines a bijection from the set of isogeny classes of simple abelian varieties over $\mathbb{F}$ split by K to the set

$$\{f : W \to \mathbb{Z} \mid \ f + \iota f = d, \quad 0 \le f(w) \le d \text{ all } w\}. \tag{5}$$

The character groups of $P(A)$ and $L(A)$ have a simple description in terms of f_A, and so computing the dimensions of $P(A)$ and $L(A)$ is only an exercise in linear algebra (cf. Wei 1993, Part I; Milne 2001, A7).

Abelian varieties with exotic Tate classes

Typically, some power of a simple abelian variety over $\mathbb{F}$ will have exotic Tate classes.

PROPOSITION 2.9. *Let K be a CM-subfield of $\mathbb{C}$, finite and Galois over $\mathbb{Q}$, with Galois group G.*

(a) *There exists a CM-field K_0 such that, if $K \supset K_0$, then*

   i) *the decomposition groups D_w in G of the p-adic primes w of K are not normal in G,*

   ii) *ι acts without fixed points on the set W of p-adic primes of K.*

(b) *Assume $K \supset K_0$, so that $D_w \neq D_{w'}$ for some $w, w' \in W$. Let A be the simple abelian variety over $\mathbb{F}$ corresponding (as in 2.8) to a function $f\colon W \to \mathbb{Z}$ such that $f(w) \neq f(w')$ and neither $f(w)$ nor $f(w')$ lie in $f(W \smallsetminus \{w, w'\})$. Then some power of A supports an exotic Tate class.*

PROOF. (a) There exists a totally real field F with Galois group S_5 over $\mathbb{Q}$ having at least three p-adic primes (Wei 1993, 1.6.9), we can take $K_0 = F \cdot Q$ where Q is any quadratic imaginary field in which p splits.

(b) We have

$$\dim P(A) \leq t + 1, \quad \text{where } |W| = 2t,$$

(Wei 1993, 1.4.4). Let

$$H = \{g \in G \mid f(gw) = f(w) \text{ for all } w \in W\}.$$

Then $C(A) \approx K^H$, and so

$$\dim L(A) = \tfrac{1}{2}(G\colon H) + 1.$$

The conditions on f imply that $H \subset D_w \cap D_{w'}$, which is properly contained in D_w. As $2t = (G\colon D_w)$, we see that $\dim L(A) > \dim P(A)$, and so $L(A) \neq P(A)$. □

The set (5) has t^d elements, of which $t(t-1)(t-2)^{d-2}$ satisfy the conditions of (2.9b). As we let K grow, the ratio $t(t-1)(t-2)^{d-2}/t^d$ tends to 1, which justifies the statement preceding the proposition.

THEOREM 2.10. *There exists a family of abelian varieties A over $\mathbb{F}$ for which the Tate conjecture $T(A)$ is true and $\mathcal{T}_\ell^*(A)$ is not generated by $\mathcal{T}_\ell^1(A)$.*

PROOF. See Milne 2001 (the proof makes use of Schoen 1988, 1998). □

3 $K3$ surfaces

The next theorem was proved by Artin and Swinnerton-Dyer (1973).

THEOREM 3.1. *The Tate conjecture holds for $K3$ surfaces over $\mathbb{F}$ that admit a pencil of elliptic curves.*

SKETCH OF PROOF. Let X be an elliptic $K3$ surface, and let $f\colon X \to \mathbb{P}^1$ be the pencil of elliptic curves. A transcendental Tate class on X gives rise to a sequence $(p_n)_{n\geq 1}$ of elements of the Tate-Shafarevich group of the generic fibre of $E = X_\eta$ of f such that $\ell p_{n+1} = p_n$ for all n. From these elements, we get a tower

$$\cdots \to P_{n+1} \to P_n \to \cdots$$

of principle homogeneous spaces for E over $\mathbb{F}(\mathbb{P}^1)$. By studying the behaviour of certain invariants attached to the P_n, Artin and Swinnerton-Dyer were able to show that no such tower can exist. □

For later work on K3 surfaces, see Nygaard 1983; Nygaard and Ogus 1985; Zarhin 1996.

ASIDE 3.2. With the proof of the theorems of Tate (2.1) and of Artin and Swinnerton-Dyer (3.1), there was considerable optimism in the early 1970s that the Tate conjecture would soon be proved for surfaces over finite fields — all one had to do was attach a sequence of algebro-geometric objects to a transcendental Tate class, and then prove that the sequence couldn't exist. However, the progress since then has been meagre. For example, we still don't know the Tate conjecture for all $K3$ surfaces over $\mathbb{F}$.

4 Algebraic classes have the Tannaka property

Let $\mathcal{S}$ be a class of algebraic varieties over $\mathbb{F}$ containing the projective spaces and closed under disjoint unions and products and passage to a connected component.

THEOREM 4.1. *Let H_W be a Weil cohomology theory on the algebraic varieties over $\mathbb{F}$ with coefficients in a field Q in the sense of Kleiman 1994, §3). Assume that for all $X \in \mathcal{S}$ the kernel of the cycle map $Z^*(X) \to H_W^{2*}(X)(*)$ consists exactly of the cycles numerically equivalent to zero. Let $X \in \mathcal{S}$, and let G_X be the algebraic subgroup of $\mathrm{GL}(H_W^*(X)) \times \mathrm{GL}(Q(1))$ fixing all algebraic classes on all powers of X. Then the Q-vector space $H_W^{2*}(X^n)(*)^{G_X}$ is spanned by algebraic classes for all n.*

PROOF. Let $\mathsf{Mot}(\mathbb{F})$ be the category of motives over $\mathbb{F}$ based on the varieties in $\mathcal{S}$ and using numerical equivalence classes of algebraic cycles as correspondences. Because the Künneth components of the diagonal are known to be algebraic, $\mathsf{Mot}(\mathbb{F})$ is a semisimple Tannakian category (Jannsen 1992). Our assumption on the cycle map implies that H_W defines a fibre functor ω on $\mathsf{Mot}(\mathbb{F})$. It follows from the definition of $\mathsf{Mot}(\mathbb{F})$, that for any variety X over $\mathbb{F}$ and $n \geq 0$,

$$Z^*_{\mathrm{num}}(X^n)_{\mathbb{Q}} \simeq \mathrm{Hom}(\mathbb{1}, h^{2*}(X^n)(*))$$

where $Z^*_{\mathrm{num}}(X^n)$ is the graded $\mathbb{Z}$-algebra of algebraic cycles modulo numerical equivalence and the subscript means that we have tensored with $\mathbb{Q}$. On applying ω to this isomorphism, we obtain an isomorphism

$$Z^*_{\mathrm{num}}(X^n)_Q \simeq \mathrm{Hom}_G(Q, H_W^{2*}(X^n)(*)) = H_W^{2*}(X^n)(*)^G$$

where $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$. Since G_X is the image of G in $\mathrm{GL}(H_W^*(X)) \times \mathrm{GL}(Q(1))$, this implies the assertion. □

This is a powerful result: once we know that some cohomology classes are algebraic, it allows us to deduce that many more are (the group fixing the classes we *know* to be algebraic contains the group fixing *all* algebraic classes, and so any class that it fixes is in the span of the algebraic classes). On applying the theorem to the smallest class $\mathcal{S}$ satisfying the conditions and containing a variety X, we obtain the following criterion:

> 4.2 Let X be an algebraic variety over $\mathbb{F}$ such that $E(X^n, \ell)$ holds for all n. In order to prove that $T(X^n, \ell)$ holds for all n, it suffices to find enough algebraic classes on the powers of X for some Frobenius map to be Zariski dense in the algebraic subgroup of $\mathrm{GL}(H^*(X, \mathbb{Q}_\ell)) \times \mathrm{GL}(\mathbb{Q}_\ell(1))$ fixing the classes.

ASIDE 4.3. Theorem 4.1 holds also for almost-algebraic classes in characteristic zero in the sense of Serre 1974, 5.2 and Tate 1994, p76.

5 On the equality of equivalence relations

Recall that an abelian variety A has sufficiently many endomorphisms if $\mathrm{End}^0(A)$ contains an étale $\mathbb{Q}$-subalgebra E of degree $2\dim A$. It is known that such an E can be chosen to be a product of CM-fields. There then exists a unique involution ι_E of E such that $\rho\circ\iota_E = \iota\circ\rho$ for any homomorphism $\rho\colon E\to\mathbb{C}$.

The next theorem (and its proof) is an abstract version of the main theorem of Clozel 1999.

THEOREM 5.1. *Let k be an algebraically closed field, and let H_W^* be a Weil cohomology theory with coefficient field Q . Let A be an abelian variety over k with sufficiently many endomorphisms, and choose a $\mathbb{Q}$-subalgebra E of $\mathrm{End}^0(A)$ as above. Assume that Q splits E (i.e., $E\otimes_{\mathbb{Q}} Q \approx Q^{[E:\mathbb{Q}]}$) and that there exists an involution ι_Q of Q such that*

- $\sigma\circ\iota_E = \iota_Q\circ\sigma$ *for all* $\sigma\colon E\to Q$,
- *there exists a Weil cohomology theory $H_{W_0}^*\subset H_W^*$ with coefficient field $Q_0 \overset{\text{def}}{=} Q^{\langle\iota_Q\rangle}$ such that $H_W^* = Q\otimes_{Q_0} H_{W_0}^*$.*

Then the kernel of the cycle class map $Z^*(X)\to H_W^{2*}(X)(*)$ consists exactly of the cycles numerically equivalent to zero.

SKETCH OF PROOF. Choose a subset Φ of $\mathrm{Hom}(E,Q)$ such that

$$\mathrm{Hom}(E,Q) = \Phi\sqcup\iota_Q\Phi$$

where

$$\iota_Q\Phi = \{\iota_Q\circ\varphi \mid \varphi\in\Phi\} = \Phi\iota_E.$$

The space $H_W^1(A)$ is free of rank one over $E\otimes_{\mathbb{Q}} Q$, and so $H_W^1(A) = \bigoplus_{\sigma\in\Phi\sqcup\iota\Phi} H_W^1(A)_\sigma$ where $H_W^1(A)_\sigma$ is the one-dimensional Q-subspace on which E acts through σ. Similarly,

$$H_W^r(A)\simeq \bigwedge\nolimits_Q^r H_W^1(A) = \bigoplus_{I,J,|I|+|J|=r} H_W^r(A)_{I,J}$$

where I and J are subsets of Φ and $\iota\Phi$ respectively, and $H_W^r(A)_{I,J}$ is the one-dimensional subspace on which $a\in E$ acts as $\prod_{\sigma\in I\sqcup J}\sigma a$. Let $\mathcal{A}_W^*(A)$ be the $\mathbb{Q}$-subalgebra of $H_W^{2*}(A)(*)$ generated by the algebraic classes. Since $\mathcal{A}_W^*(A)\subset H_{W_0}^{2*}(A)(*)$, the Q-subspace $Q\cdot\mathcal{A}_W^*(A)$ of $H_W^{2*}(A)(*)$ is stable under the involution ι_H of $H_W^{2*}(A)(*)$ defined by the Q_0-substructure $H_{W_0}^{2*}(A)(*)$. Therefore, if $H_W^{2r}(A)_{I\sqcup J}(r)$ is contained in $Q\cdot\mathcal{A}_W^*(A)$, then so also is

$$\iota_W H_W^{2r}(A)_{I\sqcup J}(r) = H_W^{2r}(A)_{\iota_Q I\sqcup\iota_Q J}(r).$$

Using this, Clozel constructs, for every nonzero algebraic class, an algebraic class of complementary degree whose product with the first class is nonzero. □

COROLLARY 5.2. *For an abelian variety A over $\mathbb{F}$, there is an infinite set of primes $\ell\neq p$ such that $E(A^n,\ell)$ is true for all n.*

PROOF. Choose a $\mathbb{Q}$-subalgebra E of $\mathrm{End}^0(A)$ as before, and let Q' be the composite of the images of E in $\mathbb{C}$ under homomorphisms $E\to\mathbb{C}$. Then Q' is a finite Galois extension of $\mathbb{Q}$ that splits E and it is a CM-field. Let S be the set of primes $\ell \neq p$ such that ι lies in the decomposition group of some ℓ-adic prime v of Q'. For example, if ι is the Frobenius element of an ℓ-adic prime of Q', then $\ell \in S$, and so S has density > 0. The Weil cohomology theory $H_W \stackrel{\text{def}}{=} H_\ell \otimes Q'_v$ satisfies the hypotheses of the theorem for A. For A^n, we can choose the $\mathbb{Q}$-algebra to be E^n acting diagonally and use the same set S. □

On combining (5.2) with (4.2) we obtain the following criterion:

> 5.3 Let A be an abelian variety over $\mathbb{F}$. In order to prove that $T(A^n)$ holds for all n, it suffices to find enough algebraic classes on powers of A for some Frobenius endomorphism to be Zariski dense in the algebraic subgroup of $\mathrm{GL}(H^*_\ell(X)) \times \mathrm{GL}(\mathbb{Q}_\ell(1))$ fixing the classes for a suitable ℓ.

6 The Hodge conjecture and the Tate conjecture

To go further, we shall need to consider the Hodge conjecture (following Deligne 1982).

For a variety X over an algebraically closed field k of characteristic zero, define

$$H^*_{\mathbb{A}}(X) = H^*_{\mathbb{A}_f}(X) \times H^*_{\mathrm{dR}}(X) \text{ where } H^*_{\mathbb{A}_f}(X) = \left(\varprojlim_m H^*(X_{\mathrm{et}}, \mathbb{Z}/m\mathbb{Z})\right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For any algebraically closed field K containing k,

$$H^*_{\mathbb{A}_f}(X_K) \simeq H^*_{\mathbb{A}_f}(X) \text{ and } H^*_{\mathrm{dR}}(X_K) \simeq H^*_{\mathrm{dR}}(X) \otimes_k K,$$

and so there is a canonical homomorphism $H^*_{\mathbb{A}}(X) \to H^*_{\mathbb{A}}(X_K)$.

Let σ be a homomorphism $k \to \mathbb{C}$.[5] An element of $H^{2*}_{\mathbb{A}}(X)(*)$ is ***Hodge relative to*** σ if its image in $H^{2*}_{\mathbb{A}}(X_{\mathbb{C}})(*)$ is a Hodge class, i.e., lies in $H^{2*}(X_{\mathbb{C}}, \mathbb{Q})(*) \subset H^{2*}_{\mathbb{A}}(X_{\mathbb{C}})(*)$ and is of type $(0,0)$.

CONJECTURE (DELIGNE). *If an element of $H^{2*}_{\mathbb{A}}(X)(*)$ is Hodge relative to one homomorphism $\sigma: k \to \mathbb{C}$, then it is Hodge relative to every such homomorphism.*

An element of $H^{2*}_{\mathbb{A}}(X)(*)$ is ***absolutely Hodge*** if it is Hodge relative to every σ. Let $\mathcal{B}^*_{\mathrm{abs}}(X)$ be the set of absolutely Hodge classes on X. Then $\mathcal{B}^*_{\mathrm{abs}}(X)$ is a graded $\mathbb{Q}$-subalgebra of $H^{2*}_{\mathbb{A}}(X)(*)$, and Deligne (1982) shows:

(a) for every regular map $f: X \to Y$, f^* maps $\mathcal{B}^*_{\mathrm{abs}}(Y)$ into $\mathcal{B}^*_{\mathrm{abs}}(X)$ and f_* maps $\mathcal{B}^*_{\mathrm{abs}}(X)$ into $\mathcal{B}^*_{\mathrm{abs}}(Y)$;
(b) for every X, $\mathcal{B}^*_{\mathrm{abs}}(X)$ contains the algebraic classes;
(c) for every homomorphism $k \to K$ of algebraically closed fields, $\mathcal{B}^*_{\mathrm{abs}}(X) \simeq \mathcal{B}^*_{\mathrm{abs}}(X_K)$;
(d) for any model X_1 of X over a subfield k_1 of k with k algebraic over k_1, $\mathrm{Gal}(k/k_1)$ acts on $\mathcal{B}^*_{\mathrm{abs}}(X)$ through a finite quotient.

Property (d) shows that the image of $\mathcal{B}^*_{\mathrm{abs}}(X)$ in $H^{2*}(X, \mathbb{Q}_\ell(*))$ consists of Tate classes.

THEOREM 6.2. *If the Tate conjecture holds for X, then all absolutely Hodge classes on X are algebraic.*

[5] Throughout, I assume that k is not too big to be embedded into $\mathbb{C}$. For fields that are "too big", one can use property (c) of $\mathcal{B}^*_{\mathrm{abs}}$ below as a definition of $\mathcal{B}^*_{\mathrm{abs}}(K)$.

PROOF. Let $\mathcal{A}^*(X)$ be the $\mathbb{Q}$-subspace of $H^{2*}_{\mathbb{A}}(X)(*)$ spanned by the classes of algebraic cycles, and consider the diagram defined by a homomorphism $k \to \mathbb{C}$,

$$\begin{tikzcd}
 & \mathcal{B}^*(X_{\mathbb{C}}) \arrow[r, "\subset"] & H^{2*}_B(X_{\mathbb{C}})(*) \arrow[r, "\subset"] & H^{2*}_{\ell}(X_{\mathbb{C}})(*) \\
\mathcal{A}^*(X) \arrow[r, "\subset"] & \mathcal{B}^*_{\mathrm{abs}}(X) \arrow[u, "\cup"] \arrow[r] & \mathcal{T}^*_{\ell}(X) \arrow[r, "\subset"] & H^{2*}_{\ell}(X)(*). \arrow[u, "\simeq"]
\end{tikzcd}$$

The four groups at upper left are finite dimensional $\mathbb{Q}$-vector spaces, and the map at top right gives an isomorphism $H^{2*}_B(X_{\mathbb{C}})(*) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq H^{2*}_\ell(X_{\mathbb{C}})(*)$. Therefore, on tensoring the $\mathbb{Q}$-vector spaces in the above diagram with $\mathbb{Q}_\ell$, we get injective maps

$$\mathcal{A}^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow \mathcal{B}^*_{\mathrm{abs}}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow \mathcal{T}^*_\ell(X).$$

If the Tate conjecture holds for X, then the composite of these maps is an isomorphism, and so the first is also an isomorphism. This implies that $\mathcal{A}^*(X) = \mathcal{B}^*_{\mathrm{abs}}(X)$. □

THEOREM 6.3. *For varieties X satisfying Deligne's conjecture, the Tate conjecture for X implies the Hodge conjecture for $X_{\mathbb{C}}$.*

PROOF. For any homomorphism $k \to \mathbb{C}$, the homomorphism $H^{2*}_{\mathbb{A}}(X)(*) \hookrightarrow H^{2*}_{\mathbb{A}}(X_{\mathbb{C}})(*)$ maps $\mathcal{B}^*_{\mathrm{abs}}(X)$ into $\mathcal{B}^*(X_{\mathbb{C}})$. When Deligne's conjecture holds for X, $\mathcal{B}^*_{\mathrm{abs}}(X) \simeq \mathcal{B}^*(X_{\mathbb{C}})$. Therefore, if $\mathcal{B}^*_{\mathrm{abs}}(X)$ consists of algebraic classes, so also does $\mathcal{B}^*(X_{\mathbb{C}})$. □

ASIDE 6.4. The Hodge conjecture is known for divisors, and the Tate conjecture is generally expected to be true for divisors. However, there is little evidence for either conjecture in higher codimensions, and hence little reason to believe them. On the other hand, Deligne believes his conjecture to be true.[6]

ASIDE 6.5. As Tate pointed out at the workshop, one reason the Tate conjecture is harder than the Hodge conjecture is that it doesn't tell you which cohomology classes are algebraic; it only tells you the $\mathbb{Q}_\ell$-span of the algebraic classes.

Deligne's theorem on abelian varieties

The following is an abstract version of the main theorem of Deligne 1982.

THEOREM 6.6. *Let k be an algebraically closed subfield of $\mathbb{C}$. Suppose that for every abelian variety A over k, we have a graded $\mathbb{Q}$-subalgebra $\mathcal{C}^*(A)$ of $\mathcal{B}^*(A_{\mathbb{C}})$ such that*

(A1) *for every regular map $f\colon A \to B$ of abelian varieties over k, f^* maps $\mathcal{C}^*(B)$ into $\mathcal{C}^*(A)$ and f_* maps $\mathcal{C}^*(A)$ into $\mathcal{C}^*(B)$;*

(A2) *for every abelian variety A, $\mathcal{C}^1(A)$ contains the divisor classes; and*

(A3) *let $f\colon \mathcal{A} \to S$ be an abelian scheme over a connected smooth (not necessarily complete) k-variety S, and let $\gamma \in \Gamma(S_{\mathbb{C}}, R^{2*}f_{\mathbb{C}*}\mathbb{Q}(*))$; if γ_t is a Hodge class for all $t \in S(\mathbb{C})$ and γ_s lies in $\mathcal{C}^*(A_s)$ for one $s \in S(k)$, then it lies in $\mathcal{C}^*(A_s)$ for all $s \in S(k)$.*

Then $\mathcal{C}^*(A) \simeq \mathcal{B}^*(A_{\mathbb{C}})$ for all abelian varieties over k.

[6]When asked at the workshop, Tate said that he believes the Tate conjecture for divisors, but that in higher codimension he has no idea. Also it worth recalling that Hodge didn't conjecture the Hodge conjecture: he raised it as a problem (Hodge 1952, p184). There is considerable evidence (and some proofs) that the classes predicted to be algebraic by the Hodge and Tate conjectures do behave as if they are algebraic, at least in some respects, but there is little evidence that they are actually algebraic.

For the proof, see the endnotes to Deligne 1982). I list three applications of this theorem.

THEOREM 6.7. *In order to prove the Hodge conjecture for abelian varieties, it suffices to prove the variational Hodge conjecture.*

PROOF. Take $\mathcal{C}^*(A)$ to be the $\mathbb{Q}$-subspace of $H_B^{2*}(A_{\mathbb{C}})(*)$ spanned by the classes of algebraic cycles on A. Clearly (A1) and (A2) hold, and (A3) is (one form of) the variational Hodge conjecture. □

The following is the original version of the main theorem of Deligne 1982.

THEOREM 6.8. *Deligne's conjecture holds for all abelian varieties A over k (hence the Tate conjecture implies the Hodge conjecture for abelian varieties).*

PROOF. Take $\mathcal{C}^*(A)$ to be $\mathcal{B}^*_{\text{abs}}(A)$. Clearly (A2) holds, and we have already noted that (A1) holds. That (A3) holds is proved in Deligne 1982. □

The theorem implies that, for an abelian variety A over an algebraically closed field k of characteristic zero, any homomorphism $k \to \mathbb{C}$ defines an isomorphism $\mathcal{B}^*_{\text{abs}}(A) \to \mathcal{B}^*(A_{\mathbb{C}})$. In view of this, I now write $\mathcal{B}^*(A)$ for $\mathcal{B}^*_{\text{abs}}(A)$ and call its elements the ***Hodge classes*** on A.

MOTIVATED CLASSES (FOLLOWING ANDRÉ 1996) Let k be an algebraically closed field, and let H_W^* be a Weil cohomology theory on the varieties over k with coefficient field Q. For a variety X over k, let L and Λ be the operators defined by a hyperplane section of X, and define

$$\mathcal{E}^*(X) = Q[L, \Lambda] \cdot \mathcal{A}_W^*(X) \subset H_W^{2*}(X)(*)$$

where $\mathcal{A}_W^*(X)$ is the $\mathbb{Q}$-subspace of $H_W^{2*}(X)(*)$ generated by algebraic classes. Then $\mathcal{E}^*(X)$ is a graded Q-subalgebra of $H_W^{2*}(X)(*)$, but these subalgebras are not (obviously) stable under direct images. However, when we define

$$\mathcal{C}^*(X) = \bigcup\nolimits_Y p_*\mathcal{E}^*(X \times Y),$$

then $\mathcal{C}^*(X)$ is a graded Q-subalgebra of $H_W^{2*}(X)(*)$, and these algebras satisfy (A1). They obviously satisfy (A2).

THEOREM 6.9. *Let k be an algebraically closed subfield of $\mathbb{C}$, and let H_W be the Weil cohomology theory $X \mapsto H_B^*(X_{\mathbb{C}})$. For every abelian variety A, $\mathcal{C}^*(A) = \mathcal{B}^*(A_{\mathbb{C}})$.*

PROOF. Clearly (A2) holds, and that (A3) holds is proved in André 1996, 0.5. □

The elements of $\mathcal{C}^*(X)$ are called ***motivated classes***.

ASIDE 6.10. As Ramakrishnan pointed out at the workshop, since proving the Hodge conjecture is worth a million dollars and the Tate conjecture is harder, it should be worth more.

7 Rational Tate classes

There are by now many papers proving that, if the Tate conjecture is true, then something else even more wonderful is true. But what if we are never able to decide whether the Tate conjecture is true? or worse, what if it turns out to be false? In this section, I suggest an alternative to the Tate conjecture for varieties over finite fields, which appears to be much more accessible, and which has some of the same consequences.

An abelian variety with sufficiently many endomorphisms over an algebraically closed field of characteristic zero will now be called a ***CM abelian variety***. Let $\mathbb{Q}^{\mathrm{al}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Then the functor $A \rightsquigarrow A_{\mathbb{C}}$ from CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ to CM abelian varieties over $\mathbb{C}$ is an equivalence of categories (see, for example, Milne 2006, §7).

Fix a p-adic prime w of $\mathbb{Q}^{\mathrm{al}}$, and let $\mathbb{F}$ be its residue field. It follows from the theory of Néron models that there is a well-defined reduction functor $A \rightsquigarrow A_0$ from CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ to abelian varieties over $\mathbb{F}$, which the Honda-Tate theorem shows to be surjective on isogeny classes.

Let $\mathbb{Q}_w^{\mathrm{al}}$ be the completion of $\mathbb{Q}$ at w. For a variety X over $\mathbb{F}$, define

$$H^*_{\mathbb{A}}(X) = H^*_{\mathbb{A}_f}(X) \times H^*_p(X) \text{ where } \begin{cases} H^*_{\mathbb{A}_f}(X) &= \left(\varprojlim_{m,p\nmid m} H^*(X_{\mathrm{et}}, \mathbb{Z}/m\mathbb{Z})\right) \otimes_{\mathbb{Z}} \mathbb{Q} \\ H^*_p(X) &= H^*_{\mathrm{crys}}(X) \otimes_{W(\mathbb{F})} \mathbb{Q}_w^{\mathrm{al}}. \end{cases}$$

For a CM abelian variety A over $\mathbb{Q}^{\mathrm{al}}$,

$$\begin{aligned} H^*_{\mathbb{A}_f}(A_K)\,(\text{not-}p) &\simeq H^*_{\mathbb{A}_f}(A_0) \\ H^*_{\mathrm{dR}}(A) \otimes_{\mathbb{Q}^{\mathrm{al}}} \mathbb{Q}_w^{\mathrm{al}} &\simeq H^*_{\mathrm{crys}}(A_0) \otimes_{W(\mathbb{F})} \mathbb{Q}_w^{\mathrm{al}}, \end{aligned}$$

and so there so there is a canonical (specialization) map $H^*_{\mathbb{A}}(A) \to H^*_{\mathbb{A}}(A_0)$.

Let $\mathcal{S}$ be a class of smooth projective varieties over $\mathbb{F}$ satisfying the following condition:

> (*) it contains the abelian varieties and projective spaces and is closed under disjoint unions, products, and passage to a connected component.

DEFINITION 7.1. A family $(\mathcal{R}^*(X))_{X\in\mathcal{S}}$ with each $\mathcal{R}^*(X)$ a graded $\mathbb{Q}$-subalgebra of $H^{2*}_{\mathbb{A}}(X)(*)$ is a ***good theory of rational Tate classes*** if

(R1) for all regular maps $f\colon X \to Y$ of varieties in $\mathcal{S}$, f^* maps $\mathcal{R}^*(Y)$ into $\mathcal{R}^*(X)$ and f_* maps $\mathcal{R}^*(X)$ into $\mathcal{R}^*(Y)$;

(R2) for all varieties X in $\mathcal{S}$, $\mathcal{R}^1(X)$ contains the divisor classes;

(R3) for all CM abelian varieties A over $\mathbb{Q}^{\mathrm{al}}$, the specialization map $H^{2*}_{\mathbb{A}}(A)(*) \to H^{2*}_{\mathbb{A}}(A_0)(*)$ sends the Hodge classes on A to elements of $\mathcal{R}^*(A_0)$;

(R4) for all varieties X in $\mathcal{S}$ and all primes l (including $l = p$), the projection $H^{2*}_{\mathbb{A}}(X)(*) \to H^{2*}_l(X)(*)$ defines an isomorphism $\mathcal{R}^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}_l \to \mathcal{T}^*_l(X)$.

Thus, (R3) says that there is a diagram

$$\begin{array}{ccc} \mathcal{B}^*(A) & \subset & H^{2*}_{\mathbb{A}}(A)(*) \\ \downarrow & & \downarrow \\ \mathcal{R}^*(A) & \subset & H^{2*}_{\mathbb{A}}(A_0)(*), \end{array}$$

and (R4) says that $\mathcal{R}^*(X)$ is simultaneously a $\mathbb{Q}$-structure on each of the $\mathbb{Q}_l$-spaces $\mathcal{T}^*_l(X)$ of Tate classes (including for $l = p$). The elements of $\mathcal{R}^*(X)$ will be called the ***rational Tate classes on*** X (for the theory $\mathcal{R}$).

The next theorem is an abstract version of the main theorem of Milne 1999b.

THEOREM 7.2. *In the definition of a good theory of rational Tate classes, the condition* (R4) *can be weakened to:*

(R4*) *for all varieties X in $\mathcal{S}$ and all primes l, the projection map $H^{2*}_{\mathbb{A}}(X) \to H^{2*}_{l}(X)(*)$ sends $\mathcal{R}^{*}(X)$ into $\mathcal{T}^{*}_{\ell}(X)$.*

In other words, if a family satisfies (R1-3), and (R4*), then it satisfies (R4). For the proof, see Milne 2007. I list three applications of it.

Any choice of a basis for a $\mathbb{Q}_l$-vector space defines a $\mathbb{Q}$-structure on the vector space. Thus, there are many choices of $\mathbb{Q}$-structures on the $\mathbb{Q}_l$-spaces $\mathcal{T}^{*}_{l}(X)$. The next theorem says, however, that there is *exactly* one family of choices satisfying the compatibility conditions (R1–4).

THEOREM 7.3. *There exists at most one good theory of rational Tate classes on $\mathcal{S}$. In other words, if $\mathcal{R}^{*}_{1}$ and $\mathcal{R}^{*}_{2}$ are two such theories, then, for all $X \in \mathcal{S}$, the $\mathbb{Q}$-subalgebras $\mathcal{R}^{*}_{1}(X)$ and $\mathcal{R}^{*}_{2}(X)$ of $H^{2*}_{\mathbb{A}}(X)(*)$ are equal.*

PROOF. It follows from (R4) that if $\mathcal{R}^{*}_{1}$ and $\mathcal{R}^{*}_{2}$ are both good theories of rational Tate classes and $\mathcal{R}^{*}_{1} \subset \mathcal{R}^{*}_{2}$, then they are equal. But if $\mathcal{R}^{*}_{1}$ and $\mathcal{R}^{*}_{2}$ satisfy (R1–4), then $\mathcal{R}^{*}_{1} \cap \mathcal{R}^{*}_{2}$ satisfies (R1–3) and (R4*), and hence also (R4). Therefore $\mathcal{R}^{*}_{1} = \mathcal{R}^{*}_{1} \cap \mathcal{R}^{*}_{2} = \mathcal{R}^{*}_{2}$. □

The following is the main theorem of Milne 1999b.

THEOREM 7.4. *The Hodge conjecture for CM abelian varieties implies the Tate conjecture for abelian varieties over $\mathbb{F}$.*

PROOF. Let $\mathcal{S}_0$ be the smallest class satisfying (*). For $X \in \mathcal{S}_0$, let $\mathcal{R}^{*}(X)$ be the $\mathbb{Q}$-subalgebra of $H^{2*}_{\mathbb{A}}(X)(*)$ spanned by the algebraic classes. The family $(\mathcal{R}^{*}(X))_{X \in \mathcal{S}_0}$ satisfies (R1), (R2), and (R4*), and the Hodge conjecture implies that it satisfies (R3). Therefore it satisfies (R4), which means that the Tate conjecture holds for abelian varieties over $\mathbb{F}$. □

The following is the main theorem of André 2006.

THEOREM 7.5. *All Tate classes on abelian varieties over $\mathbb{F}$ are motivated.*

PROOF. Let $\mathcal{S}_0$ be the smallest class satisfying (*). Fix an ℓ, and for $X \in \mathcal{S}_0$, let $\mathcal{C}^{*}(X)$ be the $\mathbb{Q}_\ell$-algebra of motivated classes in $H^{2*}_{\ell}(X)(*)$. The family $(\mathcal{C}^{*}(X))_{X \in \mathcal{S}_0}$ satisfies (R1), (R2), and (R4*) with $\mathbb{A}$ replaced by ℓ, and André shows that it satisfies (R3). Therefore, by Theorem 7.2 with $\mathbb{A}$ replaced by ℓ, it satisfies (R4). □

ASIDE 7.6. Assume that there exists a good theory of rational Tate classes for abelian varieties over $\mathbb{F}$. Then we expect that all Hodge classes on all abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at w (not necessarily CM) specialize to rational Tate classes. This will follow from knowing that every $\mathbb{F}$-point on a Shimura variety lifts to a special point, which is perhaps already known (Zink 1983, Vasiu 2003). Note that it implies the "particularly interesting" corollary of the Hodge conjecture noted in Deligne 2006, §6.

8 Reduction to the case of codimension 2

Throughout this section, K is a CM subfield of $\mathbb{C}$, finite and Galois over $\mathbb{Q}$, and $G = \mathrm{Gal}(K/\mathbb{Q})$. Recall that an abelian variety A is said to be split by K if $\mathrm{End}^0(A) \otimes_{\mathbb{Q}} K$ is isomorphic to a product of matrix algebras over K.

CM abelian varieties

8.1 Let S be a finite left G-set on which ι acts without fixed points, and let S^+ be a subset of S such that $S = S^+ \sqcup \iota S^+$. Then $E \overset{\text{def}}{=} \mathrm{Hom}(S, K)^G$ is a $\mathbb{Q}$-algebra split by K such that $\mathrm{Hom}(E, K) \simeq S$. The condition on ι implies that E is a CM-algebra, and the condition on S^+ implies that it is a CM-type on E, and so S^+ defines an isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}^{S^+}$. The quotient of $\mathbb{C}^{S^+}$ by any lattice in E is an abelian variety of CM-type (E, S^+). Thus, from such a pair (S, S^+) we obtain a CM abelian variety $A(S, S^+)$, well-defined up to isogeny, which is split by K, and every such abelian variety arises in this way (up to isogeny). For example, an abelian variety of CM-type (E, Φ) where E is split by K is isogenous to (S, Φ) where $S = \mathrm{Hom}(E, \mathbb{Q})$. Note that if $(S, S^+) = \bigsqcup (S_i, S_i^+)$, then, by construction, $A(S, S')$ is isogenous to the product of the varieties $A(S_i, S_i^+)$.

PROPOSITION 8.2. *Let G act on the set S of CM-types on K by the rule*

$$g\Phi = \Phi \circ g^{-1} \overset{\text{def}}{=} \{\varphi \circ g^{-1} \mid \varphi \in \Phi\}, \quad g \in G, \quad \Phi \in S,$$

and let S^+ be the subset of CM-types containing 1_G. The pair (S, S^+) defines an abelian variety $A(S, S^+)$, and every simple CM abelian variety split by K occurs as an isogeny factor of $A(S, S^+)$.

PROOF. Certainly S is a finite left G-set on which ι acts without fixed points, and for any CM-type Φ on K, exactly one of Φ or $\iota\Phi$ contains 1_G, and so $A(S, S')$ is defined.

Let A be a simple CM abelian variety split by K, say, of CM-type (E, Φ). Fix a homomorphism $i \colon E \to K$, and let

$$\Phi' = \{g \in G \mid g \circ i \in \Phi\}$$

— it is a CM-type on K. An element g of G fixes iE if and only if $g\Phi' = \Phi'$ (see, for example, Milne 2006, 1.10), and so $g \circ i \mapsto g\Phi'$ is a bijection of G-sets from $\mathrm{Hom}(E, K)$ onto the orbit O of Φ' in S. Moreover, $1_G \in g\Phi' \overset{\text{def}}{=} \Phi' \circ g^{-1}$ if and only if $g \in \Phi'$, i.e., $g \circ i \in \Phi$. Thus, $g \circ i \mapsto g\Phi'$ sends Φ onto $O \cap S^+$. This shows that A is isogenous to the factor $A(O, O \cap S^+)$ of $A(S, S^+)$. □

The Hodge conjecture

Let $2m = [K \colon \mathbb{Q}]$. Let $a(2^m)$ be the hyperplane arrangement $\{H_1, \ldots, H_m\}$ in $\mathbb{R}^m$ with H_i the coordinate hyperplane $x_i = 0$. The hyperplanes H_i divide $\mathbb{R}^m \smallsetminus \bigcup H_i$ into connected regions

$$R(\varepsilon_1, \ldots, \varepsilon_m) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \smallsetminus \bigcup H_i \mid \mathrm{sign}(x_i) = \varepsilon_i\}$$

indexed by the set $\{\pm\}^m$. Let $R(2^m)$ be the set of connected regions, and let $R(2^m)^+$ be the subset of those with $\varepsilon_1 = +$.

Let ρ be a faithful linear representation of G on $\mathbb{R}^m$ such that G acts transitively on the set $a(2^m)$ of coordinate hyperplanes and $\rho(\iota)$ acts as -1 on $\mathbb{R}^m$. The pair $(R(2^m), R(2^m)^+)$ then satisfies the conditions of (8.5), and so defines a CM abelian variety $A = A(G, K, \rho)$ of dimension 2^{m-1} split by K.

The next statement is the main technical result of Hazama 2003.

THEOREM 8.3. *Let $A = A(G, K, \rho)$. For every $n \geq 0$, the $\mathbb{Q}$-algebra $\mathcal{B}^*(A^n)$ is generated by the classes of degree ≤ 2.*

PROOF. See Hazama 2003, §7. □

Let F be the largest totally real subfield of K. The choice of a CM-type $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ on K determines a commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle\iota\rangle & \longrightarrow & G & \longrightarrow & \mathrm{Gal}(F/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \iota\mapsto(-,\ldots,-)} & & \downarrow{\scriptstyle \rho_\Phi} & & \downarrow & & \\
1 & \longrightarrow & \{\pm\}^m & \longrightarrow & \{\pm\}^m \rtimes S_m & \longrightarrow & S_m & \longrightarrow & 1
\end{array}$$

Here $\{\pm\}$ denotes the multiplicative group of order 2, and the symmetric group S_m acts on $\{\pm\}^m$ by permutating the factors,

$$(\sigma\varepsilon)_i = \varepsilon_{\sigma^{-1}(i)}, \quad \sigma \in S_m, \quad \varepsilon = (\varepsilon_i)_{1\le i\le m} \in \{\pm\}^m.$$

Let ϵ be the unique isomorphism of groups $\{\pm\} \to \{0, 1\}$. For $g \in G$, write

$$g \circ \varphi_i = \iota^{\epsilon(\varepsilon_i)}\varphi_{\sigma^{-1}(i)}.$$

Then ρ_Φ is the homomorphism $g \mapsto ((\varepsilon_i)_i, \sigma)$.

There is a natural action of $\{\pm\}^m \rtimes S_m$ on $\mathbb{R}^m$:

$$((\varepsilon_i)_{1\le i\le m}, \sigma)(x_i)_{1\le i\le m} = (\varepsilon_i x_{\sigma^{-1}(i)})_{1\le i\le m}.$$

By composition, we get a linear representation of G on $\mathbb{R}^m$, also denoted ρ_Φ. This acts transitively transitively on $a(2^m)$, and $\rho_\Phi(\iota)$ acts as -1.

The next statement is Hazama 2003, 6.2, but the proof there is incomplete.[7]

PROPOSITION 8.4. *Every simple CM abelian variety split by K is isogenous to a subvariety of $A(G, K, \rho_\Phi)$.*

PROOF. Because of Proposition 8.2, it suffices to show that $A(G, K, \rho_\Phi)$ is isogenous to the abelian variety $A(S, S^+)$ of (8.1). For a CM-type Φ' on K, let $\varepsilon_i(\Phi')$ equal $+$ or $-$ according as $\varphi_i \in \Phi'$ or not. Then $\Phi' \mapsto (\varepsilon_i(\Phi'))_{1\le i\le m}: S \to \{\pm\}^m \simeq R(2^m)$ is a bijection, sending S^+ onto $R(2^m)^+$. This map is G-equivariant, and so $A(S, S^+)$ is isogenous to $A(R(2^m), R(2^m)^+) \overset{\text{def}}{=} A(G, K, \rho_\Phi)$. □

The following is the main theorem of Hazama 2002, 2003.

THEOREM 8.5. *In order to prove the Hodge conjecture for CM abelian varieties over $\mathbb{C}$, it suffices to prove it in codimension 2.*

PROOF. If the Hodge conjecture holds in codimension 2, then Theorem 8.3 shows that it holds for the varieties $A(G, K, \rho_\Phi)^n$, but Proposition 8.4 shows that every CM abelian A is isogenous to a subvariety of $A(G, K, \rho_\Phi)^n$ for some K and n. It is easy to see that if the Hodge conjecture holds for an abelian variety A, then it holds for any abelian subvariety (because it is an isogeny factor). □

[7] It applies only to the simple subvariety of the abelian variety of CM-type (K, Φ). Also the description (ibid. p632) of the divisor classes is incorrect, and so the statement in Theorem 7.14 that $A_{A(2^n)}(G; K)$ has exotic Hodge classes when $n \ge 3$ should be treated with caution.

The Tate conjecture

Theorems 8.5 and 7.4 show that, in order to prove the Tate conjecture for abelian varieties over $\mathbb{F}$, it suffices to prove the Hodge conjecture in codimension 2. The following is a more natural statement.

THEOREM 8.6. *In order to prove the Tate conjecture for abelian varieties over $\mathbb{F}$, it suffices to prove it in codimension 2.*

More precisely, we shall show that if $T^2(A,\ell)$ holds for all abelian varieties A over $\mathbb{F}$ and some ℓ, then $T^r(A,\ell)$ and $E^r(A,\ell)$ hold for all abelian varieties A over $\mathbb{F}$ and all r and ℓ. We fix a p-adic prime w of $\mathbb{Q}^{\mathrm{al}}$, and use the same notations as in §7.

LEMMA 8.7. *Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$. If A is split by K, then so also is A_0. Conversely, every abelian variety over $\mathbb{F}$ split by K is isogenous to an abelian variety A_0 with A split by K.*

PROOF. Let E be an étale subalgebra of $\mathrm{End}^0(A)$ such that $[E:\mathbb{Q}] = 2\dim A$. Then E is a maximal étale subalgebra of $\mathrm{End}^0(A_0)$. Because E is split by K, so also is $\mathrm{End}^0(A_0)$. The converse follows from Tate 1968, Lemme 3. □

PROOF (OF THEOREM 8.6). Let $A = A(S,S^+)$ be the abelian variety in (8.2), which (see §7) we may regard as an abelian variety over $\mathbb{Q}^{\mathrm{al}}$. Then A_0 is an abelian variety over $\mathbb{F}$ split by K, and every simple abelian variety over $\mathbb{F}$ split by K is isogenous to an abelian subvariety of A_0 (by 8.2, 8.7). The inclusion $\mathrm{End}^0(A) \hookrightarrow \mathrm{End}^0(A_0)$ realizes $C(A_0)$ as a $\mathbb{Q}$-subalgebra of $C(A)$, and hence defines an inclusion $L(A_0) \to L(A)$ of Lefschetz groups (see §2). Consider the diagram

$$\begin{tikzcd}
\mathrm{MT}(A) \arrow[r, hook] & L(A) \\
P(A_0) \arrow[r, hook] \arrow[u, dashed] & L(A_0) \arrow[u, hook]
\end{tikzcd}$$

in which $MT(A)$ is the Mumford-Tate group of A and $P(A_0)$ is the smallest algebraic subgroup of $L(A_0)$ containing a Frobenius endomorphism of A_0. Almost by definition, $\mathrm{MT}(A)$ is the largest algebraic subgroup of $L(A)$ fixing the Hodge classes in $H_B^{2*}(A_{\mathbb{C}}^n)(*)$ for all n, and so, for any prime ℓ, $\mathrm{MT}(A)_{\mathbb{Q}_\ell}$ is the largest algebraic subgroup of $L(A)_{\mathbb{Q}_\ell}$ fixing the Hodge classes in $H_\ell^{2*}(A^n)(*)$ for all n. On the other hand, the classes in $H_\ell^{2*}(A_0^n)(*)$ fixed by $P(A_0)_{\mathbb{Q}_\ell}$ are exactly the Tate classes. The specialization isomorphism $H_\ell^{2*}(A)(*) \to H_\ell^{2*}(A_0)(*)$ is equivariant for the homomorphism $L(A_0) \to L(A)$. As Hodge classes map to Tate classes in $H_\ell^{2*}(A)(*)$ (see §6), they map to Tate classes in $H_\ell^{2*}(A_0)(*)$, and so they are fixed by $P(A_0)_{\mathbb{Q}_\ell}$. This shows that $P\,(A_0)_{\mathbb{Q}_\ell} \subset \mathrm{MT}(A)_{\mathbb{Q}_\ell}$ (inside $L(A)_{\mathbb{Q}_\ell}$), and so $P(A_0) \subset \mathrm{MT}(A)$ (inside $L(A)$). The following is the main technical result of Milne 1999a (Theorem 6.1):

> Assume that K contains a quadratic imaginary field. Then the algebraic subgroup $\mathrm{MT}(A)$ and $L(A_0)$ of $L(A)$ intersect in $P(A_0)$.

We now enlarge K so that it contains a quadratic imaginary field. Corollary 5.2 allows us to choose ℓ so that $E(A_0,\ell)$ holds. Let G be the algebraic subgroup of $L(A_0)_{\mathbb{Q}_\ell}$ fixing the algebraic classes in $H_\ell^{2*}(A_0^n)(*)$ for all n. If the Tate conjecture holds in codimension 2, then G fixes the Tate classes in $H_\ell^4(A_0^n)(2)$ (all n); therefore (by Theorem 8.3), it fixes the Hodge classes in $H_{\ell}^{2*}(A^n)(*)$ (all n), and so $G \subset \mathrm{MT}(A)_{\mathbb{Q}_\ell} \cap L(A_0)_{\mathbb{Q}_\ell} = P(A_0)_{\mathbb{Q}_\ell}$; therefore, G fixes all Tate classes in $H_{\ell}^{2*}(A_0^n)(*)$ (all n), which shows that the space of Tate classes in $H_{\ell}^{2*}(A_0^n)(*)$ (all n) is spanned by the algebraic classes (apply Theorem 4.1). Therefore, the equivalent statements in Theorem 1.5 hold for A_0^n for all r and n. It follows that the same is true of every abelian subvariety of some power A_0, which includes all abelian varieties over $\mathbb{F}$ split by K (apply Lemma 8.7). Since every abelian variety over $\mathbb{F}$ is split by some CM-field, this completes the proof. □

Optimists will now try to prove the Hodge conjecture in codimension 2 for CM abelian varieties, or, what may (or may not) be easier, the Tate conjecture in codimension 2 for abelian varieties over $\mathbb{F}$. Pessimists will try to prove the opposite. Others may prefer to look at the questions in §10.

9 The Hodge standard conjecture

Let k be an algebraically closed field, and let H_W be a Weil cohomology theory on the varieties over k. For a variety X over k, let $\mathcal{A}_W^r(X)$ be the $\mathbb{Q}$-subspace of $H_W^{2r}(X)(r)$ spanned by the classes of algebraic cycles. Let $\xi \in H_W^2(X)(1)$ be the class of a hyperplane section of X, and let $L\colon H_W^i(X) \to H_W^{i+2}(X)(1)$ be the map $\cup\xi$. The ***primitive part*** $\mathcal{A}_W^r(X)_{\mathrm{prim}}$ of $\mathcal{A}_W^r(X)$ is defined to be

$$\mathcal{A}_W^r(X)_{\mathrm{prim}} = \{z \in \mathcal{A}_W^r(X) \mid L^{\dim(X)-2r+1}z = 0\}.$$

CONJECTURE (HODGE STANDARD). *Let $d = \dim X$. For $2r \leq d$, the symmetric bilinear form*

$$(x, y) \mapsto (-1)^r x \cdot y \cdot \xi^{d-2r}\colon \mathcal{A}_W^r(X)_{prim} \times \mathcal{A}_W^r(X)_{prim} \to \mathcal{A}_W^d(X) \simeq \mathbb{Q}$$

is positive definite (Grothendieck 1969, Hdg(X)).

The next theorem is an abstract version of the main theorem of Milne 2002.

THEOREM 9.2. *The Hodge standard conjecture holds for every good theory of rational Tate classes.*

In more detail, let $(\mathcal{R}^*(X))_{X\in\mathcal{S}}$ be a good theory of rational Tate classes. For $X \in \mathcal{S}$, the cohomology class ξ of a hyperplane section of X lies in $\mathcal{R}^1(X)$, and we can define $\mathcal{R}^r(X)_{\mathrm{prim}}$ and the pairing on $\mathcal{R}^r(X)_{\mathrm{prim}}$ by the above formulas. The theorem states that this pairing

$$\mathcal{R}^r(X)_{\mathrm{prim}} \times \mathcal{R}^r(X)_{\mathrm{prim}} \to \mathbb{Q}$$

is positive definite.

For the proof, see Milne 2007. I list one application of this theorem.

THEOREM 9.3. *If there exists a good theory of rational Tate classes for which all algebraic classes are rational Tate classes, then the Hodge standard conjecture holds.*

PROOF. The bilinear form on $\mathcal{R}^r(X)_{\mathrm{prim}}$ restricts to the correct bilinear form on $\mathcal{A}^r(X)_{\mathrm{prim}}$. If the first is positive definite, then so is the second, which implies that the form on $\mathcal{A}_W^r(X)_{\mathrm{prim}}$ is positive definite for any Weil cohomology theory H_W. □

ASIDE 9.4. Let $\mathcal{S}_0$ be the smallest class satisfying (*) and let $\mathcal{S}$ be a second (possibly larger) class. If the Hodge conjecture holds for CM abelian varieties, then the family $(\mathcal{A}^r(X))_{X\in\mathcal{S}_0}$ is a good theory of rational Tate classes for $\mathcal{S}_0$; if moreover, the Tate conjecture holds for all varieties in $\mathcal{S}$, then $(\mathcal{A}^r(X))_{X\in\mathcal{S}}$ is a good theory of rational Tate classes for $\mathcal{S}$. However, the Tate conjecture alone does not imply that $(\mathcal{A}^r(X))_{X\in\mathcal{S}}$ is a good theory of rational Tate classes on $\mathcal{S}$; in particular, we don't know that the Tate conjecture implies the Hodge standard conjecture. Thus, in some respects, the existence of a good theory of rational Tate classes is a stronger statement than the Tate conjecture for varieties over $\mathbb{F}$.

10 On the existence of a good theory of rational Tate classes

I consider this only for the smallest class $\mathcal{S}_0$ satisfying (*), which, I recall, contains the abelian varieties.

CONJECTURE (RATIONALITY CONJECTURE). *Let A be a CM abelian variety over $\mathbb{Q}^{\mathrm{al}}$. The product of the specialization to A_0 of any Hodge class on A with any Lefschetz class on A_0 of complementary dimension lies in $\mathbb{Q}$.*

In more detail, a Hodge class on A is an element of γ of $H^{2*}_{\mathbb{A}}(A)(*)$ and its specialization γ_0 is an element of $H^{2*}_{\mathbb{A}}(A_0)(*)$. Thus the product $\gamma_0\cdot\delta$ of γ_0 with a Lefschetz class of complementary dimension δ lies in

$$H^{2d}_{\mathbb{A}}(A_0)(d)\simeq\mathbb{A}^p_f\times\mathbb{Q}^{\mathrm{al}}_w,\quad d=\dim(A).$$

The conjecture says that it lies in $\mathbb{Q}\subset\mathbb{A}^p_f\times\mathbb{Q}^{\mathrm{al}}_w$. Equivalently, it says that the l-component of $\gamma_0\cdot\delta$ is a rational number independent of l.

REMARK 10.2. (a) The conjecture is true for a particular γ if γ_0 is algebraic. Therefore, the conjecture is implied by the Hodge conjecture for CM abelian varieties (or even by the weaker statement that the Hodge classes specialize to algebraic classes).

(b) The conjecture is true for a particular δ if it lifts to a rational cohomology class on A. In particular, the conjecture is true if A_0 is ordinary and A is its canonical lift (because then all Lefschetz classes on A_0 lift to Lefschetz classes on A).

For an abelian variety A over $\mathbb{F}$, let $\mathcal{L}^*(A)$ be the $\mathbb{Q}$-subalgebra of $H^{2*}_{\mathbb{A}}(A)(*)$ generated by the divisor classes, and call its elements the ***Lefschetz classes*** on A.

DEFINITION 10.3. Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction to an abelian variety A_0 over $\mathbb{F}$. A Hodge class γ on A is ***locally w-Lefschetz*** if its image γ_0 in $H^{2*}_{\mathbb{A}}(A_0)(*)$ is in the $\mathbb{A}$-span of the Lefschetz classes, and it is ***w-Lefschetz*** if γ_0 is itself Lefschetz.

CONJECTURE (WEAK RATIONALITY CONJECTURE). *Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction to an abelian variety A_0 over $\mathbb{F}$. Every locally w-Lefschetz Hodge class is itself w-Lefschetz.*

THEOREM 10.5. *The following statements are equivalent:*

(a) *The rationality conjecture holds for all CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$.*
(b) *The weak rationality conjecture holds for all CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$.*
(c) *There exists a good theory of rational Tate classes on abelian varieties over $\mathbb{F}$.*

PROOF. (a) $\implies$ (b): Choose a $\mathbb{Q}$-basis $e_1, \ldots, e_t$ for the space of Lefschetz classes of codimension r on A_0, and let $f_1, \ldots, f_t$ be the dual basis for the space of Lefschetz classes of complementary dimension (here we use Milne 1999a, 5.2, 5.3). If γ is a locally w-Lefschetz class of codimension r, then $\gamma_0 = \sum c_i e_i$ for some $c_i \in \mathbb{A}$. Now

$$\langle \gamma_0 \cdot f_j \rangle = c_j,$$

which (a) implies lies in $\mathbb{Q}$.

(b) $\implies$ (c): See Milne 2007.

(c) $\implies$ (a): If there exists a good theory $\mathcal{R}$ of rational Tate classes, then certainly the rationality conjecture is true, because then $\gamma_0 \cdot \delta \in \mathcal{R}^{2d} \simeq \mathbb{Q}$. □

Two questions

QUESTION 10.6. Let A be a CM abelian variety over $\mathbb{Q}^{\mathrm{al}}$, let γ be a Hodge class on A, and let δ be a divisor class on A_0. Does (A_0, γ_0, δ) always lift to characteristic zero? That is, does there always exist a CM abelian variety A' over $\mathbb{Q}^{\mathrm{al}}$, a Hodge class γ' on A', a divisor class δ' on A' and an isogeny $A'_0 \to A_0$ sending γ'_0 to γ_0 and δ'_0 to δ?

PROPOSITION 10.7. *If Question 10.6 has a positive answer, then the rationality conjecture holds for all CM abelian varieties.*

PROOF. Let γ be a Hodge class on a CM abelian variety A of dimension d over $\mathbb{Q}^{\mathrm{al}}$. If γ has dimension ≤ 1, then it is algebraic and so satisfies the rationality conjecture. We shall proceed by induction on the codimension of γ. Assume γ has dimension $r \geq 2$, and let δ be a Lefschetz class of dimension $d - r$. We may suppose that $\delta = \delta_1 \cdot \delta_2 \cdots$ where $\delta_1, \delta_2, \ldots$ are divisor classes. Apply (10.6) to (A, γ, δ). Then $\gamma' \cdot \delta'_1$ is a Hodge class on A' of codimension $r - 1$, and

$$\gamma_0 \cdot \delta \in (\gamma' \cdot \delta'_1)_0 \cdot \delta_2 \cdot \cdots \cdot \delta_{d-r}\mathbb{Q} \subset \mathbb{Q}.$$ □

A pair (A, ν) consisting of an abelian variety A over $\mathbb{C}$ and a homomorphism ν from a CM field E to $\mathrm{End}^0(A)$ is said to be of ***Weil type*** if the tangent space to A at 0 is a free $E \otimes_{\mathbb{Q}} k$-module. For such a pair (A, ν), the space

$$W^d(A, \nu) \stackrel{\text{def}}{=} \bigwedge\nolimits_E^d H^1(A, \mathbb{Q}) \subset H^d(A, \mathbb{Q}), \text{ where } d = \dim_E H^1(A, \mathbb{Q}),$$

consists of Hodge classes (Deligne 1982, 4.4). When E is quadratic over $\mathbb{Q}$, the spaces W^d were studied by Weil (1977), and for this reason its elements are called ***Weil classes***. A ***polarization*** of an abelian variety (A, ν) of Weil type is a polarization of A whose Rosati involution stabilizes E and induces complex conjugation on it. There then exists an E-hermitian form ϕ on $H_1(A, \mathbb{Q})$ and an $f \in E^\times$ with $\bar{f} = -f$ such that $\psi(x, y) \stackrel{\text{def}}{=} \mathrm{Tr}_{E/\mathbb{Q}}(f\phi(x, y))$ is a Riemann form for λ (ibid. 4.6). We say that the Weil classes on (A, ν) are ***split*** if there exists a polarization of (A, ν) for which the E-hermitian form ϕ is split (i.e., admits a totally isotropic subspace of dimension $\dim_E H_1(A, \mathbb{Q})/2$).

QUESTION 10.8. Is it possible to prove the weak rationality conjecture for split Weil classes on CM abelian variety by considering the families of abelian varieties considered in Deligne 1982, proof of 4.8, and André 2006, §3?

A positive answer to this question implies the weak rationality conjecture because of the following result of Andre (1992) (or the results of Deligne 1982, §5).

> Let A be a CM abelian variety over $\mathbb{C}$. Then there exist CM abelian varieties B_i and homomorphisms $A \to B_i$ such that every Hodge class on A is a linear combination of the inverse images of split Weil classes on the B_i.

In the spirit of Weil 1967, I leave the questions as exercises for the interested reader.

ASIDE 10.9. In the paper in which they state their conjecture concerning the structure of the points on a Shimura variety over a finite field, Langlands and Rapoport prove the conjecture for some simple Shimura varieties of PEL-type under the assumption of the Hodge conjecture for CM-varieties, the Tate conjecture for abelian varieties over finite fields, and the Hodge standard conjecture for abelian varieties over finite fields. I've proved that the first of these conjectures implies the other two (see 7.4 and 9.3), and so we have gone from needing three conjectures to needing only one. A proof of the rationality conjecture would eliminate the need for the remaining conjecture. Probably we can get by with much less, but having come so far I would like to finish it off with no fudges.

ASIDE 10.10. Readers of the Wall Street Journal on August 1, 2007, were excited to find a headline on the front page of Section B directing them to a column on "The Secret Life of Mathematicians". The column was about the workshop, and included the following paragraph:

> Progress, though, was made. V. Kumar Murty, of the University of Toronto, said that as a result of the sessions, he'd be pursuing a new line of attack on Tate. It makes use of ideas of the J.S. Milne of Michigan, who was also in attendance, and involves Abelian varieties over finite fields, in case you want to get started yourself.

This becomes more-or-less correct when you replace "Tate" with the "weak rationality conjecture".[8]

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[8] The column on the WSJ website is available only to subscribers, but there is a summary of it on the MAA site at `http://mathgateway.maa.org/do/ViewMathNews?id=143.`.

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