The Work of John Tate

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March 18, 2012/September 23, 2012/December 3, 2012

Tate helped shape the great reformulation of arithmetic and geometry which has taken place since the 1950s.
Andrew Wiles.[1]

This is my article on Tate's work for the second volume in the book series on the Abel Prize winners. True to the epigraph, I have attempted to explain it in the context of the "great reformulation".

Contents

[1] Introduction to Tate's talk at the conference on the Millenium Prizes, 2000.

Notations

We speak of the primes of a global field where others speak of the places.

$M_S = S \otimes_R M$ for M an R-module and S and R-algebra.

$|S|$ is the cardinality of S.

$X_n = \mathrm{Ker}(x \mapsto nx : X \to X)$ and $X(\ell) = \bigcup_{m \geq 0} X_{\ell^m}$ (ℓ-primary component, ℓ a prime).

$\mathrm{Gal}(K/k)$ or $G(K/k)$ denotes the Galois group of K/k.

$\mu(R)$ is the group of roots of 1 in R.

$R^\times$ denotes the group of invertible elements of a ring R (apologies to Bourbaki).

$V^\vee$ denotes the dual of a vector space or the contragredient of a representation.

K^{al}, K^{sep}, K^{ab},$K^{\mathrm{un}}\ldots$ denote an algebraic, separable, abelian, unramified...closure of a field K.

$\mathcal{O}_K$ denotes the ring of integers in a local or global field K.

1 Hecke L-series and the cohomology of number fields

1.1 Background

KRONECKER, WEBER, HILBERT, AND RAY CLASS GROUPS

For every abelian extension L of $\mathbb{Q}$, there is an integer m such that L is contained in the cyclotomic field $\mathbb{Q}[\zeta_m]$; it follows that the abelian extensions of $\mathbb{Q}$ are classified by the subgroups of the groups $(\mathbb{Z}/m\mathbb{Z})^\times \simeq G(\mathbb{Q}[\zeta_m]/\mathbb{Q})$ (Kronecker-Weber). On the other hand, the unramified abelian extensions of a number field K are classified by the subgroups of the ideal class group C of K (Hilbert). In order to be able to state a common generalization of these two results, Weber introduced the ray class groups. A modulus $\mathfrak{m}$ for a number field K is the formal product of an ideal $\mathfrak{m}_0$ in $\mathcal{O}_K$ with a certain number of real primes of K. The corresponding ray class group $C_\mathfrak{m}$ is the quotient of the group of ideals relatively prime to $\mathfrak{m}_0$ by the principal ideals generated by elements congruent to 1 modulo $\mathfrak{m}_0$ and positive at the real primes dividing $\mathfrak{m}$. For $\mathfrak{m} = (m)\infty$ and $K = \mathbb{Q}$, $C_\mathfrak{m} \simeq (\mathbb{Z}/m\mathbb{Z})^\times$. For $\mathfrak{m} = 1$, $C_\mathfrak{m} = C$.

TAKAGI AND THE CLASSIFICATION OF ABELIAN EXTENSIONS

Let K be a number field. Takagi showed that the abelian extensions of K are classified by the ray class groups: for each modulus $\mathfrak{m}$, there is a well-defined "ray class field" $L_\mathfrak{m}$ with $G(L_\mathfrak{m}/K) \approx C_\mathfrak{m}$, and every abelian extension of K is contained in a ray class field for some modulus $\mathfrak{m}$. Takagi also proved precise decomposition rules for the primes in an extension L/K in terms of the associated ray class group. These would follow from knowing that the map sending a prime ideal to its Frobenius element gives an isomorphism $C_\mathfrak{m} \to G(L_\mathfrak{m}/K)$, but Takagi didn't prove that.

DIRICHLET, HECKE, AND L-SERIES

For a character χ of $(\mathbb{Z}/m\mathbb{Z})^\times$, Dirichlet introduced the L-series

$$L(s,\chi) = \prod_{(p,m)=1} \frac{1}{1-\chi(p)p^{-s}} = \sum_{(n,m)=1} \chi(n)n^{-s}$$

in order to prove that each arithmetic progression, a, $a+m$, $a+2m$, ... with a relatively prime to m has infinitely many primes. When χ is the trivial character, $L(s,\chi)$ differs from the zeta function $\zeta(s)$ by a finite number of factors, and so has a pole at $s = 1$. Otherwise $L(s,\chi)$ can be continued to a holomorphic function on the entire complex plane and satisfies a functional equation relating $L(s,\chi)$ and $L(1-s,\overline{\chi})$.

Hecke proved that the L-series of characters of the ray class groups $C_\mathfrak{m}$ had similar properties to Dirichlet L-series, and noted that his methods apply to the L-series of even

more general characters, now called Hecke characters (Hecke 1918, 1920).[2] The L-series of Hecke characters are of fundamental importance. For example, Deuring (1953)[3] showed that the L-series of an elliptic curve with complex multiplication is a product of two Hecke L-series.

ARTIN AND THE RECIPROCITY LAW

Let K/k be an abelian extension of number fields, corresponding to a subgroup H of a ray class group $C_{\mathfrak{m}}$. Then

$$\zeta_K(s)/\zeta_k(s) = \prod\nolimits_{\chi} L(s,\chi) \quad \text{(up to a finite number of factors)} \tag{1}$$

where χ runs through the nontrivial characters of $C_{\mathfrak{m}}/H$. From this and the results of Dirichlet and Hecke, it follows that $\zeta_K(s)/\zeta_k(s)$ is holomorphic on the entire complex plane. In the hope of extending this statement to nonabelian extensions K/k, Artin (1923)[4] introduced what are now called Artin L-series.

Let K/k be a Galois extension of number fields with Galois group G, and let $\rho: G \to \mathrm{GL}(V)$ be a representation of G on a finite dimensional complex vector space V. The Artin L-series of ρ is

$$L(s,\rho) = \prod\nolimits_{\mathfrak{p}} \frac{1}{\det(1-\rho(\sigma_{\mathfrak{p}})\mathbb{N}\mathfrak{P}^{-s} \mid V^{I_{\mathfrak{P}}})}$$

where $\mathfrak{p}$ runs through the prime ideals of K, $\mathfrak{P}$ is a prime ideal of K lying over $\mathfrak{p}$, $\sigma_{\mathfrak{p}}$ is the Frobenius element of $\mathfrak{P}$, $\mathbb{N}\mathfrak{P} = (\mathcal{O}_K:\mathfrak{P})$, and $I_{\mathfrak{P}}$ is the inertia group.

Artin observed that his L-series for one-dimensional representations would coincide with the L-series of characters on ray class groups if the following "theorem" were true:

> for the field L corresponding to a subgroup H of a ray class group $C_{\mathfrak{m}}$, the map $\mathfrak{p} \mapsto (\mathfrak{p}, L/K)$ sending a prime ideal $\mathfrak{p}$ not dividing $\mathfrak{m}$ to its Frobenius element induces an isomorphism $C_{\mathfrak{m}}/H \to G(L/K)$.

Initially, Artin was able to prove this statement only for certain extensions. After Chebotarev had proved his density theorem by a reduction to the cyclotomic case, Artin (1927)[5] proved the statement in general. He called it the reciprocity law because, when K contains a primitive mth root of 1, it directly implies the classical mth power reciprocity law.

Artin noted that $L(s,\rho)$ can be analytically continued to a meromorphic function on the whole complex plane if its character χ can be expressed in the form

$$\chi = \sum\nolimits_i n_i \operatorname{Ind} \chi_i, \quad n_i \in \mathbb{Z}, \tag{2}$$

with the χ_i one-dimensional characters on subgroups of G, because then

$$L(s,\rho) = \prod\nolimits_i L(s,\chi_i)^{n_i}$$

[2] Hecke, E., Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. I, Math. Zeit. 1, 357-376 (1918); II, Math. Zeit. 6, 11-51 (1920). [3] Deuring, Max, Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins. Nachr. Akad. Wiss. Göttingen. 1953, 85–94. [4] Artin, E. Über die Zetafunktionen gewisser algebraischer Zahlkörper. Math. Ann. 89 (1923), no. 1-2, 147–156. [5] Artin, E., Beweis des allgemeinen Reziprozitätsgesetzes. Abhandlungen Hamburg 5, 353-363 (1927).

with the $L(s,\chi_i)$ abelian L-series. Brauer (1947)[6] proved that the character of a representation can always be expressed in the form (1), and Brauer and Tate found what is probably the simplest known proof of this fact (see p.59).

To complete his program, Artin conjectured that, for every nontrivial irreducible representation ρ, $L(s,\rho)$ is holomorphic on the entire complex plane. This is called the Artin conjecture. It is known to be true if the character of ρ can be expressed in the form (2) with $n_i \geq 0$, and in a few other cases.

CHEVALLEY AND IDÈLES

Chevalley gave a purely local proof of local class field theory, and a purely algebraic proof of global class field theory, but probably his most lasting contribution was to reformulate class field theory in terms of idèles.

An idèle of a number field K is an element $(a_v)_v$ of $\prod_v K_v^\times$ such that $a_v \in \mathcal{O}_v^\times$ for all but finitely many primes v. The idèles form a group J_K, which becomes a locally compact topological group when endowed with the topology for which the subgroup

$$\prod_{v|\infty} K_v^\times \times \prod_{v \text{ finite}} \mathcal{O}_v^\times$$

is open and has the product topology.[7]

Let K be number field. In Chevalley's reinterpretation, global class field theory provides a homomorphism $\phi\colon J_K/K^\times \to G(K^{\mathrm{ab}}/K)$ that induces an isomorphism

$$J_K/(K^\times \cdot \operatorname{Nm} J_L) \longrightarrow G(L/K)$$

for each finite abelian extension L/K. For each prime v of K, local class field theory provides a homomorphism $\phi_v\colon K_v^\times \to G(K_v^{\mathrm{ab}}/K_v)$ that induces an isomorphism

$$K_v^\times/\operatorname{Nm} L^\times \to G(L/K_v)$$

for each finite abelian extension L/K_v. The maps ϕ_v and ϕ are related by the diagram:

$$\begin{array}{ccc} K_v^\times & \xrightarrow{\phi_v} & G(K_v^{\mathrm{ab}}/K_v) \\ \downarrow{\scriptstyle i_v} & & \downarrow \\ J_K & \xrightarrow{\phi} & G(K^{\mathrm{ab}}/K). \end{array}$$

Beyond allowing class field theory to be stated for infinite extensions, Chevalley's idèlic approach greatly clarified the relation between the local and global reciprocity maps.

1.2 Tate's thesis and the local constants

The modern definition is that a Hecke character is a quasicharacter of $J/K^\times$, i.e., a continuous homomorphism $\chi\colon J \to \mathbb{C}^\times$ such that $\chi(x) = 1$ for all $x \in K^\times$. We explain how to interpret χ as a map on a group of ideals, which is the classical definition.

[6] Brauer, Richard, On Artin's L-series with general group characters. Ann. of Math. (2) 48, (1947). 502–514.

[7] The original topology defined by Chevalley is not Hausdorff. It was Weil who pointed out the need for a topology in which the Hecke characters become the characters on J (Weil, A., Remarques sur des résultats recents de C. Chevalley. C. R. Acad. Sci., Paris 203, 1208-1210 1936). By the time of Tate's thesis, the correct definition seems to have been common knowledge.

For a finite set S of primes, including the infinite primes, let J^S denote the subgroup of J_K consisting of the idèles $(a_v)_v$ with $a_v = 1$ for all $v \in S$, and let I^S denote the group of fractional ideals generated by those prime ideals not in S. There is a canonical surjection $J^S \to I^S$. For each Hecke character χ, there exists a finite set S such that χ factors through $J^S \xrightarrow{\text{can.}} I^S$, and a homomorphism $\varphi: I^S \to \mathbb{C}^\times$ arises from a Hecke character if and only if there exists an integral ideal $\mathfrak{m}$ with support in S, complex numbers $(s_\sigma)_{\sigma\in\mathrm{Hom}(K,\mathbb{C})}$, and integers $(m_\sigma)_{\sigma\in\mathrm{Hom}(K,\mathbb{C})}$ such that

$$\varphi((\alpha)) = \prod_{\sigma\in\mathrm{Hom}(K,\mathbb{C})} \sigma(\alpha)^{m_\sigma}\,|\sigma(\alpha)|^{s_\sigma}$$

for all $\alpha \in K^\times$ with $(\alpha) \in I^S$ and $\alpha \equiv 1 \pmod{\mathfrak{m}}$.

HECKE'S PROOF

The classical proof uses that $\mathbb{R}^n$ is self-dual as an additive topological group, and that the discrete subgroup $\mathbb{Z}^n$ of $\mathbb{R}^n$ is its own orthogonal complement under the duality. The Poisson summation formula follows easily[8] from this: for any Schwartz function f on $\mathbb{R}^n$ and its Fourier transform $\widehat{f}$,

$$\sum\nolimits_{m\in\mathbb{Z}^n} f(m) = \sum\nolimits_{m\in\mathbb{Z}^n} \widehat{f}(m).$$

Write the L-series as a sum over integral ideals, and decompose it into a finite family of sums, each of which is over the integral ideals in a fixed element of an ideal class group. The individual series are Mellin transforms of theta series, and the functional equation follows from the transformation properties of the theta series, which, in turn, follow from the Poisson summation formula.

TATE'S PROOF

An adèle of K is an element $(a_v)_v$ of $\prod K_v$ such that $a_v \in \mathcal{O}_v$ for all but finitely many primes v. The adèles form a ring $\mathbb{A}$, which becomes a locally compact topological ring when endowed with its natural topology.

Tate proved that the ring of adèles $\mathbb{A}$ of K is self-dual as an additive topological group, and that the discrete subgroup K of $\mathbb{A}$ is its own orthogonal complement under the duality. As in the classical case, this implies an (adèlic) Poisson summation formula: for any Schwartz function f on $\mathbb{A}$ and its Fourier transform $\widehat{f}$

$$\sum\nolimits_{\gamma\in K} f(\gamma) = \sum\nolimits_{\gamma\in K} \widehat{f}(\gamma).$$

Let χ be a Hecke character of K, and let χ_v be the quasicharacter $\chi \circ i_v$ on $K_v^\times$. Tate defines local L-functions $L(\chi_v)$ for each prime v of K (including the infinite primes) as integrals over K_v, and proves functional equations for them. He writes the global L-function as an integral over J, which then naturally decomposes into a product of local L-functions. The functional equation for the global L-function follows from the functional equations of the local L-functions and the Poisson summation formula.

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[8] Let f be a Schwartz function on $\mathbb{R}$, and let $\widehat{f}$ be its Fourier transform on $\widehat{\mathbb{R}} = \mathbb{R}$. Let ϕ be the function $x + \mathbb{Z} \mapsto \sum_{n\in\mathbb{Z}} f(x+n)$ on $\mathbb{R}/\mathbb{Z}$, and let $\widehat{\phi}$ be its Fourier transform on $\widehat{\mathbb{R}/\mathbb{Z}} = \mathbb{Z}$. A direct computation shows that $\widehat{f}(n) = \widehat{\phi}(n)$ for all $n \in \mathbb{Z}$. The Fourier inversion formula says that $\phi(x) = \sum_{n\in\mathbb{Z}} \widehat{\phi}(n)\chi(x)$; in particular, $\phi(0) = \sum_{n\in\mathbb{Z}} \widehat{\phi}(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$. But, by definition, $\phi(0) = \sum_{n\in\mathbb{Z}} f(n)$.

Although, the two proofs are superficially similar, in the details they are quite different. Once Tate has developed the harmonic analysis of the local fields and of the adèle ring, including the Poisson summation formula, "an analytic continuation can be given at one stroke for all of the generalized ζ-functions, and an elegant functional equation can be established for them ... without Hecke's complicated theta-formulas".[9]

One consequence of Tate's treating all primes equally, is that the Γ-factors arise naturally as the local zeta functions of the infinite primes. By contrast, in the classical treatment, their appearance is more mysterious.

As Kudla writes:[10]

> Tate provides an elegant and unified treatment of the analytic continuation and functional equation of Hecke L-functions. The power of the methods of abelian harmonic analysis in the setting of Chevalley's adèles/idèles provided a remarkable advance over the classical techniques used by Hecke. ... In hindsight, Tate's work may be viewed as giving the theory of automorphic representations and L-functions of the simplest connected reductive group $G = \mathrm{GL}(1)$, and so it remains a fundamental reference and starting point for anyone interested in the modern theory of automorphic representations.

Tate's thesis completed the re-expression of the classical theory in terms of idèles. In this way, it marked the end of one era, and the start of a new.

NOTES. Tate completed his thesis in May 1950. It was widely quoted long before its publication in 1967. Iwasawa obtained similar results about the same time as Tate, but published nothing except for the brief notes Iwasawa 1950, 1952.[11]

LOCAL CONSTANTS

Let χ be a Hecke character, and let $\Lambda(s,\chi)$ be its completed L-series. The theorem of Hecke and Tate says that $\Lambda(s,\chi)$ admits a meromorphic continuation to the whole complex plane, and satisfies a functional equation

$$\Lambda(1-s,\chi) = W(\chi)\cdot\Lambda(s,\overline{\chi})$$

with $W(\chi)$ a complex number of absolute value 1. The number $W(\chi)$ is called the root number or the epsilon factor. It is a very interesting number. For example, for a Dirichlet character χ with conductor f, it equals $\tau(\chi)/\sqrt{\pm f}$ where $\tau(\chi)$ is the Gauss sum $\sum_{a=1}^{f}\chi(a)\mathbf{e}(a/f)$. An importance consequence of Tate's description of the global functional equation as a product of local functional equations is that he obtains an expression

$$W(\chi) = \prod\nolimits_v W(\chi_v) \tag{3}$$

of $W(\chi)$ as a product of (explicit) local root numbers $W(\chi_v)$.

Langlands pointed out[12] that his conjectural correspondence between degree n representations of the Galois groups of number fields and automorphic representations of GL_n

[9] Tate 1967, pp. 305–306. [10] Kudla, S. In: An introduction to the Langlands program. Edited by Joseph Bernstein and Stephen Gelbart. Birkhäuser Boston, Inc., Boston, MA, 2003, p.133. [11] Iwasawa, K., A Note on Functions, Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, p.322. Amer. Math. Soc., Providence, R. I., 1952; Letter to J. Dieudonné. Zeta functions in geometry, April 8, 1952, Adv. Stud. Pure Math., 21, pp.445–450, Kinokuniya, Tokyo, 1992. [12] See his "Notes on Artin L-functions" and the associated comments at http://publications.ias.edu/rpl/section/22.

requires that there be a similar decomposition for the root numbers of Artin L-series, or, more generally, for the Artin-Hecke L-series that generalize both Artin and Hecke L-series (see p.11). For a Hecke character, the required decomposition is just that of Tate. Every expression (2), p.4, of an Artin character χ as a sum of monomial characters gives a decomposition of its root number $W(\chi)$ as a product of local root numbers — the problem is to show that the decomposition is *independent* of the expression of χ as a sum.[13]

For an Artin character χ, Dwork (1956)[14] proved that there exists a decomposition (3) of $W(\chi)$ well-defined up to signs; more precisely, he proved that there exists a well-defined decomposition for $\chi(-1)W(\chi)^2$. Langlands completed Dwork's work and thereby found a local proof that there exists a well-defined decomposition for $W(\chi)$. However, he abandoned the writing up of his proof when Deligne (1973)[15] found a simpler global proof.

Tate (1977b) gives an elegant exposition of these questions, including a proof of (3) for Artin root numbers by a variant of Deligne's method, and a proof of a theorem of Fröhlich and Queyrut that $W(\chi) = 1$ when χ is the character of a representation that preserves a quadratic form.

1.3 The cohomology of number fields

TATE COHOMOLOGY

With the action of a group G on an abelian group M, there are associated homology groups $H_r(G, M)$, $r \geq 0$, and cohomology groups $H^r(G, M)$, $r \geq 0$. When G is finite, the map $m \mapsto \sum_{\sigma \in G} \sigma m$ defines a homomorphism

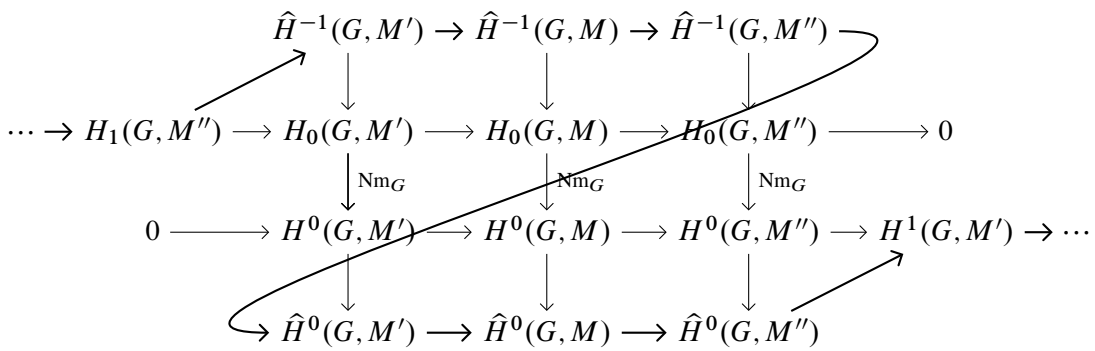
$$H_0(G, M) \stackrel{\text{def}}{=} M_G \stackrel{\text{Nm}_G}{\longrightarrow} M^G \stackrel{\text{def}}{=} H^0(G, M),$$

and Tate defined cohomology groups $\hat{H}^r(G, M)$ for all integers r by setting

$$\hat{H}^r(G, M) \stackrel{\text{def}}{=} \begin{cases} H_{-r-1}(G, M) & r < -1 \\ \text{Ker}(\text{Nm}_G) & r = -1 \\ \text{Coker}(\text{Nm}_G) & r = 0 \\ H^r(G, M) & r > 0. \end{cases}$$

[13] In fact, this is not quite true, but is true for "virtual representations" with "virtual degree 0". The decomposition of the root number of the character χ of a Galois representation is obtained by writing it as $\chi = (\chi - \dim\chi \cdot 1) + \dim\chi \cdot 1$. [14] Dwork, B., On the Artin root number. Amer. J. Math. 78 (1956), 444–472. Based on his 1954 thesis as a student of Tate. [15] Deligne, P. Les constantes des équations fonctionnelles des fonctions L. Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501–597. Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973.

The diagram



shows that a short exact sequence of G-modules gives an exact sequence of cohomology groups infinite in both directions. These groups are now called the Tate cohomology groups. Most of the usual constructions for cohomology groups (except the inflation maps) extend to the Tate groups.

NOTES. Tate's construction was included in Serre 1953,[16] Cartan and Eilenberg 1956,[17] and elsewhere. Farrell 1978[18] extended Tate's construction to infinite groups having finite virtual cohomological dimension (Tate-Farrell cohomology), and others have defined an analogous extension of Hochschild cohomology (Tate-Hochschild cohomology).

THE COHOMOLOGY GROUPS OF ALGEBRAIC NUMBER FIELDS

Let G be a finite group, let C be a G-module, and let u be an element of $H^2(G,C)$. Assume that $H^1(H,C) = 0$ for all subgroups H of G and that $H^2(H,C)$ is cyclic of order $(H:1)$ with generator the restriction of u. Then Tate (1952c) showed that cup product with u defines an isomorphism

$$x \mapsto x \cup u\colon \hat{H}^r(G,\mathbb{Z}) \to \hat{H}^{r+2}(G,C) \tag{4}$$

for all $r \in \mathbb{Z}$. He proves this by constructing an exact sequence

$$0 \to C \to C(\varphi) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

depending on the choice of a 2-cocycle φ representing u, and showing that

$$\hat{H}^r(G,C(\varphi)) = 0 = \hat{H}^r(G,\mathbb{Z}[G])$$

for all $r \in \mathbb{Z}$. Now the double boundary map is an isomorphism $\hat{H}^r(G,\mathbb{Z}) \to \hat{H}^{r+2}(G,C)$.

On taking G to be the Galois group of a finite extension L/K of number fields, C to be the idèle class group of L, and u the fundamental class of L/K,[19] one obtains for $r = -2$ the inverse of the Artin reciprocity map

$$G/[G,G] \xrightarrow{\simeq} C^G.$$

[16] Serre, Jean-Pierre, Cohomologie et arithmétique, Séminaire Bourbaki 1952/1953, no. 77. [17] Cartan, Henri; Eilenberg, Samuel. Homological algebra. Princeton University Press, Princeton, N. J., 1956 [18] Farrell, F. Thomas, An extension of Tate cohomology to a class of infinite groups. J. Pure Appl. Algebra 10 (1977/78), no. 2, 153-161. [19] Which had been discovered by Nakayama and Weil, cf. Artin and Tate, 2009, p189.

Let L/K be a finite Galois extension of global fields (e.g., number fields) with Galois group G. There is an exact sequence of G-modules

$$1 \to L^\times \to J_L \to C_L \to 1 \tag{5}$$

where J_L is the group of idèles of L and C_L is the idèle class group. Tate determined the cohomology groups of the terms in this sequence by relating them to those in the much simpler sequence

$$0 \to X \to Y \to Z \to 0. \tag{6}$$

Here Y is the free abelian group on the set of primes of L (including the infinite primes) with G acting through its action on the primes, and Z is just $\mathbb{Z}$ with G acting trivially; the map $Y \to Z$ is $\sum n_P P \mapsto \sum n_P$, and X is its kernel. Tate proved that there is a canonical isomorphism of doubly infinite exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \hat{H}^r(G,X) & \longrightarrow & \hat{H}^r(G,Y) & \longrightarrow & \hat{H}^r(G,Z) & \longrightarrow & \cdots \\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ \cdots & \longrightarrow & \hat{H}^{r+2}(G,L^\times) & \longrightarrow & \hat{H}^{r+2}(G,J_L) & \longrightarrow & \hat{H}^{r+2}(G,C_L) & \longrightarrow & \cdots. \end{array} \tag{7}$$

Tate announced this result in his Short Lecture at the 1954 International Congress, but did not immediately publish the proof.

The Tate-Nakayama theorem

Nakayama (1957) generalized Tate's isomorphism (4) by weakening the hypotheses — it suffices to require them for Sylow subgroups — and strengthening the conclusion — cup product with u defines an isomorphism

$$x \mapsto x \cup u : \hat{H}^r(G,M) \to \hat{H}^{r+2}(G, C \otimes M)$$

provided C or M is torsion-free. Building on this, Tate (1966c) proved that the isomorphism (7) holds with each of the sequences (5) and (6) replaced by its tensor product with M. In other words, he replaced the torus $\mathbb{G}_m$ implicit in (7) with an arbitrary torus defined over K. He also proved the result for any "suitably large" set of primes S — the module $L^\times$ is replaced with the group of S-units in L and J_L is replaced by the group of idèles whose components are units outside S. This result is usually referred to as the Tate-Nakayama theorem, and is widely used, for example, throughout the Langlands program including in the proof of the fundamental lemma.

Abstract class field theory: class formations

Tate's theorem (see (4) above) shows that, in order to have a class field theory over a field k, all one needs is, for each system of fields

$$k^{\text{sep}} \supset L \supset K \supset k, \quad [L:k] < \infty, \quad L/K \text{ Galois},$$

a $G(L/K)$-module C_L and a "fundamental class" $u_{L/K} \in H^2(G(L/K), C_L)$ satisfying Tate's hypotheses; the pairs $(C_L, u_{L/K})$ should also satisfy certain natural conditions when K and L vary. Then Tate's theorem then provides "reciprocity" isomorphisms

$$C_L^G \xrightarrow{\simeq} G/[G,G], \quad G = G(L/K),$$

Artin and Tate (1961, Chapter 14) formalized this by introducing the abstract notion of a class formation.

For example, for any nonarchimedean local field k, there is a class formation with $C_L = L^\times$ for any finite extension L of k, and for any global field, there is a class formation with $C_L = J_L/L^\times$. In both cases, $u_{L/K}$ is the fundamental class.

Let k be an algebraic function field in one variable with algebraically closed constant field. Kawada and Tate (1955a) show that there is a class formation for unramified extensions of K with C_L the dual of the group of divisor classes of L. In this way they obtain a "pseudo class field theory" for k, which they examine in some detail when $k = \mathbb{C}$.

THE WEIL GROUP

Weil was the first to find a common generalization of Artin L-series and Hecke L-series. For this he defined what is now known as the Weil group. The Weil group of a finite Galois extension of number fields L/K is an extension

$$1 \to C_L \to W_{L/K} \to \mathrm{Gal}(L/K) \to 1$$

corresponding to the fundamental class in $H^2(G_{L/K}, C_L)$. Each representation of $W_{L/K}$ has an L-series attached to it, and the L-series arising in this way are called Artin-Hecke L-series. Weil (1951)[20] constructed these groups, thereby discovering the fundamental class, and proved the fundamental properties of Weil groups. Artin and Tate (1961, Chapter XV) developed the theory of the Weil groups in the abstract setting of class formations, basing their definition on the existence of a fundamental class. In the latest (2009) edition of the work, Tate expanded their presentation and included a sketch of Weil's original construction (pp. 185–189).

SUMMARY

The first published exposition of class field theory in which full use of the cohomology theory is made is Chevalley 1954.[21] There Chevalley writes:

> One of the most baffling features of classical class field theory was that it appeared to say practically nothing about normal extensions that are not abelian. It was discovered by A. Weil and, from a different point of view, T. Nakayama that class field theory was actually much richer than hitherto suspected; in fact, it can now be formulated in the form of statements about normal extensions without any mention whatsoever of abelian extensions. Of course, it is true that it is only in the abelian case that these statements lead to laws of decomposition for prime ideals of the subfield and to the law of reciprocity. Nevertheless, it is clear that, by now, we know something about the arithmetic of non abelian extensions. In fact, since the work of J. Tate, it may be said that we know almost everything that may be formulated in terms of cohomology in the idèle class group, and generally a great deal about everything that can be formulated in cohomological terms.

NOTES. Tate was not the first to make use of group cohomology in class field theory. In a sense it had always been there, since crossed homomorphisms and factor systems had long been used.

[20] Weil, André, Sur la théorie du corps de classes. J. Math. Soc. Japan 3, (1951). 1–35. [21] Chevalley, C. Class field theory. Nagoya University, Nagoya, 1954.

Weil and Nakayama independently discovered the fundamental class, Weil by constructing the Weil group, and Nakayama as a consequence of his work (partly with Hochschild) to determine the cohomology groups of number fields in degrees 1 and 2. Tate's contribution was to give a remarkably simple description of all the basic cohomology groups of number fields, and to construct a general isomorphism that, in the particular case of an abelian extension and in degree -2, became the Artin reciprocity isomorphism.

1.4 The cohomology of profinite groups

Krull (1928)[22] showed that, when the Galois group of an infinite Galois extension of fields Ω/F is endowed with a natural topology, there is a Galois correspondence between the intermediate fields of Ω/F and the *closed* subgroups of the Galois group. The topological groups that arise as Galois groups are exactly the compact groups G whose open normal subgroups U form a fundamental system $\mathcal{N}$ of neighbourhoods of 1. Tate described such topological groups as being "of Galois-type", but we now say they are "profinite".

For a profinite group G, Tate (1958d) considered the G-modules M such that $M = \bigcup_{U\in\mathcal{N}} M^U$. These are the G-modules M for which the action is continuous relative to the discrete topology on M. For such a module, Tate defined cohomology groups $H^r(G,M)$, $r \geq 0$, using continuous cochains, and he showed that

$$H^r(G,M) = \varinjlim_{U\in\mathcal{N}} H^r(G/U, M^U)$$

where $H^r(G/U, M^U)$ denotes the usual cohomology of the (discrete) finite group G/U acting on the abelian group M^U. In particular, $H^r(G,M)$ is torsion for $r > 0$.

The cohomological dimension and strict cohomological dimension of a profinite group G relative to a prime number p are defined by the conditions:

$$\begin{aligned} \mathrm{cd}_p(G) \leq n &\iff H^r(G,M)(p) = 0 \text{ whenever } r > n \text{ and } M \text{ is torsion;} \\ \mathrm{scd}_p(G) \leq n &\iff H^r(G,M)(p) = 0 \text{ whenever } r > n. \end{aligned}$$

Here (p) denotes the p-primary component. The (strict) cohomological dimension of a field is the (strict) cohomological dimension of its absolute Galois group. Among Tate's theorems are the following statements:

(a) A pro p-group G is free if and only if $\mathrm{cd}_p(G) = 1$. (A pro p-group is a profinite group G such that G/U is a p-group for all $U \in \mathcal{N}$; it is free if it is of the form $\varprojlim F/N$ where F is the free group on symbols $(a_i)_{i\in I}$, say, and N runs through the normal subgroups of G containing all but finitely many of the a_i and such G/N is a finite p-group).

(b) If k is a local field other than $\mathbb{R}$ or $\mathbb{C}$, then $\mathrm{scd}_p(k) = 2$ for all $p \neq \mathrm{char}(k)$.

(c) Let $K \supset k$ be an extension of fields of transcendence degree n. Then

$$\mathrm{cd}_p(K) \leq \mathrm{cd}_p(k) + n,$$

with equality if K is finitely generated over k, $\mathrm{cd}_p(k) < \infty$, and $p \neq \mathrm{char}(k)$. In particular, if k is algebraically closed, then the p-cohomological dimension of a finitely generated K is equal to its transcendence degree over k ($p \neq \mathrm{char}(k)$).

[22] Krull, W., Galoissche Theorie der unendlichen algebraischen Erweiterungen. Math. Ann. 100, 687-698 (1928).

According to Tate 1958d, statement (c) "historically arose at [the theory's] beginning. Its conjecture and the sketch of its proof are due to Grothendieck". Indeed, from Grothendieck's point of view, the cohomology of the absolute Galois group of a field k should be interpreted as the étale cohomology of $\operatorname{Spec} k$, and the last statement of (c) is suggested by the weak Lefschetz theorem in étale cohomology.

NOTES. Tate explained the above theory in his 1958 seminar at Harvard.[23] Douady reported on Tate's work in a Bourbaki seminar in 1959,[24] and Lang included Tate's unpublished article 1958d as Chapter VII of his 1967 book.[25] Serre included the theory in his course at the Collège de France, 1962–63; see Serre 1964.[26] Tate himself published only the brief lectures Tate 2001.

1.5 *Duality theorems*

In the early 1960s, Tate proved duality theorems for modules over the absolute Galois groups of local and global fields that have become an indispensable tool in Iwasawa theory, the theory of abelian varieties, and in other parts of arithmetic geometry. The main global theorem was obtained independently by Poitou, and is now referred to as the Poitou-Tate duality theorem.

Throughout, K is a field, $\bar{K}$ is a separable closure of K, and G is the absolute Galois group $\operatorname{Gal}(\bar{K}/K)$. All G-modules are discrete (i.e., the action is continuous for the discrete topology on the module). The dual M' of such a module is $\operatorname{Hom}(M, \bar{K}^\times)$.

LOCAL RESULTS

Let K be a nonarchimedean local field, i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Local class field theory provides us with a canonical isomorphism $H^2(G, \bar{K}^\times) \simeq \mathbb{Q}/\mathbb{Z}$. Tate proved that, for every finite G-module M whose order m is not divisible by characteristic of K, the cup-product pairing

$$H^r(G, M) \times H^{2-r}(G, M') \to H^2(G, \bar{K}^\times) \simeq \mathbb{Q}/\mathbb{Z} \tag{8}$$

is a perfect duality of finite groups for all $r \in \mathbb{N}$. In particular, $H^r(G, M) = 0$ for $r > 2$. Moreover, the following holds for the Euler-Poincaré characteristic of M:

$$\frac{\left|H^0(G, M)\right| \left|H^2(G, M)\right|}{\left|H^1(G, M)\right|} = \frac{1}{(\mathcal{O}_K : m\mathcal{O}_K)}.$$

A G-module M is said to be unramified if the inertia group I in G acts trivially on M. When M is unramified and its order is prime to the residue characteristic, Tate proved that the submodules $H^1(G/I, M)$ and $H^1(G/I, M')$ of $H^1(G, M)$ and $H^1(G, M')$ are exact annihilators in the pairing (8).

Let $K = \mathbb{R}$. In this case, there is a canonical isomorphism $H^2(G, \bar{K}^\times) \simeq \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. For any finite G-module M, the cup-product pairing of Tate cohomology groups

$$\hat{H}^r(G, M) \times \hat{H}^{2-r}(G, M') \to H^2(G, \bar{K}^\times) \simeq \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}$$

is a perfect pairing for all $r \in \mathbb{Z}$. Moreover, $\hat{H}^r(G, M)$ is a finite group, killed by 2, whose order is independent of r.

[23] See Shatz, Math Reviews 0212073. [24] Douady, Adrien, Cohomologie des groupes compacts totalement discontinus (d'après des notes de Serge Lang sur un article non publie de Tate). Séminaire Bourbaki, Vol. 5, Exp. No. 189, 287–298, Soc. Math. France, Paris, 1959. [25] Lang, Serge, Rapport sur la cohomologie des groupes. W. A. Benjamin, Inc., New York-Amsterdam 1967. [26] Serre, Jean-Pierre, Cohomologie Galoisienne. Cours au Collège de France, 1962-1963. Seconde édition. Lecture Notes in Mathematics 5 Springer-Verlag, Berlin-Heidelberg-New York.

GLOBAL RESULTS

Let K be a global field, and let M be a finite G-module whose order is not divisible by the characteristic of K. Let

$$H^r(K_v, M) = \begin{cases} H^r(G_{K_v}, M) & \text{if } v \text{ is nonarchmedean} \\ \hat{H}^r(G_{K_v}, M) & \text{otherwise.} \end{cases}$$

The local duality results show that $\prod_v H^0(K_v, M)$ is dual to $\bigoplus_v H^2(K_v, M')$ and that $\prod'_v H^1(K_v, M)$ is dual to $\prod'_v H^1(K_v, M')$ — the $'$ means that we are taking the restricted product with respect to the subgroups $H^1(G_{K_v}/I_v, M^{I_v})$.

In the table below, the homomorphisms at right are the duals of the homomorphisms at left with M replaced by M', i.e., $\beta^r(M) = \alpha^{2-r}(M')^*$ with $-^* = \text{Hom}(-, \mathbb{Q}/\mathbb{Z})$:

$$\begin{aligned} H^0(K,M) &\xrightarrow{\alpha^0} \textstyle\prod_v H^0(K_v,M) & \textstyle\bigoplus_v H^2(K_v,M) &\xrightarrow{\beta^2} H^0(K,M')^* \\ H^1(K,M) &\xrightarrow{\alpha^1} \textstyle\prod'_v H^1(K_v,M) & \textstyle\prod'_v H^1(K_v,M) &\xrightarrow{\beta^1} H^1(K,M')^* \\ H^2(K,M) &\xrightarrow{\alpha^2} \textstyle\bigoplus_v H^2(K_v,M) & \textstyle\prod_v H^0(K_v,M) &\xrightarrow{\beta^0} H^2(K,M')^*. \end{aligned}$$

The Poitou-Tate duality theorem states that there is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(K,M) & \xrightarrow{\alpha^0} & \prod_v H^0(K_v,M) & \xrightarrow{\beta^0} & H^2(K,M')^* \\ & & & & & & \downarrow \\ & & H^1(K,M')^* & \xleftarrow{\beta^1} & \prod'_v H^1(K_v,M) & \xleftarrow{\alpha^1} & H^1(K,M) \\ & & \downarrow & & & & \\ & & H^2(K,M) & \xrightarrow{\alpha^2} & \bigoplus_v H^2(K_v,M) & \xrightarrow{\beta^2} & H^0(K,M')^* \longrightarrow 0. \end{array} \tag{9}$$

(with explicit descriptions for the unnamed arrows). Moreover, for $r \geq 3$, the map

$$H^r(K,M) \to \textstyle\prod_{v \text{ real}} H^r(K_v,M)$$

is an isomorphism. In fact, the statement is more general in that one replaces the set of all primes with a nonempty set S containing the archimedean primes in the number field case (there is then a restriction on the order of M).

For the Euler-Poincaré characteristic, Tate proved that

$$\frac{\left|H^0(G,M)\right|\left|H^2(G,M)\right|}{\left|H^1(G,M)\right|} = \frac{1}{|M|^{r_1+2r_2}} \prod_{v|\infty} \left|M^{G_v}\right| \tag{10}$$

where r_1 and r_2 are the numbers of real and complex primes.

NOTES. Tate announced the above results (with brief indications of proof) in his talk at the 1962 International Congress except for the last statement on the Euler-Poincaré characteristic, which was announced in Tate 1966e. Tate's proofs of the local statements were included in Serre 1964.[26] Later Tate (1966f) proved a duality theorem for an abstract class formation, which included both the local and global duality results, and in which the exact sequence (9) arises as a sequence of Exts. This proof, as well as proofs of the formulas for the Euler-Poincaré characteristics, are included in Milne 1986.[27]

[27] Milne, J. S. Arithmetic duality theorems. Perspectives in Mathematics, 1. Academic Press, Inc., Boston, MA, 1986.

1.6 Expositions

The notes of the famous Artin-Tate seminar on class field theory have been a standard reference on the topic since they first became available in 1961. They have recently been republished in slightly revised form by the American Mathematical Society. Tate made important contributions, both in his article on global class field theory and in the exercises, to another classic exposition of algebraic number theory, namely, the proceeding of the 1965 Brighton conference.[28] His talk on Hilbert's ninth problem, which asked for "a proof of the most general reciprocity law in any number field", illuminates the problem and the work done on it (Tate 1976a). Tate's contribution to the proceedings of the Corvallis conference, gave a modern account of the Weil group and an explanation of the hypothetical nonabelian reciprocity law in terms of the more general Weil-Deligne group (Tate 1979).

2 Abelian varieties and curves

In the course of proving the Riemann hypothesis for curves and abelian varieties in the 1940s, Weil rewrote the foundations of algebraic geometry, including the theory of abelian varieties. This made it possible to do algebraic geometry in a rigorous fashion over arbitrary base fields. In the late 1950s, Grothendieck rewrote the foundations again, developing the more natural and flexible language of schemes.

2.1 The Riemann hypothesis for curves

After Hasse proved the Riemann hypothesis for elliptic curves over finite fields in 1930, he and Deuring realized that, in order to extend the proof to curves of higher genus, one should replace the endomorphisms of the elliptic curve by correspondences. However, they regarded correspondences as objects in a double field, and this approach didn't lead to a proof until Roquette 1953[29] (Roquette was a student of Hasse). In the meantime Weil had realized that everything needed for the proof could be found already in the work of the Italian geometers on correspondences, at least in characteristic zero. In order to give a rigorous proof, he laid the foundations for algebraic geometry over arbitrary fields,[30] and completed the proof of the Riemann hypothesis for all curves over finite fields in 1945.[31][32]

The key point of Weil's proof is that the inequality of Castelnuovo-Severi continues to hold in characteristic p, i.e., for a divisor D on the product of two complete nonsingular curves C and C' over an algebraically closed field,

$$[D \cdot D] \leq 2dd' \tag{11}$$

where $d = [D \cdot (P \times C')]$ and $d' = [D \cdot (C \times P')]$ are the degrees of D over C and C' respectively. Mattuck and Tate (1958a) showed that it is possible to derive (11) directly and

[28] Algebraic number theory. Proceedings of an instructional conference organized by the London Mathematical Society. Edited by J. W. S. Cassels and A. Fröhlich, Academic Press, London; Thompson Book Co., Inc., Washington, D.C. 1967 [29] Roquette, Peter, Arithmetischer Beweis der Riemannschen Vermutung in Kongruenzfunktionenkörpern beliebigen Geschlechts. J. Reine Angew. Math. 191, (1953). 199–252. [30] The main lacunae at the time were a rigorous intersection theory taking account of the phenomenon of pure inseparability and the construction of the Jacobian variety in nonzero characteristic. [31] Weil, André. Sur les courbes algébriques et les variétés qui s'en déduisent. Publ. Inst. Math. Univ. Strasbourg 7 (1945). [32] Much has been written on these events. I've found the following particularly useful: Schappacher, Norbert, The Bourbaki Congress at El Escorial and other mathematical (non)events of 1936. The Mathematical Intelligencer, Special issue International Congress of Mathematicians Madrid August 2006, 8-15.

easily from the Riemann-Roch theorem for surfaces, for which they were able to appeal to Zariski 1952[33] or to a sheaf-theoretic proof of Serre which is sketched in Zariski 1956.[34]

The Mattuck-Tate proof is the most attractive geometric proof of Weil's theorem. Grothendieck[35] simplified it further by showing that the Castelnuovo-Severi inequality can most naturally be derived from the Hodge index theorem for surfaces, which itself can be derived directly from the Riemann-Roch theorem.

Hodge proved his index theorem for smooth projective varieties over $\mathbb{C}$. That it should hold for such varieties in nonzero characteristic is known as Grothendieck's "Hodge standard conjecture", whose proof Grothendieck calls one of the "most urgent tasks in algebraic geometry".[36] In the more than forty years since Grothendieck formulated the conjecture, almost no progress has been made towards its proof — even in characteristic zero, there exists no algebraic proof in dimensions greater than 2.

THE TATE MODULE OF AN ABELIAN VARIETY

Let A be an abelian variety over a field k. For a prime l, let $A(l) = \bigcup A(k^{\text{sep}})_{l^n}$ where $A(k^{\text{sep}})_{l^n} = \text{Ker}(A(k^{\text{sep}}) \xrightarrow{l^n} A(k^{\text{sep}}))$. Then $A \rightsquigarrow A(l)$ is a functor from abelian varieties over k to l-divisible groups equipped with an action of $\text{Gal}(k^{\text{sep}}/k)$. When $l \neq \text{char}(k)$, $A(l) \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2\dim A}$, and Weil used $A(l)$ to study the endomorphisms of A. Tate observed that it is more convenient to work with

$$T_l A = \varprojlim A(k^{\text{sep}})_{l^n},$$

which is a free $\mathbb{Z}_l$-module of rank $2\dim A$ when $l \neq \text{char}(k)$ — this is now called the Tate module of A.

2.2 *Heights on abelian varieties*

THE NÉRON-TATE (CANONICAL) HEIGHT

Let K be a number field, and normalize the absolute values $|\cdot|_v$ of K so that the product formula holds:

$$\prod\nolimits_v |a|_v = 1 \text{ for all } v \in K^\times.$$

The logarithmic height of a point $P = (a_0{:}\ldots{:}a_n)$ of $\mathbb{P}^n(K)$ is defined to be

$$h(P) = \log\left(\prod\nolimits_v \max\{|a_0|_v, \ldots, |a_n|_v\}\right).$$

The product formula shows that this is independent of the representation of P.

Let X be a projective variety. A morphism $f: X \to \mathbb{P}^n$ from X into projective space defines a height function $h_f(P) = h(f(P))$ on X. In a Short Communication at the 1958 International Congress, Néron conjectured that, in certain cases, the height is given by a quadratic form.[37] Tate proved this for abelian varieties by a simple direct argument.

[33] Zariski, Oscar, Complete linear systems on normal varieties and a generalization of a lemma of Enriques–Severi. Ann. of Math. (2) 55, (1952). 552–592. [34] Zariski, Oscar, Scientific report on the Second Summer Institute, III Algebraic Sheaf Theory, Bull. Amer. Math. Soc. 62 (1956), 117–141. [35] Grothendieck, A. Sur une note de Mattuck-Tate. J. Reine Angew. Math. 200 1958 208–215. [36] Grothendieck, A. Standard conjectures on algebraic cycles. 1969 Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968) pp. 193–199 Oxford Univ. Press, London. [37] Only the title, *Valeur asymptotique du nombre des points rationnels de hauteur borndée sur une courbe elliptique*, of Néron's communication is included in the Proceedings. The sentence paraphrases one from: Lang, Serge. Les formes bilinéaires de Néron et Tate. Séminaire Bourbaki, 1963/64, Fasc. 3, Exposé 274.

Let A be an abelian variety over a number field K. A nonconstant map $f: A \to \mathbb{P}^n$ of A into projective space is said to be symmetric if the inverse image D of a hyperplane is linearly equivalent to $(-1)^* D$. For a symmetric embedding f, Tate proved that there exists a unique *quadratic* map $\hat{h}: A(K) \to \mathbb{R}$ such that $\hat{h}(P) - h_f(P)$ is bounded on $A(K)$. To say that $\hat{h}$ is quadratic means that $\hat{h}(2P) = 4\hat{h}(P)$ and that the function

$$P, Q \mapsto \frac{1}{2}\left(\hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)\right) \tag{12}$$

is bi-additive on $A(K) \times A(K)$.

Note first that there exists at most one function $\hat{h}: A(K) \to \mathbb{R}$ such that (a) $\hat{h}(P) - h_f(P)$ is bounded on $A(K)$, and (b) $\hat{h}(2P) = 4\hat{h}(P)$ for all $P \in A(K)$. Indeed, if $\hat{h}$ satisfies (a) with bound B, then

$$\left|\hat{h}(2^n P) - h_f(2^n P)\right| \leq B$$

for all $P \in A(K)$ and all $n \geq 0$. If in addition it satisfies (b), then

$$\left|\hat{h}(P) - \frac{h_f(2^n P)}{4^n}\right| \leq \frac{B}{4^n}$$

for all n, and so

$$\hat{h}(P) = \lim_{n \to \infty} \frac{h_f(2^n P)}{4^n}. \tag{13}$$

Tate used the equation (13) to define $\hat{h}$, and applied results of Weil on abelian varieties to verify that it is quadratic.

Let A' be the dual abelian variety to A. For a map $f: A \to \mathbb{P}^n$ corresponding to a divisor D, let $\varphi_f: A(K) \to A'(K)$ be the map sending P to the point on A represented by the divisor $(D + P) - D$. Tate showed that there is a unique bi-additive pairing

$$\langle\ ,\ \rangle: A'(K) \times A(K) \to \mathbb{R} \tag{14}$$

such that, for every symmetric f, the function $\langle \varphi_f(P), P\rangle + 2h_f(P)$ is bounded on $A(K)$.

Néron (1965)[38] found his own construction of $\hat{h}$, which is much longer than Tate's, but which has the advantage of expressing $\hat{h}$ as a sum of local heights. The height function $\hat{h}$, is now called the Néron-Tate, or canonical, height. It plays a fundamental role in arithmetic geometry.

NOTES. Tate explained his construction in his course on abelian varieties at Harvard in the fall of 1962, but did not publish it. However, it was soon published by others.[39]

VARIATION OF THE CANONICAL HEIGHT OF A POINT DEPENDING ON A PARAMETER

Let T be an algebraic curve over $\mathbb{Q}^{\mathrm{al}}$, and let $E \to T$ be an algebraic family of elliptic curves parametrised by T. Let $P: T \to E$ be a section of E/T, and let $\hat{h}_t$ be the Néron-Tate height on the fibre E_t of E/T over a closed point t of T. Tate (1983a) proves that the map $t \mapsto \hat{h}_t(P_t)$ is a height function on the curve T for a certain divisor class $q(P)$ on

[38] Néron, A., Quasi-fonctions et hauteurs sur les variétés abéliennes. Ann. of Math. (2) 82 1965 249–331.

[39] Lang, S., see footnote 37, and Diophantine approximations on toruses. Amer. J. Math. 86 1964 521–533; Manin, Ju. I., The Tate height of points on an abelian variety, its variants and applications. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 28 1964 1363–1390.

T; moreover, the degree of $q(P)$ is the Néron-Tate height of P regarded as a point on the generic fibre of E/T.

As Tate noted "The main obstacle to extending the theorem in this paper to abelian varieties seems to be the lack of a canonical compactification of the Néron model in higher dimensions." After Faltings compactified the moduli stack of abelian varieties,[40] one of his students, William Green, extended Tate's theorem to abelian varieties.[41]

HEIGHT PAIRINGS VIA BIEXTENSIONS

Let A be an abelian variety over a number field K, and let A' be its dual. The classical Néron-Tate height pairing is a pairing

$$A(K)\times A'(K)\to\mathbb{R}$$

whose kernels are precisely the torsion subgroups of $A(K)$ and $A'(K)$. In order, for example, to state a p-adic version of the conjecture of Birch and Swinnerton-Dyer, it is necessary to define a $\mathbb{Q}_p$-valued height pairing,

$$A(K)\times A'(K)\to\mathbb{Q}_p.$$

When A has good ordinary or multiplicative reduction at the p-adic primes, Mazur and Tate (1983b) use the expression of the duality between A and A' in terms of biextensions, and exploit the local splittings of these biextensions, to define such pairings. They compare their definition with other suggested definitions. It is not known whether the pairings are nondegenerate modulo torsion.

2.3 The cohomology of abelian varieties

THE LOCAL DUALITY FOR ABELIAN VARIETIES

Let A be an abelian variety over a field k. A principal homogeneous space over A is a variety V over k together with a regular map $A\times V\to V$ such that, for every field K containing k for which $V(K)$ is nonempty, the pairing $A(K)\times V(K)\to V(K)$ makes $V(K)$ into a principal homogeneous space for $A(K)$ in the usual sense. The isomorphism classes of principal homogeneous spaces form a group, which Tate (1958b) named the Weil-Châtelet group, and denoted $\mathrm{WC}(A/k)$.

For a finite extension k of $\mathbb{Q}_p$, local class field theory provides a canonical isomorphism $H^2(k,\mathbb{G}_m)\simeq\mathbb{Q}/\mathbb{Z}$. Tate (ibid.) defines an "augmented" cup-product pairing

$$H^r(k,A)\times H^{1-r}(k,A')\to H^2(k,\mathbb{G}_m)\simeq\mathbb{Q}/\mathbb{Z},\tag{15}$$

and proves that it is a perfect duality for $r=1$. In other words, the discrete group $\mathrm{WC}(A/k)$ is canonically dual to the compact group $A'(k)$. Later, he showed that (15) is a perfect duality for all r. In the case $k=\mathbb{R}$, he proved that $H^1(\mathbb{R},A)$ is canonically dual to $A'(\mathbb{R})/A'(\mathbb{R})^\circ=\pi_0(A'(\mathbb{R}))$.

NOTES. The above results are proved in Tate 1958b, 1959b, or 1962d. The analogous statements for local fields of characteristic p are proved in Milne 1970.[42]

[40] Faltings, G. Arithmetische Kompaktifizierung des Modulraums der abelschen Varietäten. Workshop Bonn 1984 (Bonn, 1984), 321–383, Lecture Notes in Math., 1111, Springer, Berlin, 1985. [41] Green, William, Heights in families of abelian varieties. Duke Math. J. 58 (1989), no. 3, 617–632. [42] Milne, J. S. Weil-Châtelet groups over local fields. Ann. Sci. École Norm. Sup. (4) 3 1970 273–284; ibid. 5 (1972), 261-264.

PRINCIPAL HOMOGENEOUS SPACES OVER ABELIAN VARIETIES

Lang and Tate (1958c) explain the relation between the set $\mathrm{WC}(A/k)$ of isomorphism classes of principal homogeneous spaces over a group variety A and the Galois cohomology group $H^1(k,A)$. Briefly, there is a canonical injective map $\mathrm{WC}(A/k) \to H^1(k,A)$ which Weil's descent theorems show to be surjective. This generalizes results of Châtelet.

Let K be a field complete with respect to a discrete valuation with residue field k, and let A be an abelian variety over K with good reduction to an abelian variety $\bar{A}$ over k. Then, for any integer m prime to the characteristic of k, Lang and Tate (ibid.) prove that there is a canonical exact sequence

$$0 \to H^1(k,\bar{A})_m \to H^1(k,A)_m \to \mathrm{Hom}(\mu_m(k), \bar{A}(k^{\mathrm{sep}})_m) \to 0.$$

In the final section of the article, they study abelian varieties over global fields. In particular, they prove the weak Mordell-Weil theorem.

As Cassels wrote,[43] the article Tate 1962b provides "A laconic but useful review of the existing state of knowledge [on principal homogeneous spaces for abelian varieties] for different types of groundfield."

THE CONJECTURE OF BIRCH AND SWINNERTON-DYER

For an elliptic curve A over $\mathbb{Q}$, Mordell showed that the group $A(\mathbb{Q})$ is finitely generated. It is easy to compute the torsion subgroup of $A(\mathbb{Q})$, but there is at present no proven algorithm for computing its rank $r(A)$. Computations led Birch and Swinnerton-Dyer to conjecture that $r(A)$ is equal to the order of the zero at 1 of the L-series of A, and further work led to a more precise conjecture. Tate (1966e) formulated the analogues of their conjectures for an abelian variety A over a global field K.

Let v be a nonarchimedean prime of K, and let $\kappa(v)$ be the corresponding residue field. If A has good reduction at v, then it gives rise to an abelian variety $A(v)$ over $\kappa(v)$. The characteristic polynomial of the Frobenius endomorphism of $A(v)$ is a polynomial $P_v(T)$ of degree $2d$ with coefficients in $\mathbb{Z}$ such that, when we factor it as $P_v(T) = \prod_i (1 - a_i T)$, then $\prod_i (1 - a_i^m)$ is the number of points on $A(v)$ with coordinates in the finite field of degree m over $\kappa(v)$. For any finite set S of primes of K including the archimedean primes and those where A has bad reduction, we define the L-series $L_S(s,A)$ by the formula

$$L_S(A,s) = \textstyle\prod_{v \notin S} P_v(A, Nv^{-s})^{-1}$$

where $Nv = [\kappa(v)]$. The product converges for $\Re(s) > 3/2$, and it is conjectured that $L_S(A,s)$ can be analytically continued to a meromorphic function on the whole complex plane. This is known in the function field case, and over $\mathbb{Q}$ for elliptic curves. The analogue of the first conjecture of Birch and Swinnerton-Dyer for A is that

$$L_S(A,s) \text{ has a zero of order } r(A) \text{ at } s = 1. \tag{16}$$

Let ω be a nonzero global differential d-form on A. As $\Gamma(A, \Omega_A^d)$ has dimension 1, ω is uniquely determined up to multiplication by an element of $K^\times$. For each nonarchimedean prime v of K, let μ_v be the Haar measure on K_v for which $\mathcal{O}_v$ has measure 1, and for each archimedean prime, take μ_v to be the usual Lebesgue measure on K_v. Define

$$\mu_v(A,\omega) = \int_{A(K_v)} |\omega|_v \mu_v^d$$

[43] Math. Reviews 0138625.

Let μ be the measure $\prod\mu_v$ on the adèle ring $\mathbb{A}_K$ of K, and set $|\mu| = \int_{\mathbb{A}_K/K}\mu$. For any finite set S of primes of K including all archimedean primes and those nonarchimedean primes for which A has bad reduction or such that ω does not reduce to a nonzero differential d-form on $A(v)$, we define

$$L_S^*(s,A) = L_S(s,A)\frac{|\mu|^d}{\prod_{v\in S}\mu_v(A,\omega)}.$$

The product formula shows that this is independent of the choice of ω. The asymptotic behaviour of $L_S^*(s,A)$ as $s\to 1$, which is all we are interested in, doesn't depend on S. The analogue of the second conjecture of Birch and Swinnerton-Dyer is that

$$\lim_{s\to 1}\frac{L_S^*(s,A)}{(s-1)^{r(A)}} = \frac{[Ш(A)]\cdot|D|}{[A'(K)_{\text{tors}}]\cdot[A(K)_{\text{tors}}]} \tag{17}$$

where $Ш(A)$ is the Tate-Shafarevich group of A,

$$Ш(A) \stackrel{\text{def}}{=} \text{Ker}\left(H^1(K,A)\to\prod\nolimits_v H^1(K_v,A)\right),$$

which is conjectured to be finite, and D is the discriminant of the height pairing (14), which is known to be nonzero.

GLOBAL DUALITY

In his talk at the 1962 International Congress, Tate stated the local duality theorems reviewed above (p.18), and he announced some global theorems which we now discuss.

In their computations, Birch and Swinnerton-Dyer found that the order of the Tate-Shafarevich group predicted by (17) is always a square. Cassels and Tate conjectured independently that the explanation for this is that there exists an alternating pairing

$$Ш(A)\times Ш(A)\to\mathbb{Q}/\mathbb{Z} \tag{18}$$

that annihilates only the divisible subgroup of $Ш(A)$. Cassels proved this for an elliptic curve over a number field.[44] For an abelian variety A and its dual abelian variety A', Tate proved that there exists a canonical pairing

$$Ш(A)\times Ш(A')\to\mathbb{Q}/\mathbb{Z} \tag{19}$$

that annihilates only the divisible subgroups; moreover, for a divisor D on A and the homomorphism $\varphi_D\colon A\to A'$, $a\mapsto[D_a-D]$, it defines, the pair $(\alpha,\varphi_D(\alpha))$ maps to zero under (19) for all $\alpha\in Ш(A)$. The pairing (19), or one of its several variants, is now called the Cassels-Tate pairing.

For an elliptic curve A over a number field k such that $Ш(A)$ is finite, Cassels determined the Pontryagin dual of the exact sequence

$$0\to Ш(A)\to H^1(k,A)\to\bigoplus\nolimits_v H^1(k_v,A)\to Ш(A)\to 0 \tag{20}$$

[44] Cassels, J. W. S. Arithmetic on curves of genus 1. IV. Proof of the Hauptvermutung. J. Reine Angew. Math. 211 1962 95–112.

(regarded as a sequence of discrete groups). Assume that Ш(A) is finite. Using Tate's local duality theorem (see p.18) for an elliptic curve, Cassels (1964)[45] showed that the dual of (20) takes the form

$$0 \leftarrow Ш \leftarrow \Theta \leftarrow \prod\nolimits_v A(k_v)' \leftarrow \widetilde{A(k)} \leftarrow 0 \tag{21}$$

for a certain explicit Θ and with $\widetilde{A(k)}$ equal to the closure of $A(k)$ in $\prod_v A(k_v)'$. Tate proved the same statement for abelian varieties over number fields, except that, in (21), it is necessary to replace A with its dual A'. So modified, the sequence (21) is now called the Cassels-Tate dual exact sequence.

Let A and B be isogenous elliptic curves over a number field. Then $L_S(s, A) = L_S(s, B)$ and $r(A) = r(B)$, and so the first conjecture of Birch and Swinnerton-Dyer is true for A if and only if it is true for B. Cassels proved the same statement for the second conjecture.[46] This amounts to showing that a certain product of terms doesn't change in passing from A to B (even though the individual terms may change). Using his duality theorems and the formula (10), p.14, for the Euler-Poincaré characteristic, Tate (1966e, 2.1) proved the same result for abelian varieties over number fields.

Tate's global duality theorems were widely used, even before there were published proofs. Since 1994, the duality theorems have been used in cryptography.

NOTES. Tate's results are more general and complete than stated above; in particular, he works with a nonempty set S of primes of k (not necessarily the complete set). Proofs of the theorems of Tate in this subsection can be found in Milne 1986.[27]

2.4 *Serre-Tate liftings of abelian varieties*

In a talk at the 1964 Woods Hole conference, Tate discussed some results of his and Serre on the lifting of abelian varieties from characteristic p.

For an abelian scheme A over a ring R, let A_n denote the kernel of $A \xrightarrow{p^n} A$ regarded as a finite group scheme over R, and let $A(p)$ denote the direct system

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots$$

of finite group schemes. Let R be an artinian local ring with residue field k of characteristic $p \neq 0$. An abelian scheme A over R defines an abelian variety $\bar{A}$ over k and a system of finite group schemes $A(p)$ over R. Serre and Tate prove that the functor

$$A \rightsquigarrow (\bar{A}, A(p))$$

is an equivalence of categories (Serre-Tate theorem). In particular, to lift an abelian variety A from k to R amounts to lifting the system of finite group schemes $A(p)$.

This has many important consequences.

- ◇ Let A and B be abelian schemes over a complete local noetherian ring R with residue field a field k of characteristic $p \neq 0$. A homomorphism $f: \bar{A} \to \bar{B}$ of abelian varieties over k lifts to a homomorphism $A \to B$ of abelian schemes over R if and only if $f(p)$: $\bar{A}(p) \to \bar{B}(p)$ lifts to R. For the artinian quotients of R, this is part of the above statement, and the statement for R follows by passing to the limit over the artinian quotients of R and applying a theorem of Grothendieck.

[45] Cassels, J. W. S., Arithmetic on curves of genus 1. VII. The dual exact sequence. J. Reine Angew. Math. 216 1964 150–158. [46] Cassels, J. W. S. Arithmetic on curves of genus 1. VIII. On conjectures of Birch and Swinnerton-Dyer. J. Reine Angew. Math. 217 1965 180–199.

- ◇ Let A be an abelian variety over a perfect field k of characteristic $p \neq 0$. If A is ordinary, then
$$A_n \approx (\mathbb{Z}/p^n\mathbb{Z})^{\dim A} \times (\mu_{p^n})^{\dim A},$$
and so each A_n has a canonical lifting to a finite group scheme over the ring $W(k)$ of Witt vectors of k. Thus A has a canonical lifting to an abelian scheme over $W(k)$ (at least formally, but the existence of polarizations implies that the formal abelian scheme is an abelian scheme). Deligne has used this to give a "linear algebra" description of the category of ordinary abelian varieties over a finite field similar to the classical description of abelian varieties over $\mathbb{C}$.[47]
- ◇ Over a ring R in which p is nilpotent, the infinitesimal deformation theory of A is equivalent to the infinitesimal deformation theory of $A(p)$. For example, when A is ordinary, this implies that the local deformation space of an ordinary abelian variety A over k has a natural structure of a formal torus over $W(k)$ of relative dimension $\dim(A)^2$.

NOTES. Lifting results were known to Hasse and Deuring for elliptic curves. The canonical lifting of an ordinary abelian variety was found by Serre, prompting Tate to prove the general result. Lubin, Serre, and Tate 1964b contains a sketch of the proofs. The liftings of A obtained from liftings of the system $A(p)$ are sometimes called Serre-Tate liftings, especially in the ordinary case. Messing 1972[48] includes a proof of the Serre-Tate theorem.

2.5 *Mumford-Tate groups and the Mumford-Tate conjecture*

In 1965, Mumford gave a talk at the AMS Summer Institute[49] whose results he described as being "partly joint work with J. Tate". In it, he attached a reductive group to an abelian variety, and stated a conjecture. The first is now called the Mumford-Tate group, and the second is the Mumford-Tate conjecture.

Let A be a complex abelian variety of dimension g. Then $V \overset{\text{def}}{=} H_1(A,\mathbb{Q})$ is a $\mathbb{Q}$-vector space of dimension $2g$ whose tensor product with $\mathbb{R}$ acquires a complex structure through the canonical isomorphism
$$H_1(A,\mathbb{Q})_{\mathbb{R}} \simeq \mathrm{Tgt}_0(A).$$
Let $u\colon U^1 \to \mathrm{GL}(V_{\mathbb{R}})$ be the homomorphism describing this complex structure, where $U^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The Mumford-Tate group of A is defined to be the smallest algebraic subgroup H of GL_V such that $H(\mathbb{R})$ contains $u(U^1)$.[50] Then H is a reductive algebraic group over $\mathbb{Q}$, which acts on $H^*(A^r,\mathbb{Q})$, $r \in \mathbb{N}$, through the isomorphisms
$$\begin{aligned} H^*(A^r,\mathbb{Q}) &\simeq \textstyle\bigwedge^* H^1(A^r,\mathbb{Q}),\\ H^1(A^r,\mathbb{Q}) &\simeq rH^1(A,\mathbb{Q}),\\ H^1(A,\mathbb{Q}) &\simeq \mathrm{Hom}(V,\mathbb{Q}).\end{aligned}$$
It can be characterized as the algebraic subgroup of GL_V that fixes exactly the Hodge tensors in the spaces $H^*(A^r,\mathbb{Q})$, i.e., the elements of the $\mathbb{Q}$-spaces
$$H^{2p}(A^r,\mathbb{Q}) \cap \bigoplus H^{p,p}(A^r).$$

[47] Deligne, Pierre. Variétés abéliennes ordinaires sur un corps fini. Invent. Math. 8 1969 238–243. [48] Messing, William. The crystals associated to Barsotti-Tate groups: with applications to abelian schemes. Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin-New York, 1972. [49] Mumford, David. Families of abelian varieties. 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965) pp. 347–351 Amer. Math. Soc., Providence, R.I. [50] Better, it should be thought of as the pair (H,u).

Let A be an abelian variety over a number field k, and, for a prime number l, let

$$V_l A = \mathbb{Q}_l \otimes_{\mathbb{Z}_l} T_l A$$

where $T_l A$ is the Tate module of A (p.16). Then

$$V_l A \simeq \mathbb{Q}_l \otimes_{\mathbb{Q}} H_1(A_{\mathbb{C}}, \mathbb{Q}).$$

The Galois group $G(k^{\mathrm{al}}/k)$ acts on $A(k^{\mathrm{al}})$, and hence there is a representation

$$\rho_l : G(k^{\mathrm{al}}/k) \to \mathrm{GL}(V_l A)$$

The Zariski closure $H_l(A)$ of $\rho_l(G(k^{\mathrm{al}}/k))$ is an algebraic group in $\mathrm{GL}_{V_l A}$. Although $H_l(A)$ may change when k is replaced by a finite extension, its identity component $H_l(A)^\circ$ does not and can be thought of as determining the image of ρ_l up to finite groups. The Mumford-Tate conjecture states that

$$H_l(A)^\circ = (\text{Mumford-Tate group of } A_{\mathbb{C}})_{\mathbb{Q}_l} \text{ inside } \mathrm{GL}_{V_l A} \simeq (\mathrm{GL}_{H_1(A_{\mathbb{C}},\mathbb{Q})})_{\mathbb{Q}_l}.$$

In particular, it posits that the $\mathbb{Q}_l$-algebraic groups $H_l(A)^\circ$ are independent of l in the sense that they all arise by base change from a single algebraic group over $\mathbb{Q}$. In the presence of the Mumford-Tate conjecture, the Hodge and Tate conjectures for A are equivalent. Much is known about the Mumford-Tate conjecture.

Let H be the Mumford-Tate group of an abelian variety A, and let $u : U^1 \to H(\mathbb{R})$ be the above homomorphism. The centralizer K of u in $H(\mathbb{R})$ is a maximal compact subgroup of $H(\mathbb{R})$, and the quotient manifold $X = H(\mathbb{R})/K$ has a unique complex structure for which $u(z)$ acts on the tangent space at the origin as multiplication by z. With this structure X is isomorphic to a bounded symmetric domain, and it supports a family of abelian varieties whose Mumford-Tate groups "refine" that of A. The quotients of X by congruence subgroups of $H(\mathbb{Q})$ are connected Shimura varieties.

The notion of a Mumford-Tate group has a natural generalization to an arbitrary polarizable rational Hodge structure. In this case the quotient space X is a homogeneous complex manifold, but it is not necessarily a bounded symmetric domain. The complex manifolds arising in this way were called Mumford-Tate domains by Green, Griffiths, and Kerr.[51] As these authors say: "Mumford-Tate groups have emerged as the principal symmetry groups in Hodge theory."

2.6 *Abelian varieties over finite fields (Weil, Tate, Honda theory)*

Consider the category whose objects are the abelian varieties over a field k and whose morphisms are given by

$$\mathrm{Hom}^0(A, B) \stackrel{\text{def}}{=} \mathrm{Hom}(A, B) \otimes \mathbb{Q}.$$

Weil's results[52] imply that this is a semisimple abelian category whose endomorphism algebras are finite dimensional $\mathbb{Q}$-algebras. Thus, to describe the category up to equivalence, it suffices to list the isomorphism classes of simple objects and, for each class, describe the

[51] Green, Mark; Griffiths, Phillip; Kerr, Matt, Mumford-Tate domains. Boll. Unione Mat. Ital. (9) 3 (2010), no. 2, 281–307. [52] Weil, André. Variétés abéliennes et courbes algébriques. Actualités Sci. Ind., no. 1064 = Publ. Inst. Math. Univ. Strasbourg 8 (1946). Hermann & Cie., Paris, 1948.

endomorphism algebra of an object in the class. This the theory of Weil, Tate, and Honda does when k is finite. Briefly: Weil showed that there is a well-defined map from isogeny classes of simple abelian varieties to conjugacy classes of Weil numbers, Tate proved that the map is injective and determined the endomorphism algebra of each simple class, and Honda used the theory of Shimura and Taniyama to prove that the map is surjective.

In more detail, let k be a field with $q = p^a$ elements. Each abelian variety A over k admits a Frobenius endomorphism π_A, which acts on the k^{al}-points of A as $(a_0:a_1:\ldots) \mapsto (a_0^q:a_1^q:\ldots)$. Weil proved that the image of π_A in $\mathbb{C}$ under any homomorphism $\mathbb{Q}[\pi_A] \to \mathbb{C}$ is a Weil q-integer, i.e., it is an algebraic integer with absolute value $q^{\frac{1}{2}}$ (this is the Riemann hypothesis). Thus, attached to every simple abelian variety A over k, there is a conjugacy class of Weil q-integers. Isogenous simple abelian varieties give the same conjugacy class.

Tate (1966b) proved that a simple abelian A is determined up to isogeny by the conjugacy class of π_A, and moreover, that $\mathbb{Q}[\pi_A]$ is the centre of $\mathrm{End}^0(A)$. Since $\mathrm{End}^0(A)$ is a division algebra with centre the field $\mathbb{Q}[\pi_A]$, class field theory shows that its isomorphism class is determined by its invariants at the primes v of $\mathbb{Q}[\pi_A]$. These Tate determined as follows:

$$\mathrm{inv}_v(\mathrm{End}^0(A)) = \begin{cases} \frac{1}{2} & \text{if } v \text{ is real,} \\ \dfrac{\mathrm{ord}_v(\pi_A)}{\mathrm{ord}_v(q)}[\mathbb{Q}[\pi_A]_v:\mathbb{Q}_p] & \text{if } v|p, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$2\dim A = [\mathrm{End}^0(A):\mathbb{Q}[\pi_A]]^{\frac{1}{2}} \cdot [\mathbb{Q}[\pi_A]:\mathbb{Q}].$$

The abelian varieties of CM-type over $\mathbb{C}$ are classified up to isogeny by their CM-types, and every such abelian variety has a model over $\mathbb{Q}^{\mathrm{al}}$. When we choose a p-adic prime of $\mathbb{Q}^{\mathrm{al}}$, an abelian variety A of CM-type over $\mathbb{Q}^{\mathrm{al}}$ specializes to an abelian variety $\bar{A}$ over a finite field of characteristic p. The Shimura-Taniyama formula determines $\pi_{\bar{A}}$ up to a root of 1 in terms of the CM-type of A. Using this, Honda proved that every Weil q-number arises from an abelian variety, possibly after a finite extension of the base field.[53] An application of Weil restriction of scalars completes the proof.

2.7 *Good reduction of Abelian Varieties*

The language of Weil's foundations of algebraic geometry is ill-suited to the study of algebraic varieties in mixed characteristic. For example, it makes it cumbersome to prove even that an algebraic variety over a number field has good reduction at almost all primes of the field.[54] Serre and Tate (1968a) use schemes and Néron's theory of minimal models[55] to simplify and sharpen known results for abelian varieties, and to extend some statements from elliptic curves to abelian varieties.

Let R be a discrete valuation ring with field of fractions K and perfect residue field k. For an abelian variety A over K, Néron proved that the functor sending a smooth R-scheme X to $\mathrm{Hom}(X_K, A)$ is represented by a smooth group scheme $\widetilde{A}$ of finite type over R. Using this, Serre and Tate prove the following criterion:

[53] Honda, Taira, Isogeny classes of abelian varieties over finite fields. J. Math. Soc. Japan 20 1968 83–95.
[54] See the proof of Theorem 26 of Shimura, Goro, Reduction of algebraic varieties with respect to a discrete valuation of the basic field. Amer. J. Math. 77, (1955). 134–176. [55] Néron, André. Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. Inst. Hautes Études Sci. Publ.Math. No. 21 1964 128 pp.

> If A has good reduction, then the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module $A(K^{\mathrm{sep}})_m$ is unramified for all integers m prime to $\mathrm{char}(k)$; conversely, if $A(K^{\mathrm{sep}})_m$ is unramified for infinitely many m prime to $\mathrm{char}(k)$, then A has good reduction.

The necessity was known earlier, and the sufficiency was known to Ogg and Shafarevich. in the case of elliptic curves. Because it is a direct consequence of the existence of Néron's models, Serre and Tate call it the "Néron-Ogg-Shafarevich criterion". It is of fundamental importance.

Serre and Tate say that an abelian variety has *potential good reduction* if it acquires good reduction after a finite extension of the base field, and they prove a number of results about such varieties. For example, when R is strictly henselian, there is a smallest extension L of K in K^{al} over which such an abelian variety A has good reduction, namely, the extension of K generated by the coordinates of the points of order m for any $m \geq 3$ prime to $\mathrm{char}(k)$. Moreover, just as for elliptic curves, the notion of the conductor of an abelian variety is well defined.

Let (A,i) be an abelian variety over a number field K with complex multiplication by E. By this we mean that E is a CM field of degree $2\dim A$ over $\mathbb{Q}$, and that i is a homomorphism of $\mathbb{Q}$-algebras $E \to \mathrm{End}^0(A)$. Serre and Tate apply their earlier results to show that such an abelian variety A acquires good reduction everywhere over a cyclic extension L of K; moreover, L can be chosen to have degree m or $2m$ where m is the least common multiple of the images of the inertia groups acting on the torsion points of A.

Let (A,i) and E be as in the last paragraph, and let C_K be the idèle class group of K. Shimura and Taniyama (1961, 18.3)[56] show there exists a (unique) homomorphism $\rho: C_K \to (\mathbb{R}\otimes_{\mathbb{Q}} E)^\times$ with the following property: for each $\sigma: E \to \mathbb{C}$, let χ_σ be the Hecke character

$$\chi_\sigma: C_K \xrightarrow{\rho} (\mathbb{R}\otimes_{\mathbb{Q}} E)^\times \xrightarrow{1\otimes\sigma} \mathbb{C}^\times;$$

then the L-series $L(s,A)$ coincides with the product $\prod_\sigma L(s,\chi_\sigma)$ of the L-series of the χ_σ, except possibly for the factors corresponding to a finite number of primes of K.[57] Serre and Tate make this more precise by showing that the conductor of A is the product of the conductors of the χ_σ (which each equals the conductor of ρ). In particular, the support of the conductor of each χ_σ equals the set of primes where A has bad reduction, from which it follows that $L(s,A)$ and $\prod_\sigma L(s,\chi_\sigma)$ coincide exactly.

2.8 CM abelian varieties and Hilbert's twelfth problem

A CM-type on a CM field E is a subset Φ of $\mathrm{Hom}(E,\mathbb{C})$ such that $\Phi \sqcup \bar{\Phi} = \mathrm{Hom}(E,\mathbb{C})$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, let $\sigma\Phi = \{\sigma\circ\varphi \mid \varphi\in\Phi\}$. Then $\sigma\Phi$ is also a CM-type on E. The reflex field of (E,Φ) is the subfield F of $\mathbb{C}$ such that an automorphism σ of $\mathbb{C}$ fixes F if and only if $\sigma\Phi = \Phi$. It is easy to see that F is a CM-subfield of $\mathbb{Q}^{\mathrm{al}} \subset \mathbb{C}$.

Let (A,i) be an abelian variety over $\mathbb{C}$ with complex multiplication by E. Then E acts on the tangent space of A at 0 through a CM-type Φ, and (A,i) is said to be of CM-type (E,Φ). For $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, $\sigma(A,i)$ is of CM-type $\sigma\Phi$, and it follows that $\sigma(A,i)$ is isogenous to (A,i) if and only if σ fixes the reflex field F. Fix a polarization λ of A whose Rosati involution acts as complex multiplication on E. For an integer $m \geq 1$, let

[56] Shimura, Goro; Taniyama, Yutaka Complex multiplication of abelian varieties and its applications to number theory. Publications of the Mathematical Society of Japan, 6 The Mathematical Society of Japan, Tokyo 1961.

[57] Rather, this is Serre and Tate's interpretation of what they prove; Shimura and Taniyama express their results in terms of ideals.

$\mathcal{S}(m)$ be the set of isomorphism classes of quadruples (A', λ', i', η) such that (A', λ', i') is isogenous to (A, λ, i) and η is a level m-structure on (A', i'). According to the preceding observation, $\mathrm{Aut}(\mathbb{C}/F)$ acts on the set $\mathcal{S}(m)$. Shimura and Taniyama prove that this action factors through $\mathrm{Aut}(F^{\mathrm{ab}}/F)$, and they describe it explicitly. In this way, they generalized the theory of complex multiplication from elliptic curves to abelian varieties, and they provided a partial solution to Hilbert's twelfth problem for F.

In one respect the result of Shimura and Taniyama falls short of generalizing the elliptic curve case: for an elliptic curve, the reflex field F is a complex quadratic extension of $\mathbb{Q}$; since one knows how complex conjugation acts on CM elliptic curves and their torsion points, the elliptic curve case provides a description of how the *full* group $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on CM elliptic curves and their torsion points. Shimura asked whether there was a similar result for abelian varieties, but concluded rather pessimistically that "In the higher-dimensional case, however, no such general answer seems possible."[58]

Grothendieck's theory of motives suggests the framework for an answer. The Hodge conjecture implies the existence of Tannakian category of CM-motives over $\mathbb{Q}$, whose motivic Galois group is an extension

$$1 \to S \to T \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to 1$$

of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ (regarded as a pro-constant group scheme) by the Serre group S (a certain pro-torus). Étale cohomology defines a section λ of $T \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ over the finite adèles. The pair (T, λ) (tautologically) describes the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on the CM abelian varieties and their torsion points. Deligne's theorem on Hodge classes on abelian varieties allows one to construct the pair (T, λ) without assuming the Hodge conjecture. To answer Shimura's question, it remains to give a direct explicit description of (T, λ).

Langlands's work on the zeta functions of Shimura varieties led him to define a certain explicit cocycle,[59] which Deligne recognized as conjecturally being that describing the pair (T, λ).

Tate was inspired by this to commence his own investigation of Shimura's question. He gave a simple direct construction of a map f that he conjectured describes how $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on the CM abelian varieties and their torsion points, and proved this up to signs. More precisely, he proved it up to a map e with values in an adèlic group such that $e^2 = 1$. See Tate 1981c.

It was soon checked that Langlands's and Tate's conjectural descriptions of how $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on the CM abelian varieties and their torsion points coincided, and a few months later Deligne proved that their conjectural descriptions are indeed correct.[60]

3 Rigid analytic spaces

After Hensel introduced the p-adic number field $\mathbb{Q}_p$ in the 1890s, there were attempts to develop a theory of analytic functions over $\mathbb{Q}_p$, the most prominent being that of Krasner. The problem is that every disk D in $\mathbb{Q}_p$ can be written as a disjoint union of arbitrarily many

[58] Shimura, Goro. On abelian varieties with complex multiplication. Proc. London Math. Soc. (3) 34 (1977), no. 1, 65–86. [59] See §5 of Langlands, R. P. Automorphic representations, Shimura varieties, and motives. Ein Märchen. Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 205–246, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979 [60] Deligne, P., Motifs et groupe de Taniyama, pp.261–279 in Hodge cycles, motives, and Shimura varieties. Lecture Notes in Mathematics, 900. Springer-Verlag, Berlin-New York, 1982.

open-closed smaller disks, and so there are too many functions on D that can be represented locally by power series. Outside a small group of mathematicians, p-adic analysis attracted little attention until the work of Dwork and Tate in late 1950s. In February, 1958, Tate sent Dwork a letter in which he stated a result concerning elliptic curves, and challenged Dwork to find a proof using p-adic analysis. In answering the letter, Dwork found "the first suggestion of a connection between p-adic analysis and the theory of zeta functions."[61] By November, 1959, Dwork had found his famous proof of the rationality of the zeta function $Z(V,T)$ of an algebraic variety V over a finite field, a key point of which is to express $Z(V,T)$, which initially is a power series with integer coefficients, as a quotient of two p-adically entire functions.[62]

In 1959 also, Tate discovered that, suitably normalized, certain classical formulas allow one to express many elliptic curves E over a nonarchimedean local field K as a quotient $E(K) = K^\times/q^{\mathbb{Z}}$. This persuaded him that there should exist a category in which E itself, not just its points, is a quotient; in other words, that there exists a category in which E, as an "analytic space", is the quotient of $K^\times$, as an "analytic space", by the discrete group $q^{\mathbb{Z}}$. Two years later, Tate constructed the correct category of "rigid analytic spaces", thereby founding a new subject in mathematics (with its own Math. Reviews number 14G22).

3.1 The Tate curve

Let E be an elliptic curve over $\mathbb{C}$. The choice of a differential ω realizes $E(\mathbb{C})$ as the quotient $\mathbb{C}/\Lambda \simeq E(\mathbb{C})$ of $\mathbb{C}$ by the lattice of periods of ω. More precisely, it realizes the complex analytic manifold E^{an} as the quotient of the complex analytic manifold $\mathbb{C}$ by the action of the discrete group Λ.

For an elliptic curve E over a p-adic field K, there is no similar description of $E(K)$ because there are no nonzero discrete subgroups of K (if $\lambda \in K$, then $p^n\lambda \to 0$ as $n \to \infty$). However, there is an alternative uniformization of elliptic curves over $\mathbb{C}$. Let Λ be the lattice $\mathbb{Z}+\mathbb{Z}\tau$ in $\mathbb{C}$. Then the exponential map $\underline{e}\colon \mathbb{C} \to \mathbb{C}^\times$ sends $\mathbb{C}/\Lambda$ isomorphically onto $\mathbb{C}^\times/q^{\mathbb{Z}}$ where $q = \underline{e}(\tau)$, and so $\mathbb{C}^\times/q^{\mathbb{Z}} \simeq E^{\mathrm{an}}$ (as analytic spaces). If $\mathrm{Im}(\tau) > 0$, then $|q| < 1$, and the elliptic curve E_q is given by the equation

$$Y^2Z + XYZ = X^3 - b_2XZ^2 - b_3Z^3, \tag{22}$$

where

$$\begin{cases} b_2 = 5\sum_{n=1}^{\infty} \dfrac{n^3q^n}{1-q^n} = 5q + 45q^2 + 140q^3 + \cdots \\ b_3 = \sum_{n=1}^{\infty} \dfrac{7n^5+5n^3}{12}\dfrac{q^n}{1-q^n} = q + 23q^2 + 154q^3 + \cdots \end{cases} \tag{23}$$

are power series with integer coefficients. The discriminant and modular invariant of E_q are given by the usual formulas

$$\Delta = q\prod\nolimits_{n\geq 1}(1-q^n)^{24} \tag{24}$$

$$j(E_q) = \frac{(1+48b_2)^3}{q\prod_{n\geq 1}(1-q^n)^{24}} = \frac{1}{q} + 744 + 196884q + \cdots. \tag{25}$$

[61] Katz and Tate, 1999, p.343; Dwork, Bernard. A deformation theory for the zeta function of a hypersurface. 1963 Proc. Internat. Congr. Mathematicians (Stockholm, 1962) pp. 247–259 Inst. Mittag-Leffler, Djursholm.

[62] Dwork, Bernard. On the rationality of the zeta function of an algebraic variety. Amer. J. Math. 82 1960 631–648.

Now let K be a field complete with respect to a nontrivial nonarchimedean valuation with residue field of characteristic $p \neq 0$, and let q be an element of $K^\times$ with $|q| < 1$. The series (23) converge in K, and Tate discovered[63] that (22) is an elliptic curve E_q such that $K'/q^{\mathbb{Z}} \simeq E_q(K')$ for all finite extension K' of K. It follows from certain power series identities, valid over $\mathbb{Z}$, that the discriminant and modular invariant of E_q are given by (24) and (25). Every $j \in K^\times$ with $|j| < 1$ arises from a q (determined by (25), which allows q to be expressed as a power series in $1/j$ with integer coefficients). The function field $K(E_q)$ of E_q consists of the quotients F/G of Laurent series

$$F = \sum_{-\infty}^{\infty} a_n z^n, \quad G = \sum_{-\infty}^{\infty} b_n z^n, \quad a_n, b_n \in K,$$

converging for all nonzero z in $\mathbb{C}_p$, such that the F/G is invariant under $q^{\mathbb{Z}}$:

$$F(qz)/G(qz) = F(z)/G(z).$$

The elliptic curves E over K with $|j(E)| < 1$ that arise in this way are exactly those whose reduced curve has a node with tangents that are rational over the base field. They are now called Tate (elliptic) curves.

Tate's results were contained in a 1959 manuscript, which he did not publish until 1995, but there soon appeared several summaries of his results in the literature, and Roquette[64] gave a very detailed account of the theory. The Tate curve has found many applications, for example, to Tate's isogeny conjecture (Serre 1968;[65] Tate 1995, p.180) and to the study of elliptic modular curves near a cusp (Deligne and Rapoport[66]). Mumford[67] generalized Tate's construction to curves of higher genus, and McCabe[68] and Raynaud[69] generalized it to abelian varieties of higher dimension.

3.2 Rigid analytic spaces

Tate's idea that his p-adic uniformization of elliptic curves indicated the existence of a general theory of p-adic analytic spaces was radically new. For example, Grothendieck was initially very negative.[70] However, when Tate began to work out his theory in the fall of 1961, Grothendieck, who was visiting Harvard at the time, became very optimistic,[71] and was very supportive.

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[63] "I still remember the thrill and amazement I felt when it occurred to me that the classical formulas for such an isomorphism over $\mathbb{C}$ made sense p-adically when properly normalized." Tate 2008. [64] Roquette, Peter, Analytic theory of elliptic functions over local fields. Hamburger Mathematische Einzelschriften (N.F.), Heft 1 Vandenhoeck & Ruprecht, Göttingen 1970 [65] Serre, Jean-Pierre. Abelian l-adic representations and elliptic curves. W. A. Benjamin, Inc., New York-Amsterdam 1968. [66] Deligne, P.; Rapoport, M. Les schémas de modules de courbes elliptiques. Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 143–316. Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973. [67] Mumford, David. An analytic construction of degenerating curves over complete local rings. Compositio Math. 24 (1972), 129–174. [68] McCabe, John. p-adic theta functions. Ph.D. thesis, Harvard, 1968, 222 pages. [69] Raynaud, Michel. Variétés abéliennes et géométrie rigide. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 473–477. Gauthier-Villars, Paris, 1971. [70] "Tate has written to me about his elliptic curve stuff, and has asked me if I had any ideas for a global definition of analytic varieties over complete valuation fields. I must admit that I have absolutely not understood why his results might suggest the existence of such a definition, and I remain skeptical. Nor do I have the impression of having understood his theorem at all; it does nothing more than exhibit, via brute formulas, a certain isomorphism of analytic groups." Grothendieck, letter to Serre, August 18, 1959. [71] "Sooner or later it will be necessary to subsume ordinary analytic spaces, rigid analytic spaces, formal schemes, and maybe even schemes themselves into a single kind of structure for which all these usual theorems will hold." Grothendieck, letter to Serre, October 19, 1961.

Let K be a field complete with respect to a nontrivial nonarchmedean valuation, and let $\bar{K}$ be its algebraic closure. Tate began by introducing a new class of K-algebras. The Tate algebra $T_n = K\{X_1, \dots, X_n\}$ consists of the formal power series in $K[[X_1, \dots, X_n]]$ that are convergent on the unit ball,

$$B^n = \left\{(c_i)_{1 \le i \le n} \in \bar{K} \mid |c_i| \le 1\right\}.$$

Thus the elements of T_n are the power series

$$f = \sum a_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}, \quad a_{i_1 \cdots i_n} \in K, \quad \text{such that } a_{i_1 \cdots i_n} \to 0 \text{ as } (i_1, \dots, i_n) \to \infty.$$

Tate (1962c) shows that T_n is a Banach algebra for the norm $\|f\| = \sup |a_{i_1 \cdots i_n}|$, and that the ideals $\mathfrak{a}$ of T_n are closed and finitely generated. A quotient $T_n/\mathfrak{a}$ of T_n is a Banach algebra whose topology is independent of its presentation (because every homomorphism of such algebras is continuous). Such quotients are called affinoid (or Tate) K-algebras, and the category of affine rigid analytic spaces is the opposite of the category of affinoid K-algebras.

We need a geometric interpretation of this category. Tate showed that T_n is Jacobson (i.e., every prime ideal is an intersection of maximal ideals), and that the map $A \mapsto \max(A)$ sending an affinoid algebra to its set of maximal ideals is a functor: a homomorphism $\varphi: A \to B$ of affinoid algebras defines a map $\varphi^\circ: \max(B) \to \max(A)$. The set $\max(A)$ has the Zariski topology, which is very coarse, and a canonical topology induced from that of K. When K is algebraically closed, $\max(T_n) \simeq B^n$, and, by definition, $\max(A)$ can be realized as a closed subset of $\max(T_n)$ for some n.

Let $X = \max(A)$. One would like to define a sheaf $\mathcal{O}_X$ on X such that, for every open subset U isomorphic to B^n, $\mathcal{O}_X(U) \simeq T_n$. As noted at the start of this section, this is impossible. However, Tate's realized that it is possible to achieve something like this by allowing only certain "admissible" open subsets and certain "admissible" coverings. He defined an *affine subset* of X to be a subset Y such that the functor of affinoid K-algebras

$$B \rightsquigarrow \left\{\varphi: A \to B \mid \varphi^\circ(\max(B)) \subset Y\right\}$$

is representable (say, by $A \to A(Y)$). A subset Y of X is a *special affine subset* of X if there exist two finite families (f_i) and (g_j) of elements of A such that

$$Y = \left\{x \in X \,\middle|\, |f_i(x)| \le 1, \quad \left|g_j(x)\right| \ge 1, \quad \text{all } i, j\right\}.$$

Every special affine subset is affine. Tate's acyclicity theorem (Tate 1962c, 8.2) says that, for every finite covering $(X_i)_{i \in I}$ of X by special affines, the Čech complex of the presheaf $Y \mapsto A(Y)$,

$$0 \to A \to \prod_{i_0} A(X_{i_0}) \to \prod_{i_0 < i_1} A(X_{i_0} \cap X_{i_1}) \to \cdots \to \prod_{i_0 < \cdots < i_p} A(X_{i_0} \cap \cdots \cap X_{i_p}) \to \cdots,$$

is exact. In particular, $Y \mapsto A(Y)$ satisfies the sheaf condition on such coverings.

Using Tate's acyclicity theorem it is possible to define a collection of admissible open subsets of $X = \max(A)$ and admissible coverings of them for which there exists a functor $\mathcal{O}_X$ satisfying the sheaf conditions and such that $\mathcal{O}_X(Y) = A(Y)$ for any affine subset. Although the admissible open subsets and coverings don't form a topology in the usual sense, they satisfy the conditions necessary for them to support a sheaf theory — in fact, they form a Grothendieck topology. So, in this sense, Tate recovers analytic continuation.

For the final step, extending the category of affine rigid analytic spaces to a category of global rigid analytic spaces, Tate followed suggestions of Grothendieck. This step has since been clarified and simplified; see, for example, Bosch 2005,[72] especially 1.12.

Tate reported on his work in a series of letters to Serre, who had them typed by IHES as the notes Tate 1962c. These notes were distributed to a number of mathematicians and libraries. They soon attracted the attention of the German school of complex analytic geometers, who were able to transfer many of their arguments and results to the new setting (e.g., Kiehl 1967[73]). Already by 1984 to give a comprehensive account of the theory required a book of over 400 pages (Bosch, Güntzer, Remmert 1984[74]). Tate did not publish his work, but eventually the editors of "Mir" published a Russian translation of his notes (Tate 1969a), and the editors of "Inventiones" published the original (Tate 1971).

There have been a number of extensions of Tate's theory. For example, following a suggestion of Grothendieck, Raynaud showed that it is possible to realize a rigid analytic space over a field K as the "generic fibre" of a formal scheme over the valuation ring of K. One problem with rigid analytic spaces is that, while they are adequate for the study of coherent sheaves, they have too few points for the study of locally constant sheaves — for example, there exist nonzero such sheaves whose stalks are all zero. Berkovich found a solution to this problem by enlarging the underlying set of a rigid analytic space without altering the sheaf of functions so that the spaces now support an étale cohomology theory (Berkovich 1990;[75] 1993[76]).

Rigid analytic spaces are now part of the landscape of arithmetic geometry: just as it is natural to regard the $\mathbb{R}$-points of a $\mathbb{Q}$-variety as a real analytic space, it has become natural to regard the $\mathbb{Q}_p$-points of the variety as a rigid analytic space. They have found numerous applications, for example, in the solution by Harbater and Raynaud of Abyhankar's conjecture on the étale fundamental groups of curves, and in the Langlands program (see 5.1).

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[72] Bosch, S., Lectures on Formal and Rigid Geometry, Preprint 378 of the SFB Geometrische Strukturen in der Mathematik, Múnster, 2005. [73] Kiehl, Reinhardt. Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie. Invent. Math. 2 1967 191–214; Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. Ibid. 256–273. [74] Bosch, S.; Güntzer, U.; Remmert, R. Non-Archimedean analysis. A systematic approach to rigid analytic geometry. Grundlehren der Mathematischen Wissenschaften, 261. Springer-Verlag, Berlin, 1984. [75] Berkovich, Vladimir G. Spectral theory and analytic geometry over non-Archimedean fields. Mathematical Surveys and Monographs, 33. American Mathematical Society, Providence, RI, 1990. [76] Berkovich, Vladimir G. Etale cohomology for non-Archimedean analytic spaces. Inst. Hautes Études Sci. Publ. Math. No. 78 (1993), 5–161.

4 The Tate conjecture

This stuff is too beautiful not to be true
Tate[77]

The Hodge conjecture says that a rational cohomology class on a nonsingular projective variety over $\mathbb{C}$ is algebraic if it is of type (p,p). The Tate conjecture says that an ℓ-adic cohomology class on a nonsingular projective variety over a finitely generated field k is in the span of the algebraic classes if it is fixed by the Galois group. (A field is finitely generated if it is finitely generated as a field over its prime field.)

4.1 Beginnings

In the last section of his talk at the 1962 International Congress, Tate states several conjectures.

4.1.1. *For every abelian variety A over a global field k and prime $\ell \neq \mathrm{char}(k)$, $Ш(A/k)(\ell)$ is finite.*

Let k be a global function field, and let k_0 be its finite field of constants, so that $k = k_0(C)$ for a complete nonsingular curve C over k_0. An elliptic curve A over k is the generic fibre of a map $X \to C$ with X a complete nonsingular surface over k_0, which may be taken to be a minimal. Tate showed that, in this case, (4.1.1) is equivalent to the following conjecture.

$$\begin{array}{ccc} X & \longleftarrow & A \\ \downarrow & & \downarrow \\ C & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

4.1.2. *Let $q = |k_0|$. The $\mathbb{Z}_\ell$-submodule of $H^2_{\mathrm{et}}(X_{k_0^{\mathrm{al}}}, \mathbb{Z}_\ell)$ on which the Frobenius map acts as multiplication by q is exactly the submodule generated by the algebraic classes.*

As Tate notes, (4.1.2) makes sense for any complete nonsingular surface over k_0, and that, so generalized, it is equivalent to the following statement.[78]

4.1.3. *Let X be a complete nonsingular surface over a finite field. The order of the pole of $\zeta(X,s)$ at $s=1$ is equal to the number of algebraically independent divisors on X.*

Mumford pointed out that (4.1.3) implies that elliptic curves over a finite field are isogenous if and only if they have the same zeta function, and he proved this using results of Deuring[79] on the lifting to characteristic 0 of the Frobenius automorphism.

In his talk at the 1964 Woods Hole conference, Tate vastly generalized these conjectures.

[77] As a thesis topic, Tate gave me the problem of proving a formula that he and Mike Artin had conjectured concerning algebraic surfaces over finite fields (Conjecture C below). One day he ran into me in the corridors of 2 Divinity Avenue and asked how it was going. "Not well" I said, "In one example, I computed the left hand side and got p^{13}; for the other side, I got p^{17}; 13 is not equal to 17, and so the conjecture is false." For a moment, Tate was taken aback, but then he broke into a grin and said "That's great! That's really great! Mike and I must have overlooked some small factor which you have discovered." He took me off to his office to show him. In writing it out in front of him, I discovered a mistake in my work, which in fact proved that the conjecture is correct in the example I considered. So I apologized to Tate for my carelessness. But Tate responded: "Your error was not that you made a mistake — we all make mistakes. Your error was not realizing that you must have made a mistake. This stuff is too beautiful not to be true." Benedict Gross tells of a similar experience, but as he writes: "John was so encouraging, saying that everyone made mistakes, and the key was to understand them and to keep thinking about the problem. I felt that one of his greatest talents as an advisor was to make his students feel like we were partners in a great enterprise, modern number theory." [78] Assuming the Weil conjectures, which weren't proved until 1973. [79] Deuring, Max. Die Typen der Multiplikatorenringe elliptischer Funktionenkörper. Abh. Math. Sem. Hansischen Univ. 14, (1941). 197–272.

4.2 Statement of the Tate conjecture

For a connected nonsingular projective variety V over a field k, we let $\mathcal{Z}^r(V)$ denote the $\mathbb{Q}$-vector space of algebraic cycles on V of codimension r, i.e., the $\mathbb{Q}$-vector space with basis the irreducible closed subsets of V of dimension $\dim V - r$. We let $H^r_{\text{et}}(V, \mathbb{Q}_\ell(s))$ denote the étale cohomology group of V with coefficients in the "Tate twist" $\mathbb{Q}_\ell(s)$ of $\mathbb{Q}_\ell$. There are cycle maps

$$c^r : \mathcal{Z}^r(V) \to H^{2r}_{\text{et}}(V, \mathbb{Q}_\ell(r)).$$

Assume that $\ell \neq \text{char}(k)$. Let $\bar{k}$ be an algebraically closed field containing k, and let $G(\bar{k}/k)$ be the group of automorphisms of $\bar{k}$ fixing k. Then $G(\bar{k}/k)$ acts on $H^{2r}_{\text{et}}(V_{\bar{k}}, \mathbb{Q}_\ell(r))$, and the *Tate conjecture*[80] (Tate 1964a, Conjecture 1) is the following statement:[81]

> $T^r(V)$: When k is finitely generated, the $\mathbb{Q}_\ell$-space spanned by $c^r(\mathcal{Z}^r(V_{\bar{k}}))$ consists of the elements of $H^{2r}_{\text{et}}(V_{\bar{k}}, \mathbb{Q}_\ell(r))$ fixed by some open subgroup of $G(\bar{k}/k)$.

Suppose for simplicity that $\bar{k}$ is an algebraic closure of k. For any finite extension k' of k in $\bar{k}$, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}^r(V_{\bar{k}}) & \xrightarrow{c^r} & H^{2r}_{\text{et}}(V_{\bar{k}}, \mathbb{Q}_\ell(r)) \\ \uparrow & & \uparrow \\ \mathcal{Z}^r(V_{k'}) & \xrightarrow{c^r} & H^{2r}_{\text{et}}(V_{k'}, \mathbb{Q}_\ell(r)), \end{array}$$

and the image of the right hand map is $H^{2r}_{\text{et}}(V_{\bar{k}}, \mathbb{Q}_\ell(r))^{G(\bar{k}/k')}$. As $\mathcal{Z}^r(V_{\bar{k}}) = \bigcup_{k'} \mathcal{Z}^r(V_{k'})$, we see that

$$c^r(\mathcal{Z}^r(V_{\bar{k}})) \subset \bigcup_{k'} H^{2r}_{\text{et}}(V_{\bar{k}}, \mathbb{Q}_\ell(r))^{G(\bar{k}/k')}.$$

The content of the Tate conjecture is that the first set spans the space on the right. If an element of $\mathcal{Z}^r(V_{\bar{k}})$ is fixed by $G(\bar{k}/k')$, then it lies in $\mathcal{Z}^r(V_{k'})$, and so $T^r(V)$ implies that

$$c^r(\mathcal{Z}^r(V_{k'}))\mathbb{Q}_\ell = H^{2r}_{\text{et}}(V_{\bar{k}}, \mathbb{Q}_\ell(r))^{G(\bar{k}/k')}; \tag{26}$$

conversely, if (26) holds for all (sufficiently large) k', then $T^r(V)$ is true.

When asked about the origin of the Tate conjecture, Tate responded (Tate 2011):

> Early on I somehow had the idea that the special case about endomorphisms of abelian varieties over finite fields might be true. A bit later I realized that a generalization fit perfectly with the function field version of the Birch and Swinnerton-Dyer conjecture. Also it was true in various particular examples which I looked at and gave a heuristic reason for the Sato-Tate distribution. So it seemed a reasonable conjecture.

I discuss each of these motivations in turn.

[80] In the literature, a number of variants of $T^r(V)$, not obviously equivalent to it, are also called the Tate conjecture. It is not always easy to discern what an author means by the "Tate conjecture". [81] Since Atiyah and Hirzebruch had already found their counterexample to an integral Hodge conjecture, Tate was not tempted to state his conjecture integrally.

4.3 Homomorphisms of abelian varieties

Let A be an abelian variety over a field k, let $\bar{k}$ be an algebraically closed field containing k, and let $G(\bar{k}/k)$ denote the group of automorphisms of $\bar{k}$ over k. For $\ell \neq \mathrm{char}(k)$,

$$A \rightsquigarrow T_\ell A = \varprojlim A(\bar{k})_{\ell^n}$$

is a functor from abelian varieties over k to $\mathbb{Z}_\ell$-modules equipped with an action of $G(\bar{k}/k)$. The (Tate) *isogeny conjecture* is the following statement:

> $H(A,B)$: For abelian varieties A, B over a finitely generated field k, the canonical map
>
> $$\mathbb{Z}_\ell \otimes \mathrm{Hom}(A,B) \to \mathrm{Hom}(T_\ell A, T_\ell B)^{G(\bar{k}/k)}$$
>
> is an isomorphism.

It follows from Weil's theory of correspondences and the interpretation of divisorial correspondences as homomorphisms, that, for varieties V and W,

$$\mathrm{NS}(V \times W) \simeq \mathrm{NS}(V) \oplus \mathrm{NS}(W) \oplus \mathrm{Hom}(A,B) \tag{27}$$

where A is the Albanese variety of V, B is the Picard variety of W, and NS denotes the Néron-Severi group. On comparing (27) with the decomposition of $H^2(V \times W, \mathbb{Q}(1))$ given by the Künneth formula, we find that, for varieties V and W over a finitely generated field k,

$$T^1(V \times W) \iff T^1(V) + T^1(W) + H(A,B). \tag{28}$$

When V is a curve, $T^1(V)$ is obviously true, and so, for elliptic curves E and E',

$$T^1(E \times E') \iff H(E,E').$$

At the time Tate made his conjecture, $H(E,E')$ was known for elliptic curves over a finite field as a consequence of work of Deuring (see above), and $H(E,E)$ was known for elliptic curves over number fields with at least one real prime (Serre 1964).[82]

Tate (1966b) proved $H(A,B)$ for all abelian varieties over finite fields (see below). As we discussed in (2.6), this has implication for the classification of abelian varieties over finite fields (and even cryptography).

Zarhin extended Tate's result to fields finitely generated over $\mathbb{F}_p$, and Faltings proved $H(A,B)$ for all abelian varieties over number fields in the same article in which he proved Mordell's conjecture. In fact, $H(A,B)$ has now been proved in all generality (Faltings et al. 1994).[83]

Tate's theorem proves that T^1 is true for surfaces over finite fields that are a product of curves (by (28)). When Artin and Swinnerton-Dyer (1973)[84] proved T^1 for elliptic $K3$ surfaces over finite fields, there was considerable optimism that T^1 would soon be proved for all surfaces over finite fields. However, there has been little progress in the years since then. By contrast, the Hodge conjecture is easily proved for divisors.

[82] Serre, Jean-Pierre, Groupes de Lie l-adiques attachés aux courbes elliptiques. Colloque de Clermond-Ferrand 1964, Les Tendances Géom. en Algébre et Théorie des Nombres pp. 239–256 Éditions du Centre National de la Recherche Scientifique, Paris, 1966. [83] Rational points. Papers from the seminar held at the Max–Planck-Institut für Mathematik, Bonn, 1983/1984. Edited by Gerd Faltings and Gisbert Wüstholz. Aspects of Mathematics, E6. Friedr. Vieweg & Sohn, Braunschweig, 1984. [84] Artin, M.; Swinnerton-Dyer, H. P. F. , The Shafarevich-Tate conjecture for pencils of elliptic curves on K3 surfaces. Invent. Math. 20 (1973), 249–266.

TATE'S PROOF OF $H(A,B)$ OVER A FINITE FIELD

It suffices to prove the statement with $A = B$. As the map

$$\mathbb{Z}_\ell \otimes \mathrm{End}(A) \to \mathrm{End}(T_\ell A)^{G(\bar{k}/k)}$$

is injective, the problem is to construct enough endomorphisms of A. I briefly outline Tate's proof.

(a) If $H(A,A)$ is true for one prime $\ell \neq \mathrm{char}(k)$, then it is true for all. This allows Tate to choose an ℓ that is well adapted to his arguments.

(b) A polarization on A defines a skew-symmetric pairing $V_\ell A \times V_\ell A \to \mathbb{Q}_\ell$. Let W be a maximal isotropic subspace of $V_\ell A$ that is stable under $G(\bar{k}/k)$, and let

$$X_n = (T_\ell A \cap W) + \ell^n T_\ell A.$$

There is an infinite sequence of isogenies

$$\cdots \to B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 = A$$

such that the image of $T_\ell B_n$ in $T_\ell A$ is X_n. Using a theorem of Weil, Tate shows that each B_n has a polarization *of the same degree* as the original polarization on A. As k is finite, this implies that the B_n fall into finitely many isomorphism classes. An isomorphism $B_n \to B_{n'}$, $n \neq n'$, gives a nontrivial isogeny $A \to A$.

(c) Having constructed one endomorphism of A not in $\mathbb{Z}$, Tate makes adroit use of the semisimplicity of the rings involved (and his choice of ℓ) to complete the proof.

4.4 Relation to the conjectures of Birch and Swinnerton-Dyer

The original conjectures of Birch and Swinnerton-Dyer were stated for elliptic curves over $\mathbb{Q}$. Tate re-stated them more generally (see 2.3).

> (A) For an abelian variety A over a global field K, the function $L(s,A)$ has a zero of order $r = \mathrm{rank}A(K)$ at $s = 1$.
>
> (B) Moreover,
>
> $$L^*(s,A) \quad \sim \quad \frac{|Ш(A)| \cdot |D|}{|A(K)_{\mathrm{tors}}| \cdot |A'(K)_{\mathrm{tors}}|}(s-1)^r \text{ as } s \to 1.$$

Let $f: V \to C$ be a proper map with fibres of dimension 1, where V (resp. C) is a nonsingular projective surface (resp. curve) over a finite field k. The generic fibre of f is a curve over the global field $k(C)$, and we let $A(f)$ denote its Jacobian variety (an abelian variety over $k(C)$). A comparison of the invariants of V with the invariants of $A(f)$ yields the following statement:

$$\text{Conjecture } T^1 \text{ holds for } V \iff \text{Conjecture (A) holds for } A(f).$$

In examining the situation further, Artin and Tate (Tate 1966e) were led to make the following (Artin-Tate) conjecture:

(C) For a projective smooth geometrically-connected surface V over a finite field k, the Brauer group $\mathrm{Br}(V)$ of V is finite, and

$$P_2(q^{-s}) \quad \sim \quad \frac{|\mathrm{Br}(V)|\cdot|D|}{q^{\alpha(X)}\,|\mathrm{NS}(V)_{\mathrm{tors}}|^2}(1-q^{1-s})^{\rho(V)} \quad \text{as} \quad s\to 1$$

where $P_2(T)$ is the characteristic polynomial of the Frobenius automorphism acting on $H^2(V_{k^{\mathrm{al}}},\mathbb{Q}_\ell)$, D is the discriminant of the intersection pairing on $\mathrm{NS}(V)$, $\rho(V)$ is the rank of$\mathrm{NS}(V)$, and $\alpha(V)=\chi(X,\mathcal{O}_X)-1+\dim(\mathrm{PicVar}(V))$.

Naturally, they also conjectured:

(d) Let $f\colon V\to C$ be a proper map, as above, and assume that f has connected geometric fibres and a smooth generic fibre. Then Conjecture (B) holds for $A(f)$ over $k(C)$ if and only if Conjecture (C) holds for V over k.

Tate explains that he gave this conjecture "only a small letter (d) as label, because it is of a more elementary nature than (B) and (C)", and indeed, it has been proved. Artin and Tate checked it directly when f is smooth and has a section, Gordon (1979)[85] checked it when the generic fibre has a rational cycle of degree 1, and Milne (1982)[86] checked it when this condition holds only locally. However, ultimately the proof of (d) came from a different direction, by combining the following two statements:

- ◇ Conjecture C holds for a surface V over a finite field if and only if $\mathrm{Br}(V)(\ell)$ is finite for some prime ℓ (Tate 1966e ignoring the p part; Milne 1975[87] complete statement);
- ◇ Conjecture A holds for an abelian variety A over a global field of nonzero characteristic if and only if $Ш(A)(\ell)$ is finite for some ℓ (Kato and Trihan 2003[88]).

In the situation of (d), $\mathrm{Br}(V)(\ell)$ is finite if and only if $Ш(A(f))(\ell)$ is finite.[89]

The known cases of Conjecture B over function fields have proved useful in the construction of lattice packings.[90]

4.5 *Poles of zeta functions*

Throughout this subsection, V is a nonsingular projective variety over a field k. We regard algebraic cycles on V as elements of the $\mathbb{Q}$-vector spaces $\mathcal{Z}^r(V)$.

Algebraic cycles D and D' are said to be numerically equivalent if $D\cdot E = D'\cdot E$ for all algebraic cycles E on V of complementary dimension, and they are ℓ-homologically equivalent if they have the same class in $H^{2r}(V_{k^{\mathrm{al}}},\mathbb{Q}_\ell(r))$. In his Woods Hole talk, Tate asked whether the following statement is always true:

$E^r(V)$: Numerical equivalence coincides with ℓ-homological equivalence for algebraic cycles on V of codimension r.

[85] Gordon, W. J. Linking the conjectures of Artin-Tate and Birch–Swinnerton-Dyer. Compositio Math. 38 (1979), no. 2, 163–199. [86] Comparison of the Brauer group with the Tate-Shafarevich group, J. Fac. Sci. Univ. Tokyo (Shintani Memorial Volume) IA 28 (1982), 735-743. [87] Milne, J. S., On a conjecture of Artin and Tate. Ann. of Math. (2) 102 (1975), no. 3, 517–533. [88] Kato, Kazuya; Trihan, Fabien, On the conjectures of Birch and Swinnerton-Dyer in characteristic $p>0$. Invent. Math. 153 (2003), no. 3, 537–592. [89] Cf. Liu, Qing; Lorenzini, Dino; Raynaud, Michel, On the Brauer group of a surface. Invent. Math. 159 (2005), no. 3, 673–676. [90] "One of the most exciting developments has been Elkies' (sic) and Shioda's construction of lattice packings from Mordell-Weil groups of elliptic curves over function fields. Such lattices have a greater density than any previously known in dimensions from about 54 to 4096." Preface to Conway, J. H.; Sloane, N. J. A. Sphere packings, lattices and groups. Second edition. Springer-Verlag, New York, 1993.

This is now generally regarded as a folklore conjecture (it is also a consequence of Grothendieck's standard conjectures). Note that, like the Tate conjecture, $E^r(V)$ is an existence statement for algebraic cycles: for an algebraic cycle D, it says that there exists an algebraic cycle E of complementary dimension such that $D \cdot E \neq 0$ if there exists a cohomological cycle with this property.

Let $\mathcal{A}^r$ denote the image of $\mathcal{Z}^r(V)$ in $H^{2r}(V, \mathbb{Q}_\ell(r))$, and let $\mathcal{N}^r$ denote the subspace of classes numerically equivalent to zero. Thus, $\mathcal{A}^r$ (resp. $\mathcal{A}^r/\mathcal{N}^r$) is the $\mathbb{Q}$-space of algebraic classes of codimension r modulo homological equivalence (resp. modulo numerical equivalence). In particular, $\mathcal{A}^r/\mathcal{N}^r$ is independent of ℓ.

Now assume that k is finitely generated. We need to consider also the following statement:

> $S^r(V)$: The map $H^{2r}(V_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))^{G(k^{\mathrm{al}}/k)} \to H^{2r}(V_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))_{G(k^{\mathrm{al}}/k)}$ induced by the identity map is bijective.

When k is finite, this means that 1 occurs semisimply (if at all) as an eigenvalue of the Frobenius map acting on $H^{2r}(V_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))$.

An elementary argument suffices to prove that the following three statements are equivalent (for a fixed variety V, integer r, and prime ℓ):

(a) $T^r + E^r$;
(b) $T^r + T^{\dim V - r} + S^r$;
(c) $\dim_{\mathbb{Q}}(\mathcal{A}^r/\mathcal{N}^r) = \dim_{\mathbb{Q}_l} H^{2r}(V_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))^{G(k^{\mathrm{al}}/k)}$.

When k is finite, each statement is equivalent to:

(d) the order of the pole of $\zeta(V,s)$ at $s = r$ is equal to $\dim_{\mathbb{Q}}(\mathcal{A}^r/\mathcal{N}^r)$.

See Tate 1979, 2.9. Note that (d) is independent of ℓ.

Tate (1964a, Conjecture 2) conjectured the following general version of (d):

> $P^r(V)$: Let V be a nonsingular projective variety over a finitely generated field k. Let d be the transcendence degree of k over the prime field, augmented by 1 if the prime field is $\mathbb{Q}$. Then the $2r$th component $\zeta^{2r}(V,s)$ of the zeta function of V has a pole of order $\dim_{\mathbb{Q}}(\mathcal{A}^r(V))$ at the point $s = d + r$.

This is also known as the Tate conjecture. For a discussion of the known cases of $P^r(V)$, see Tate 1964a, 1994a.

When k is a global function field, statements (a), (b), (c) are independent of ℓ, and are equivalent to the statement that $\zeta_S^{2r}(V,s)$ has a pole of order $\dim_{\mathbb{Q}}(\mathcal{A}^r(V))$ at the point $s = d + r$; here $\zeta_S^{2r}(V,s)$ omits the factors at a suitably large finite set S of primes. This follows from Lafforgue's proof of the global Langlands correspondence and other results in Langlands program by an argument that will, in principle, also work over number fields. — see Lyons 2009.[91]

In the presence of E^r, Conjecture $T^r(V)$ is equvalent to $P^r(V)$ if and only if the order of the pole of $\zeta^{2r}(V,s)$ at $s = r$ is $\dim_{\mathbb{Q}_l} H^{2r}(V_{k^{\mathrm{al}}}, \mathbb{Q}_\ell(r))^{G(k^{\mathrm{al}}/k)}$. This is known for some Shimura varieties.

[91] Lyons, Christopher. A rank inequality for the Tate conjecture over global function fields. Expo. Math. 27 (2009), no. 2, 93–108.

THE SATO-TATE CONJECTURE

Let A be an elliptic curve over $\mathbb{Q}$. For a prime p of good reduction, the number N_p of points on A mod p can be written

$$N_p = p + 1 - a_p$$
$$a_p = 2\sqrt{p}\cos\theta_p, \quad 0 \le \theta_p \le \pi.$$

When A has complex multiplication over $\mathbb{C}$, it is easily proved that the θ_p are uniformly distributed in the interval $0 \le \theta \le \pi$ as $p \to \infty$. In the opposite case, Mikio Sato found computationally that the θ_p appeared to have a density distribution $\frac{2}{\pi}\sin^2\theta$.

Tate proved that, for a power of an elliptic curve, the $\mathbb{Q}$-algebra of algebraic cycles is generated modulo homological equivalence by divisor classes. Using this, he computed that, for an elliptic curve A over $\mathbb{Q}$ without complex multiplication,

$$\operatorname{rank}(\mathcal{A}^i(A^m)) = \binom{m}{i}^2 - \binom{m}{i-1}\binom{m}{i+1},$$

from which he deduced that Sato's distribution is the only symmetric density distribution for which the zeta functions of the powers of A have their zeros and poles in agreement with the Conjecture $P^r(V)$.

The conjecture that, for an elliptic curve over $\mathbb{Q}$ without complex multiplication, the θ_p are distributed as $\frac{2}{\pi}\sin^2\theta$ is known as the Sato-Tate conjecture. It has been proved only recently, as the fruit of a long collaboration (Richard Taylor, Michael Harris, Laurent Clozel, Nicholas Shepherd-Barron, Thomas Barnet-Lamb, David Geraghty). As did Tate, they approach the conjecture through the analytic properties of the zeta functions of the powers of A.[92]

Needless to say, the Sato-Tate conjecture has been generalized to motives. Langlands has pointed out that his functoriality conjecture contains a very general form of the Sato-Tate conjecture.[93]

4.6 *Relation to the Hodge conjecture*

For a variety V over $\mathbb{C}$, there is a well-defined cycle map

$$c^r : \mathcal{Z}^r(V) \to H^{2r}(V, \mathbb{Q})$$

(cohomology with respect to the complex topology). Hodge proved that there is a decomposition

$$H^{2r}(V,\mathbb{Q})_{\mathbb{C}} = \bigoplus\nolimits_{p+q=2r} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p}.$$

In Hodge 1950,[94] he observed that the image of c^r is contained in

$$H^{2r}(V,\mathbb{Q}) \cap V^{r,r}$$

[92] The proof was completed in: T.Barnet-Lamb, D.Geraghty, M.Harris and R.Taylor, A family of Calabi-Yau varieties and potential automorphy II. P.R.I.M.S. 47 (2011), 29-98. For expository accounts, see: Carayol, Henri, La conjecture de Sato-Tate (d'après Clozel, Harris, Shepherd-Barron, Taylor). Séminaire Bourbaki. Vol. 2006/2007. Astérisque No. 317 (2008), Exp. No. 977, ix, 345–391. Clozel, L. The Sato-Tate conjecture. Current developments in mathematics, 2006, 1–34, Int. Press, Somerville, MA, 2008. [93] Langlands, Robert P. Reflexions on receiving the Shaw Prize. On certain L-functions, 297–308, Clay Math. Proc., 13, Amer. Math. Soc., Providence, RI, 2011. [94] Hodge, W. V. D., The topological invariants of algebraic varieties. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 182–192. Amer. Math. Soc., Providence, R. I., 1952.

and asked whether this $\mathbb{Q}$-module is exactly the image of c^r. This has become known as the Hodge conjecture.[95]

In his original article (Tate 1964), Tate wrote:

> I can see no direct logical connection between [the Tate conjecture] and Hodge's conjecture that a rational cohomology class of type (p, p) is algebraic.... However, the two conjectures have an air of compatibility.

Pohlmann (1968)[96] proved that the Hodge and Tate conjectures are equivalent for CM abelian varieties, Piatetski-Shapiro (1971)[97] proved that the Tate conjecture for abelian varieties in characteristic zero implies the Hodge conjecture for abelian varieties, and Milne (1999)[98] proved that the Hodge conjecture for CM abelian varieties implies the Tate conjecture for abelian varieties over finite fields.

The relation between the two conjectures has been greatly clarified by the work of Deligne. He defines the notion of an absolute Hodge class on a (complete smooth) variety over a field of characteristic zero, and conjectures that every Hodge class on a variety over $\mathbb{C}$ is absolutely Hodge. The Tate conjecture for a variety implies that all absolute Hodge classes on the variety are algebraic. Therefore, in the presence of Deligne's conjecture, the Tate conjecture implies the Hodge conjecture. As Deligne has proved his conjecture for abelian varieties, this gives another proof of Piatetski-Shapiro's theorem.

The twin conjectures of Hodge and Tate have a status in algebraic and arithmetic geometry similar to that of the Riemann hypothesis in analytic number theory. A proof of either one for any significantly large class of varieties would be a major breakthrough. On the other hand, whether or not the Hodge conjecture is true, it is known that Hodge classes behave in many ways as if they were algebraic.[99] There is some fragmentary evidence that the same is true for Tate classes in nonzero characteristic.[100]

5 Lubin-Tate theory and Barsotti-Tate group schemes

5.1 Formal group laws and applications

Let R be a commutative ring. By a formal group law over R, we shall always mean a one-parameter commutative formal group law, i.e., a formal power series $F \in R[[X, Y]]$ such that

- $F(X, Y) = X + Y +$terms of higher degree,
- $F(F(X, Y), Z) = F(X, F(Y, Z))$
- $F(X, Y) = F(Y, X)$.

These conditions imply that there exists a unique $i_F(X) \in X \cdot R[[X]]$ such that $F(X, i_F(X)) = 0$. A homomorphism $F \to G$ of formal group laws is a formal power series $f \in XR[[X]]$ such that $f(F(X, Y)) = G(f(X), f(Y))$.

[95] Hodge actually asked the question with $\mathbb{Z}$-coefficients. [96] Pohlmann, Henry. Algebraic cycles on abelian varieties of complex multiplication type. Ann. of Math. (2) 88 1968 161–180. [97] Piatetski-Shapiro, I. I., Interrelations between the Tate and Hodge hypotheses for abelian varieties. (Russian) Mat. Sb. (N.S.) 85(127) (1971), 610–620. [98] Milne, J. S., Lefschetz motives and the Tate conjecture. Compositio Math. 117 (1999), no. 1, 45–76. [99] Deligne, P., Hodge cycles on abelian varieties (notes by J.S. Milne). In: Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Math. 900, Springer-Verlag, 1982, pp. 9–100. Cattani, Eduardo; Deligne, Pierre; Kaplan, Aroldo. On the locus of Hodge classes. J. Amer. Math. Soc. 8 (1995), no. 2, 483–506. [100] Milne, J. S. Rational Tate classes. Mosc. Math. J. 9 (2009), no. 1, 111–141.

The formal group laws form a $\mathbb{Z}$-linear category. Let $c(f)$ be the first-degree coefficient of an endomorphism f of F. If R is an integral domain of characteristic zero, then $f \mapsto c(f)$ is an injective homomorphism of rings $\mathrm{End}_R(F) \to R$. See Lubin 1964.[101]

Let F be a formal group law over a field k of characteristic $p \neq 0$. A nonzero endomorphism f of F has the form

$$f = aX^{p^h} + \text{terms of higher degree}, \quad a \neq 0,$$

where h is a nonnegative integer, called the height of f. The height of the multiplication-by-p map is called the height of F.

LUBIN-TATE FORMAL GROUP LAWS AND LOCAL CLASS FIELD THEORY

Let K be a nonarchimedean local field, i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Local class field theory provides us with a homomorphism (the local reciprocity map)

$$\mathrm{rec}_K : K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

such that, for every finite abelian extension L of K in K^{ab}, rec_K induces an isomorphism

$$(-, L/K) : K^\times / \mathrm{Nm}\, L^\times \to \mathrm{Gal}(L/K);$$

moreover, every open subgroup $K^\times$ of finite index arises as the norm group of a (unique) finite abelian extension. This statement shows that the finite abelian extensions of K are classified by the open subgroups of $K^\times$ of finite index, but leaves open the following problem:

> Let L/K be the abelian extension corresponding to an open subgroup H of $K^\times$ of finite index; construct generators for L and describe how $K^\times/H$ acts on them.

Lubin and Tate (1965a) found an elegantly simple solution to this problem.

The choice of a prime element π determines a decomposition $K^\times = \mathcal{O}_K^\times \times \langle \pi \rangle$, and hence (by local class field theory) a decomposition $K^{\mathrm{ab}} = K_\pi \cdot K^{\mathrm{un}}$. Here K_π is a totally ramified extension of K with the property that π is a norm from every finite subextension. Since K^{un} is well understood, the problem then is to find generators for the subfields of K_π and to describe the isomorphism

$$(-, K_\pi/K) : \mathcal{O}_K^\times \to \mathrm{Gal}(K_\pi/K)$$

given by reciprocity map. Let $\mathcal{O} = \mathcal{O}_K$, let $\mathfrak{p} = (\pi)$ be the maximal ideal in $\mathcal{O}$, and let $q = (\mathcal{O}:\mathfrak{p})$.

Let $f \in \mathcal{O}[[T]]$ be a formal power series such that

$$\begin{cases} f(T) &= \pi T + \text{ terms of higher degree} \\ f(T) &\equiv T^q \text{ modulo } \pi\mathcal{O}[[T]], \end{cases}$$

for example, $f = \pi T + T^q$ is such a power series. An elementary argument shows that, for each $a \in \mathcal{O}$, there is a unique formal power series $[a]_f \in \mathcal{O}_K[[T]]$ such that

$$\begin{cases} [a]_f(T) &= aT + \text{ terms of higher degree} \\ [a]_f \circ f &= f \circ [a]_f. \end{cases}$$

[101] Lubin, Jonathan, One-parameter formal Lie groups over p-adic integer rings. Ann. of Math. (2) 80 1964 464–484.

Let

$$X_m = \{x \in K^{\mathrm{al}} \mid |x| < 1,\ \overbrace{(f \circ \cdots \circ f)}^{m}(x) = 0\}.$$

Then Lubin and Tate (1965a) prove:

(a) the field $K[X_m]$ is the totally ramified abelian extension of K with norm group $U_m \times \langle \pi \rangle$ where $U_m = 1 + \mathfrak{p}_K^m$;

(b) the map

$$\begin{aligned} \mathcal{O}^\times / U_m &\to \mathrm{Gal}(K[X_m]/K) \\ u &\mapsto \big(x \mapsto [u^{-1}]_f(x)\big) \end{aligned}$$

is an isomorphism.

For example, if $K = \mathbb{Q}_p$, $\pi = p$, and $f(T) = (1+T)^p - 1$, then

$$X_m = \{\zeta - 1 \mid \zeta^{p^m} = 1\} \simeq \mu_{p^m}(K^{\mathrm{al}})$$

and $[u]_f(\zeta - 1) = \zeta^{u^{-1}} - 1$.

Lubin and Tate (1965a) show that, for each f as above, there is a unique formal group law F_f admitting f as an endomorphism. Then X_m can be realized as a group of "torsion points" on F_f, which endows it with the structure of an $\mathcal{O}$-module for which it is isomorphic to $\mathcal{O}/\mathfrak{p}^m$. From this the statements follow in a straightforward way.

The proof of the above results does not use local class field theory. Using the Hasse-Arf theorem, one can show that $K_\pi \cdot K^{\mathrm{un}} = K^{\mathrm{ab}}$, and deduce local class field theory. Alternatively, using local class field theory, one can show that $K_\pi \cdot K^{\mathrm{un}} = K^{\mathrm{ab}}$, and deduce the Hasse-Arf theorem. In either case, one finds that the isomorphism in (b) is the local reciprocity map.

The F_f are called *Lubin-Tate formal group laws*, and the above theory is called *Lubin-Tate theory*.

Deformations of formal group laws (Lubin-Tate spaces)

Let F be a formal group law of height h over a perfect field k of characteristic $p \neq 0$. We consider local artinian k-algebras A with residue field k. A deformation of F over such an A is a formal group law F_A over A such that $F_A \equiv F \bmod \mathfrak{m}_A$. An isomorphism of deformations is an isomorphism of $\varphi\colon F_A \to G_A$ of formal group laws such that $\varphi(T) \equiv T \bmod \mathfrak{m}_A$.

Let W denote the ring of Witt vectors with residue field k. Lubin and Tate (1966d) prove that there exists a formal group $\mathcal{F}(t_1, \dots, t_{h-1})(X, Y)$ over $W[[t_1, \dots, t_{h-1}]]$ with the following properties:

- $\mathcal{F}(0, \dots, 0) \equiv F \bmod \mathfrak{m}_W$;
- for any deformation F_A of F, there is a unique homomorphism $W[[t_1, \dots, t_{h-1}]] \to A$ sending $\mathcal{F}$ to F_A.

The results of Lubin and Tate are actually stronger, but this seems to be the form in which they are most commonly used.

In particular, the above result identifies the space of deformations of F with the formal scheme $\mathrm{Spf}(W[[t_1, \dots, t_{h-1}])$. These spaces are now called Lubin-Tate deformation spaces. Drinfeld showed that, by adding (Drinfeld) level structures, it is possible to construct towers of deformation spaces, called Lubin-Tate towers.[102] These play an important role in the

[102] Drinfeld, V. G. Elliptic modules. (Russian) Mat. Sb. (N.S.) 94(136) (1974), 594–627, 656.

study of the moduli varieties of abelian varieties with PEL-structure and in the Langlands program. For example, it is known that both the Jacquet-Langlands correspondence and the local Langlands correspondence for GL_n can be realized in the étale cohomology of a Lubin-Tate tower (or, more precisely, in the étale cohomology of the Berkovich space that is attached to the rigid analytic space which is the generic fibre of the Lubin-Tate tower).[103]

5.2 *Finite flat group schemes*

A group scheme G over a scheme S is finite and flat if $G = \mathrm{Spec}(A)$ with A locally free of finite rank as a sheaf of $\mathcal{O}_S$-modules. When A has constant rank r, G is said to have order r. A finite flat group scheme of prime order is necessarily commutative.

In his course on Formal Groups at Harvard in the fall of 1966, Tate discussed the following classification problem:

> let R be a local noetherian ring with residue field of characteristic $p \neq 0$; assume that R contains the $(p-1)$st roots of 1, i.e., that $R^\times$ contains a cyclic subgroup of order $p-1$; determine the finite flat group schemes of order p over R.

When R is complete, Tate found that such group schemes correspond to pairs of elements (a,c) of R such that $ac \in p \cdot R^\times$; two pairs (a,c) and (a_1,c_1) correspond to isomorphic groups if and only if $a_1 = u^{p-1}a$ and $c_1 = u^{1-p}c$ for some $u \in R^\times$.

These results were extended and completed in Tate and Oort 1970a. Let

$$\Lambda_p = \mathbb{Z}[\zeta, \frac{1}{p(p-1)}] \cap \mathbb{Z}_p$$

where ζ is a primitive $(p-1)$st root of 1 in $\mathbb{Z}_p$. Tate and Oort define a sequence $w_1 = 1, w_2, \ldots, w_p$ of elements of Λ_p in which $w_1, \ldots, w_{p-1}$ are units and $w_p = pw_{p-1}$. Then, given an invertible $\mathcal{O}_S$-module L and sections a of $L^{\otimes(p-1)}$ and b of $L^{\otimes(1-p)}$ such that $a \otimes b = w_p$, they show that there is a group scheme $G^L_{a,b}$ such that, for every S-scheme T,

$$G^L_{a,b}(T) = \{x \in \Gamma(T, L \otimes_{\mathcal{O}_S} \mathcal{O}_T) \mid x^{\otimes p} = a \otimes x\},$$

and the multiplication on $G^L_{a,b}(T)$ is given by

$$x_1 \cdot x_2 = x_1 + x_2 + \frac{b}{w_{p-1}} \otimes D_p(x_1 \otimes 1, 1 \otimes x_2),$$

where

$$D_p(X_1, X_2) = \frac{w_{p-1}}{1-p} \sum_{i=1}^{p-1} \frac{X_1^i}{w_i} \frac{X_2^{p-i}}{w_{p-i}} \in \Lambda_p[X_1, X_2].$$

Every finite flat group scheme of order p over S is of the form $G^L_{a,b}$ for some triple (L,a,b), and $G^L_{a,b}$ is isomorphic to $G^{L_1}_{a_1,b_1}$ if and only if there exists an isomorphism from L to

[103] Carayol, H. Nonabelian Lubin-Tate theory. Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), 15–39, Perspect. Math., 11, Academic Press, Boston, MA, 1990. Boyer, P. Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale. Invent. Math. 138 (1999), no. 3, 573–629. Harris, Michael; Taylor, Richard The geometry and cohomology of some simple Shimura varieties. With an appendix by Vladimir G. Berkovich. Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ, 2001.

L_1 carrying a to a_1 and b to b_1. The Cartier dual of $G^L_{a,b}$ is $G^{L^{-1}}_{b,a}$. The proofs of these statements make ingenious use of the action of $\mathbb{F}_p^\times$ on $\mathcal{O}_G$.

Tate and Oort apply their result to give a classification of finite flat group schemes of order p over the ring of integers in a number field in terms of idèle class characters. In particular, they show that the only such group schemes over $\mathbb{Z}$ are the constant group scheme $\mathbb{Z}/p\mathbb{Z}$ and its dual μ_p.

In the years since Tate and Oort wrote their article, the classification of finite flat commutative group schemes over various bases has been intensively studied, and some of the results were used in the proof of the modularity conjecture for elliptic curves (hence of Fermat's last theorem). Tate (1997a) has given a beautiful exposition of the basic theory of finite flat group schemes, including the results of Raynaud 1974[104] extending the above theory to group schemes of type $(p,\dots,p)$.

5.3 Barsotti-Tate groups (p-divisible groups)

Let A be an abelian variety over a field k. In his study of abelian varieties and their zeta functions, Weil used the ℓ-primary component $A(\ell)$ of the group $A(k^{\text{sep}})$ for ℓ a prime distinct from $\text{char}(k)$. This is an ℓ-divisible group isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2\dim A}$ equipped with an action of $\text{Gal}(k^{\text{sep}}/k)$. For $p = \text{char}(k)$, it is natural to replace $A(\ell)$ with the direct system

$$A(p)\colon \quad A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_\nu \hookrightarrow A_{\nu+1} \hookrightarrow \cdots$$

where A_ν is the finite group *scheme* $\text{Ker}(p^\nu\colon A \to A)$.

In the mid sixties, Serre and Tate[105] defined a *p-divisible group* of height h over a ring R to be a direct system $G = (G_\nu, i_\nu)_{\nu\in\mathbb{N}}$ where, for each $\nu \geq 0$, G_ν is a finite group scheme over R of order $p^{\nu h}$ and the sequence

$$0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{p^\nu} G_{\nu+1}$$

is exact. An abelian scheme A over R defines a p-divisible group $A(p)$ over R of height $2\dim(A)$.

The dual of a p-divisible group $G = (G_\nu, i_\nu)$ is the system $G' = (G'_\nu, i'_\nu)$ where G'_ν is the Cartier dual of G_ν and i'_ν is the Cartier dual of the map $G_{\nu+1} \xrightarrow{p} G_\nu$. It is again a p-divisible group.

Tate developed the basic theory p-divisible group in his article for the proceedings of the 1966 Driebergen conference (Tate 1967c) and in a series of ten lectures at the Collège de France in 1965-1966.[106] He showed that p-divisible groups generalize formal Lie groups in the following sense: let R be a complete noetherian local ring with residue field k of characteristic $p > 0$; an n-dimensional commutative formal Lie group Γ over R can be defined to be a family $f(Y,Z) = (f_i(Y,Z))_{1\leq i\leq n}$ of n power series in $2n$ variables satisfying the conditions in the first paragraph of this section; if such a group Γ is divisible (i.e., $p\colon \Gamma \to \Gamma$ is an isogeny), then one can define the kernel G_ν of $p^\nu\colon \Gamma \to \Gamma$ as a group scheme over R; Tate shows that $\Gamma(p) = (G_\nu)_{\nu\geq 1}$ is a p-divisible group $\Gamma(p)$, and that the functor $\Gamma \rightsquigarrow \Gamma(p)$ is an equivalence from the category of divisible commutative formal Lie groups over R to the category of connected p-divisible groups over R.

[104] Raynaud, Michel, Schémas en groupes de type (p,...,p). Bull. Soc. Math. France 102 (1974), 241–280.

[105] Usually this is credited to Tate alone, but Tate writes: "We were both contemplating them. I think it was probably Serre who first saw clearly the simple general definition and its relation to formal groups of finite height." The dual of a p-divisible group is often called the Serre dual.

[106] See Serre, OEuvres, II pp.321–324.

The main theorem of Tate 1967c states the following:

> Let R be an integrally closed, noetherian, integral domain whose field of fractions K is of characteristic zero, and let G and H be p-divisible groups over R. Then every homomorphism $G_K \to H_K$ of the generic fibres extends uniquely to a homomorphism $G \to H$.

In other words, the functor $G \rightsquigarrow G_K$ is fully faithful. This was extended to rings R of characteristic $p \neq 0$ by de Jong in 1998.[107]

Since their introduction, p-divisible groups have become an essential tool in the study of abelian schemes. We have already seen in (2.4) one application of p-divisible groups to the problem of lifting abelian varieties. Another application was to the proof of the Mordell conjecture (Faltings 1983[108]).

In his talk at the 1970 International Congress, Grothendieck renamed p-divisible groups "Barsotti-Tate groups". Today, both terms are used.

5.4 Hodge-Tate decompositions

Now let R be a complete discrete valuation ring of unequal characteristics, and let K be its field of fractions. Let K^{al} be an algebraic closure of K, and let C be the completion of K^{al}. As Serre noted:[109]

> One of the most surprising results of Tate's theory is the fact that *the properties of p-divisible groups are intimately related to the structure of C as a Galois module over* $\mathrm{Gal}(K^{\mathrm{al}}/K)$.

Let $\mathcal{G} = \mathrm{Gal}(K^{\mathrm{al}}/K)$, and let V be a C-vector space on which $\mathcal{G}$ acts semi-linearly. The Tate twist $V(i)$, $i \in \mathbb{Z}$, is V with $\mathcal{G}$ acting by

$$(\sigma, v) \mapsto \chi(\sigma)^i \cdot \sigma v, \quad \chi \text{ the } p\text{-adic cyclotomic character.}$$

Tate proved that $H^0(\mathcal{G}, C) = K$ and $H^1(\mathcal{G}, C) \approx K$, and that $H^q(\mathcal{G}, C(i)) = 0$ for $q = 0, 1$ and $i \neq 0$.[110] Using these statements, he proved that, for a p-divisible group G over R, there is a canonical isomorphism

$$\mathrm{Hom}(TG, C) \simeq t_{G'}(C) \oplus t_G^\vee(C)(-1). \tag{29}$$

where $TG = \varprojlim_\nu G_\nu(K^{\mathrm{al}})$ is the Tate module of G, t_G is the tangent space to G at zero, and G' is the dual p-divisible group. In particular TG determines the dimension of G, a fact that is used in the proof of the main theorem in the last subsection.

When G is the p-divisible group of an abelian scheme A over R, (29) can be written as:

$$H^1_{\mathrm{et}}(A_C, \mathbb{Q}_p) \otimes C \simeq H^1(A_C, \Omega^0_{A_C/C}) \oplus H^0(A_C, \Omega^1_{A_C/C})(-1).$$

This result led Tate to make the following (Hodge-Tate) conjecture:[111]

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[107] de Jong, A. J., Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic. Invent. Math. 134 (1998), no. 2, 301–333. Erratum: ibid. 138 (1999), no. 1, 225. [108] Faltings, G. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. Invent. Math. 73 (1983), no. 3, 349–366. Erratum: Ibid. (1984), no. 2, 381. [109] Œ uvres, II, p.322. [110] Sen and Ax simplified and generalized Tate's proof that $C^G = K$, and the result is now known as the Ax-Sen-Tate theorem. [111] Tate 1967c, p.180; see also Serre's summary of Tate's lectures, Œuvres, II, p.324.

For every nonsingular projective variety X over K, there exists a canonical (Hodge-Tate) decomposition

$$H^n_{\text{et}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq \bigoplus\nolimits_{p+q=n} H^{p,q}(X_C)(-p) \tag{30}$$

where $H^{p,q}(X_C) = H^q(X, \Omega^p_{X/K}) \otimes_K C$. This decomposition is compatible with the action of $\mathrm{Gal}(K^{\mathrm{al}}/K)$.

Tate's conjecture launched a new subject in mathematics, called p-adic Hodge theory. The isomorphism (30) can be regarded as a statement about the étale cohomology of X_C regarded as a module over $\mathrm{Gal}(K^{\mathrm{al}}/K)$. About 1980, Fontaine stated a series of successively stronger conjectures, beginning with the Hodge-Tate conjecture, that describe the structure of these Galois modules.[112] Most of Fontaine's conjectures have now been proved. The Hodge-Tate conjecture itself was proved by Faltings in 1988.[113]

6 Elliptic curves

Although elliptic curves are just abelian varieties of dimension one, their study is quite different. Throughout his career, Tate has returned to the study of elliptic curves.

6.1 Ranks of elliptic curves over global fields

Mordell proved that, for an elliptic curve E over $\mathbb{Q}$, the group $E(\mathbb{Q})$ is finite generated. At one time, it was widely conjectured that the rank of $E(\mathbb{Q})$ is bounded, but, as Cassels 1966 pointed out, this is implausible.[114] Tate and Shafarevich (1967d)[115] made it even less plausible by proving that, for elliptic curves over the global field $\mathbb{F}_p(t)$, the ranks are unbounded. Their examples are quadratic twists of a supersingular elliptic curve with coefficients in $\mathbb{F}_p$; in particular, they are isotrivial (i.e., have $j \in \mathbb{F}_p$). More recently, it has been shown that the ranks are unbounded even among the nonisotrivial elliptic curves over $\mathbb{F}_p(t)$.[116] Meanwhile, the largest known rank for an elliptic curve over $\mathbb{Q}$ is 28.[117]

6.2 Torsion points on elliptic curves over $\mathbb{Q}$

Beppo Levi constructed elliptic curves E over $\mathbb{Q}$ having each of the groups

$$\begin{aligned} \mathbb{Z}/n\mathbb{Z} \qquad & n = 1, 2, \dots, 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \qquad & n = 2, 4, 6, 8, \end{aligned}$$

[112] See Fontaine, Jean-Marc. Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. Ann. of Math. (2) 115 (1982), no. 3, 529–577, and many other articles. [113] Faltings, Gerd. p-adic Hodge theory. J. Amer. Math. Soc. 1 (1988), no. 1, 255–299. [114] "It has been widely conjectured that there is an upper bound for the rank depending only on the groundfield. This seems to me implausible because the theory makes it clear that an abelian variety can only have high rank if it is defined by equations with very large coefficients." p.257 of Cassels, J. W. S., Diophantine equations with special reference to elliptic curves. J. London Math. Soc. 41 1966 193–291. [115] For a long time I was puzzled as to how this article came to be written, because I was not aware that Shafarevich had been allowed to travel to the West, but Tate writes: "sometime during the year 1965–66, which I spent in Paris, Shafarevich appeared. There must have been a brief period when the Soviets relaxed their no-travel policy.... Shafarevich was in Paris for a month or so, and the paper grew out of some discussion we had. We both liked the idea of our having a joint paper, and I was happy to have it in Russian." [116] Ulmer, Douglas Elliptic curves with large rank over function fields. Ann. of Math. (2) 155 (2002), no. 1, 295–315. [117] Elkies, see http://web.math.hr/~duje/tors/tors.html.

as the torsion subgroup of $E(\mathbb{Q})$, and he conjectured that this exhausted the list of possible such groups.[118]

Consistent with this, Mazur and Tate (1973c) show that there is no elliptic curve over $\mathbb{Q}$ with a rational point of order 13, or, equivalently, that the curve $X_1(13)$ that classifies the elliptic curves with a chosen point of order 13 has no rational points (except for its cusps). Ogg found a rational point of order 19 on the Jacobian J of $X_1(13)$, and Mazur and Tate show that J has exactly 19 rational points. They then deduce that $X_1(13)$ has no noncuspidal rational point by examining how it sits in its Jacobian.

The interest of their article is more in its methods than in the result itself.[119] The ring $\mathbb{Z}[\sqrt[3]{1}]$ acts on J, and Mazur and Tate perform a 19-descent by studying the flat cohomology of the exact sequence of group schemes

$$0 \to F \to J \xrightarrow{\pi} J \to 0$$

on $\operatorname{Spec}\mathbb{Z}\smallsetminus(13)$, where π is one of the factors of 19 in $\mathbb{Z}[\sqrt[3]{1}]$. In a major work, Mazur developed these methods further, and completely proved Levi's conjecture.[120]

The similar problem for an arbitrary number field K is probably beyond reach, but, following work of Kamienny, Merel (1996[121]) proved that, for a fixed number field K, the order of the torsion subgroup of $E(K)$ for E an elliptic curve over K is bounded by a constant that depends only the degree of K over $\mathbb{Q}$.

6.3 *Explicit formulas and algorithms*

The usual Weierstrass form of the equation of an elliptic curve is valid only in characteristics $\neq 2,3$. About 1965 Tate wrote out the complete form, valid in all characteristics, and even over $\mathbb{Z}$. For an elliptic curve over a nonarchimedean local field with perfect residue field, he wrote out an explicit algorithm (known as Tate's algorithm) for computing the minimal model of the curve and determining the Kodaira type of the special fibre. Ogg's formula then gives the conductor of the curve. The handwritten manuscript containing these formulas was invaluable to people working in the field. A copy, which had been sent to Cassels, was included, essentially verbatim, in the proceedings of the Antwerp conference (Tate 1975b).[122]

6.4 *Analogues at p of the conjecture of Birch and Swinnerton-Dyer*

For an elliptic curve E over $\mathbb{Q}$, the conjecture of Birch and Swinnerton-Dyer states that

$$L(s,E) \quad\sim\quad \Omega \prod_{p \text{ bad}} c_p \frac{|Ш(E)|\cdot R}{|E(\mathbb{Q})_{\text{tors}}|^2}(s-1)^r \quad \text{as} \quad s \to 1$$

[118] Beppo Levi, Sull'equazione indeterminata del 3° ordine, Rom. 4. Math. Kongr. 2, 173-177 1909. (Talk at the 1908 ICM.) [119] About the same time, J. Blass found a more elementary proof of the same result. [120] Mazur, B. Modular curves and the Eisenstein ideal. Inst. Hautes Études Sci. Publ. Math. No. 47 (1977), 33–186. [121] Merel, Loïc. Bornes pour la torsion des courbes elliptiques sur les corps de nombres. Invent. Math. 124 (1996), no. 1-3, 437–449. [122] Tate writes: "Early in that summer [1965], Weil had told me of the idea that all elliptic curves over $\mathbb{Q}$ are modular [and that the conductor of the elliptic curve equals the conductor of the corresponding modular form]. That motivated Swinnerton-Dyer to make a big computer search for elliptic curves over $\mathbb{Q}$ with not too big discriminant, in order to test Weil's idea. But of course it was necessary to be able to compute the conductor to do that test. That was my main motivation."

where r is the rank of $E(\mathbb{Q})$, $Ш(E)$ is the Tate-Shafarevich group, R is the discriminant of the height pairing on $E(\mathbb{Q})$, Ω is the real period of E, and the $c_p = (E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p))$. When E has good ordinary or multiplicative reduction at p, there is a p-adic zeta function $L_p(s, E)$, and Mazur, Tate, and Teitelbaum (1986) investigated whether the behaviour of $L_p(s, E)$ near $s = 1$ is similarly related to the arithmetic invariants of E.[123] They found it is, but with one major surprise: there is an "exceptional" case in which $L_p(s, E)$ is related to an extended version of $E(\mathbb{Q})$ rather than $E(\mathbb{Q})$ itself. Supported by numerical evidence, they conjectured:

> BSD(p). When E has good ordinary or nonsplit multiplicative reduction at a prime p, the function $L_p(s, E)$ has a zero at $s = 1$ of order at least $r = \operatorname{rank} E(\mathbb{Q})$, and $L_p^{(r)}(1, E)$ is equal to a certain expression involving $|Ш(E)|$ and a p-adic regulator $R_p(E)$. When E has split multiplicative reduction, it is necessary to replace r with $r + 1$.

The L-function $L_p(s, E)$ is the p-adic Mellin transform of a p-adic measure obtained from modular symbols. The p-adic regulator is the discriminant of the canonical p-adic height pairing (augmented in the exceptional case). Much more is known about BSD(p) than the original conjecture of Birch and Swinnerton-Dyer. For example, Kato[124] has proved the following statement:

> The function $L_p(s, E)$ has a zero at $s = 1$ of order at least r (at least $r + 1$ in the exceptional case). When the order of the zero equals its conjectured value, then the p-primary component of $Ш(E)$ is finite and $R_p(E) \neq 0$.

In the exceptional case, $E_{\mathbb{Q}_p}$ is a Tate elliptic curve and $L_p(1, E) = 0$. On comparing their conjecture in this case with the original conjecture of Birch and Swinnerton-Dyer, the authors were led to the conjecture

$$L_p^{(1)}(1, E) = \frac{\log_p(q)}{\operatorname{ord}_p(q)} \frac{L(1, E)}{\Omega}$$

where q is the period of the Tate curve $E_{\mathbb{Q}_p}$ and Ω is the real period of E. This became known as the Mazur-Tate-Teitelbaum conjecture. It was proved by Greenberg and Stevens in 1993[125] for $p \neq 2, 3$, and by several authors in general.

Mazur and Tate (1987) state "refined" conjectures that avoid any mention of p-adic L-functions and, in particular, avoid the problem of constructing these functions. Let E be an elliptic curve over $\mathbb{Q}$. For a fixed integer $M > 0$, they use modular symbols to construct an element θ in the group algebra $\mathbb{Q}[(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\}]$. Let R be a subring of $\mathbb{Q}$ containing the coefficients of θ and such that the order the torsion subgroup of $E(\mathbb{Q})$ is invertible in R. The analogue of an L-function having a zero of order r at $s = 1$ is that θ lie in the rth power of the augmentation ideal I of the group algebra $R[(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\}]$. Assume that M is not divisible by p^2 for any prime p at which E has split multiplicative reduction. Then Mazur and Tate conjectured:

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[123] The authors assume that E is modular — at the time, it was not known that all elliptic curves over $\mathbb{Q}$ are modular. [124] Kato, Kazuya. p-adic Hodge theory and values of zeta functions of modular forms. Cohomologies p-adiques et applications arithmétiques. III. Astérisque No. 295 (2004), ix, 117–290. [125] Greenberg, Ralph; Stevens, Glenn, p-adic L-functions and p-adic periods of modular forms. Invent. Math. 111 (1993), no. 2, 407–447.

> Let r be the rank of $E(\mathbb{Q})$, and let r' be the number of primes dividing M at which E has split multiplicative reduction. Then $\theta \in I^{r+r'}$, and there is a formula (involving the order of Ш(E)) for the image of θ in $I^{r+r'}/I^{r+r'+1}$.

Again, the authors provide numerical evidence for their conjecture. Tate's student, Ki-Seng Tan, restated the Mazur-Tate conjecture for an elliptic curve over a global field, and he proved that part of the new conjecture is implied by the conjecture of Birch and Swinnerton-Dyer.[126]

In the first article discussed above, Mazur, Tate, Teitelbaum gave explicit formulas relating the canonical p-adic height pairings to a p-adic sigma function, and used the sigma function to study the height pairings. Mazur and Tate (1991), present a detailed construction of the p-adic sigma function for an elliptic curve E with good ordinary reduction over a p-adic field K, and they prove the properties used in the earlier article. In contrast to the classical sigma function, which is defined on the universal covering of E, the p-adic sigma function is defined on the formal group of E.

Finally, the article Mazur, Stein, and Tate 2006 studies the problem of efficiently computing of p-adic heights for an elliptic curve E over a global field K. This amounts to efficiently computing the sigma function, which in turn amounts to efficiently computing the p-adic modular form E_2.

6.5 *Jacobians of curves of genus one*

For a curve C of genus one over a field k, the Jacobian variety J of C is an elliptic curve over k. The problem is to compute a Weierstrass equation for J from an equation for C.

Weil (1954)[127] showed that, when C is defined by an equation $Y^2 = f(X)$, $\deg f = 4$, then the Weierstrass equation of J can be computed using the invariant theory of the quartic of f, which goes back to Hermite.

An et al. (2001)[128] showed how formulas from classical invariant theory give Weierstrass equations for J and for the map $C \to J$ when $\mathrm{char}(k) \neq 2, 3$ and C is a double cover of $\mathbb{P}^1$, a plane cubic, or a space quartic.

Tate and Rodrigues-Villegas found the Weierstrass equations over fields of characteristic 2 and 3, where the classical invariant theory no longer applies. Together with Artin they extended their result to an arbitrary base scheme S (Artin, Rodriguez-Villegas, Tate 2005). Specifically, let C be the family of curves over a scheme S defined, as a subscheme of $\mathbb{P}^2_S$, by a cubic $f \in \Gamma(S, \mathcal{O}_S)[X, Y, Z]$, and assume that the ten coefficients of f have no common zero. Then there is a Weierstrass equation

$$g\colon \quad Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3, \quad a_i \in \Gamma(S, \mathcal{O}_S),$$

whose coefficients are given explicitly in terms of the coefficients of f, such that the functor $\mathrm{Pic}^0_{C/S}$ is represented by the smooth locus of the subscheme of $\mathbb{P}^2_S$ defined by g. A key ingredient of the proof is a characterization, over sufficiently good base schemes, of the group algebraic spaces that can be described by a Weierstrass equation.

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[126] Tan, Ki-Seng, Refined theorems of the Birch and Swinnerton-Dyer type. Ann. Inst. Fourier (Grenoble) 45 (1995), no. 2, 317–374. [127] Weil, André. Remarques sur un mémoire d'Hermite. Arch. Math. (Basel) 5, (1954). 197–202. [128] An, Sang Yook; Kim, Seog Young; Marshall, David C.; Marshall, Susan H.; McCallum, William G.; Perlis, Alexander R. Jacobians of genus one curves. J. Number Theory 90 (2001), no. 2, 304–315.

6.6 Expositions

In 1961, Tate gave a series of lectures at Haverford College titled "Rational Points on Cubic Curves" intended for bright undergraduates in mathematics. Notes were taken of the lectures, and these were distributed in mimeographed form. The book, Silverman and Tate 1992, is a revision, and expansion, of the notes.

In the spring of 1960, the fall of 1967, 1975,..., Tate gave courses on the arithmetic of elliptic curves, whose informal notes have influenced later expositions.

7 The K-theory of number fields

7.1 K-groups and symbols

Grothendieck defined $K_0(X)$ for X a scheme in order to be able to state his generalization of the Riemann-Roch theorem. The topologists soon adapted Grothendieck's definition to topological spaces, and extended it to obtain groups K_n for all $n \in \mathbb{N}$.

For a commutative ring R, $K_0(R)$ is just the Grothendieck group of the category of finitely generated projective R-modules. In 1962, Bass and Schanuel[129] defined $K_1(R)$, and in 1967, Milnor[130] defined $K_2(R)$. In the early 1970s, several authors suggested definitions for the higher K-groups, which largely coincided when this could be checked. Quillen's definition[131] was the most flexible, and it is his that has been adopted.

The Steinberg group $\mathrm{ST}(R)$ of a ring R is the group with generators

$$x_{ij}(r), \quad i,j = 1,2,3,\ldots, \quad i \neq j, \quad r \in R$$

and relations

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$$

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & \text{if } i \neq l \text{ and } j \neq k \\ x_{il}(rs) & \text{if } i \neq l \text{ and } j = k. \end{cases}$$

The elementary matrices $E_{ij}(r) = I + re_{ij}$ in $\mathrm{GL}(R)$ satisfy these relations, and so there is a homomorphism $x_{ij}(r) \mapsto E_{ij}(r)\colon \mathrm{ST}(R) \to \mathrm{GL}(R)$. The groups $K_1(R)$ and $K_2(R)$ can be defined by the exact sequence

$$1 \to K_2(R) \to \mathrm{ST}(R) \to \mathrm{GL}(R) \to K_1(R) \to 1.$$

Let F be a field. A *symbol* on F with values in a commutative group C is defined to be a bimultiplicative map

$$(\ ,\ )\colon F^\times \times F^\times \to C$$

such that $(a, 1-a) = 1$ whenever $a \neq 0, 1$. Matsumoto (1969)[132] showed that the natural map

$$\{\ ,\ \}\colon F^\times \times F^\times \to K_2F$$

[129] Bass, H.; Schanuel, S. The homotopy theory of projective modules. Bull. Amer. Math. Soc. 68 1962 425–428. [130] During a course at Princeton University; published as: Milnor, John. Introduction to algebraic K-theory. Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J., 1971. [131] Quillen, Daniel. Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973. [132] Matsumoto, Hideya, Sur les sous-groupes arithmétiques des groupes semi-simples déployés. Ann. Sci. École Norm. Sup. (4) 2 1969 1–62. (1968 thesis ENS).

is a universal symbol, i.e., that $K_2(F)$ is the free abelian group on the pairs $\{a,b\}$, $a,b \in F^\times$, subject only to the relations

$$\begin{aligned} \{aa',b\} &= \{a,b\}\{a',b\} && \text{all } a,a',b \in F^\times \\ \{a,bb'\} &= \{a,b\}\{a,b'\} && \text{all } a,b,b' \in F^\times \\ \{a,1-a\} &= 1 && \text{all } a \neq 0,1 \text{ in } F^\times. \end{aligned}$$

EXAMPLES OF SYMBOLS

(a) The tame (Hilbert) symbol. Let v be a discrete valuation of F, with residue field $\kappa(v)$. Then
$$(a,b)_v = (-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}} \mod \mathfrak{m}_v$$
is a symbol on F with values in $\kappa(v)^\times$.

(b) The Galois symbol (Tate 1970b, §1). For m not divisible by $\mathrm{char}(F)$, $H^1(F,\mu_m) \simeq F^\times/F^{\times m}$, and the cup-product pairing
$$H^1(F,\mu_m) \times H^1(F,\mu_m) \to H^2(F,\mu_m \otimes \mu_m)$$
gives a symbol on F with values in $H^2(F,\mu_m \otimes \mu_m)$. When F contains the mth roots of 1,
$$H^2(F,\mu_m \otimes \mu_m) \simeq H^2(F,\mu_m) \otimes \mu_m \simeq \mathrm{Br}(F)_m \otimes \mu_m$$
and the symbol was known classically.

(c) The differential symbol (Tate ibid.). For $p = \mathrm{char}(F)$,
$$f,g \mapsto \frac{df}{f} \wedge \frac{dg}{g} : F^\times \times F^\times \to \Omega^2_{F/\mathbb{F}_p}$$
is a symbol.

(d) On $\mathbb{C}$ there are no continuous symbols, but on $\mathbb{R}$ there is the symbol
$$(a,b)_\infty = \begin{cases} 1 & \text{if } a>0 \text{ or } b>0 \\ -1 & \text{otherwise.} \end{cases}$$

7.2 The group K_2F for F a global field

Tate recognized that the study of the K_2 of a global field is related to classical objects in number fields, and sheds new light on them. He largely initiated the study of the K-groups of global fields and their rings of integers.

For a field F, $K_0F \simeq \mathbb{Z}$ is without particular interest. On the other hand, $K_1F \simeq F^\times$. For a number field, there is an exact sequence

$$0 \to U_F \to F^\times \xrightarrow{(\mathrm{ord}_v)} \bigoplus\nolimits_v \mathbb{Z} \to C_F \to 0$$

where v runs over the finite primes of F. Dirichlet proved that $U_F \approx \mu(F) \times \mathbb{Z}^{r_1+r_2-1}$, where r_1 and r_2 are the numbers of real and complex primes, and Dedekind proved that the class group C_F is finite. Thus understanding K_1F involves understanding the two basic objects in classical algebraic number theory.

Let F be a global field. For a noncomplex prime v of F, let $\mu_v = \mu(F_v)$ and let $m_v = |\mu_v|$. For a finite prime v of F, $\mathrm{Br}(F_v) \simeq \mathbb{Q}/\mathbb{Z}$, and so the Galois symbol with $m = m_v$ gives a homomorphism $\lambda_v : K_2F_v \to \mu_v$. Similarly, $(\ ,\ )_\infty$ gives a homomorphism $\lambda_v : K_2F_v \to \mu_v$ when v is real. The λ_v can be combined with the obvious maps $K_2F \to K_2F_v$ to give the homomorphism λ_F in the sequence

$$0 \to \mathrm{Ker}(\lambda_F) \to K_2F \xrightarrow{\lambda_F} \bigoplus\nolimits_v \mu_v \to \mu_F \to 0;$$

the direct sum is over the noncomplex primes of F, and the map from it sends $(x_v)_v$ to $\prod_v \frac{m_v}{m_F} x_v$ where $m_F = |\mu(F)|$. A product formula,

$$\prod (a,b)_v^{\frac{m_v}{m_F}} = 1$$

shows that the sequence is a complex, and Moore (1969)[133] showed that the cokernel of λ_F is μ_F. Thus, to compute K_2F, it remains to identify $\mathrm{Ker}(\lambda_F)$.

For $\mathbb{Q}$, Tate proved that $\mathrm{Ker}(\lambda_F)$ is trivial, and then observed that most of his argument was already contained in Gauss's first proof of the quadratic reciprocity law.

For a global field F, Bass and Tate proved that $\mathrm{Ker}(\lambda_F)$ is finitely generated, and that it is finite of order prime to the characteristic in the function field case. Garland (1971) proved that it is also finite in the number field case.

In the function field case, Tate proved that

$$|\mathrm{Ker}(\lambda_F)| = (q-1)\cdot(q^2-1)\cdot\zeta_F(-1). \tag{31}$$

For a number field, the Birch-Tate conjecture says that

$$|\mathrm{Ker}(\lambda_F)| = \pm w_2(F)\cdot\zeta_F(-1) \tag{32}$$

where $w_2(F)$ is the largest integer m such $\mathrm{Gal}(F^{\mathrm{al}}/F)$ acts trivially on $\mu_m(F^{\mathrm{al}}) \otimes \mu_m(F^{\mathrm{al}})$ (Birch 1971;[134] Tate 1970b). The odd part of this conjecture was proved by Wiles.[135]

When Quillen defined the higher K-groups, he proved that

$$K_2(\mathcal{O}_F) = \mathrm{Ker}\left(K_2(F) \to \bigoplus\nolimits_{v \text{ finite}} \mu_v\right)$$

and so there is an exact sequence

$$0 \to \mathrm{Ker}(\lambda_F) \to K_2(\mathcal{O}_F) \to \bigoplus\nolimits_{v \text{ real}} \mu_v.$$

Thus the computation of $\mathrm{Ker}(\lambda_F)$ is closely related to that of $K_2(\mathcal{O}_F)$. Lichtenbaum[136] generalized the Birch-Tate conjecture to the following statement: for all totally real number fields F[137]

$$\frac{|K_{4i-2}(\mathcal{O}_F)|}{|K_{4i-1}(\mathcal{O}_F)|} = |\zeta_F(1-2i)|, \quad \text{all } i \geq 1.$$

[133] Moore, Calvin C. Group extensions of p-adic and adelic linear groups. Inst. Hautes Études Sci. Publ. Math. No. 35 1968 157–222. [134] Birch, B. J. K_2 of global fields. 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), pp. 87–95. Amer. Math. Soc., Providence, R.I., 1971. [135] Wiles, A. The Iwasawa conjecture for totally real fields. Ann. of Math. (2) 131 (1990), no. 3, 493–540. [136] Lichtenbaum, Stephen. On the values of zeta and L-functions. I. Ann. of Math. (2) 96 (1972), 338–360. [137] The first test of the conjecture was for $F = \mathbb{Q}$ and $i = 1$. Since $\zeta_{\mathbb{Q}}(-1) = -1/12$ and $K_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, the conjecture predicts that $|K_3(\mathbb{Z})|$ has 24 elements, but Lee and Szczarba showed that it has 48 elements. When a seminar speaker at Harvard mentioned this, and scornfully concluded that the conjecture was false, Tate responded from the audience "Only for 2". In fact, Lichtenbaum's conjecture is believed to be correct up to a power of 2.

THE GALOIS SYMBOL

Tate proved (31) by using Galois symbols. For a global field F, he proved that the map

$$K_2F \to H^2(F, \mu_m^{\otimes 2}) \tag{33}$$

defined by the Galois symbol induces an isomorphism

$$K_2F/(K_2F)^m \simeq H^2(F, \mu_m^{\otimes 2}) \tag{34}$$

when m is not divisible by $\mathrm{char}(F)$, and wrote "I don't know whether ... this holds for all fields" Tate (1970b, p.208). Merkurjev and Suslin (1982)[138] proved that it does hold for all fields.

Tate noted that the isomorphism (34) gives little information on $\mathrm{Ker}(\lambda_F)$ because $\bigcap_m (K_2F)^m$ is a subgroup of $\mathrm{Ker}(\lambda_F)$ of index at most 2, and $\mathrm{Ker}(\lambda_F) \subset (K_2F)^m$ for all m not divisible by 8. He then defined more refined Galois symbols, which are faithful.

Fix a prime $\ell \neq \mathrm{char}(F)$, and let $\mathbb{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}(F^{\mathrm{al}}))$. This is a free $\mathbb{Z}_\ell$-module of rank 1 with an action of $\mathrm{Gal}(F^{\mathrm{al}}/F)$, and we let $H^r(F, \mathbb{Z}_\ell(1)^{\otimes 2})$ denote the Galois cohomology group defined using continuous cocycles (natural topology on both $\mathrm{Gal}(F^{\mathrm{al}}/F)$ and $\mathbb{Z}_\ell(1)^{\otimes 2}$). Tate proves that the maps (33) with $m = \ell^n$ lift to a map

$$K_2F \to H^2(F, \mathbb{Z}_\ell(1)^{\otimes 2}),$$

and that this map induces an isomorphism

$$K_2F(\ell) \to H^2(F, \mathbb{Z}_\ell(1)^{\otimes 2})_{\mathrm{tors}}.$$

As K_2F is torsion, with no $\mathrm{char}(F)$-torsion, this gives a purely cohomological description of K_2F.

NOTES. The results of Tate in this subsection were announced in Tate 1970b, and proved in Tate 1973b, 1976b, or in Tate's appendix to Bass and Tate 1973a.

7.3 The Milnor K-groups

Milnor (1970)[139] defines the (Milnor) K-groups of a field F as follows: regard $F^\times$ as a $\mathbb{Z}$-module; then $K_*^M F$ is the quotient of the tensor algebra of $F^\times$ by the ideal generated by the elements

$$a \otimes (1-a), \quad a \neq 0, 1.$$

This means that, for $n \geq 2$, $K_n^M F$ is the quotient of $(K_1F)^{\otimes n}$ by the subgroup generated by the elements

$$a_1 \otimes \cdots \otimes a_n, \quad a_i + a_{i+1} = 1 \text{ for some } i.$$

There is a canonical homomorphism $K_*F \to K_*^M F$ which induces isomorphisms $K_iF \to K_i^M F$ for $i \leq 2$. In the same article, Milnor defined for each discrete valuation v on F, a homomorphism

$$\partial_v : K_*F \to K_*\kappa(v)$$

[138] Merkurjev, A. S.; Suslin, A. A. K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 5, 1011–1046, 1135–1136. [139] Milnor, John. Algebraic K-theory and quadratic forms. Invent. Math. 9 1969/1970 318–344.

of degree -1, where $\kappa(v)$ is the residue field.

Milnor (ibid.) quotes a theorem of Tate: for a global field F,

$$K_n^M F/2K_n^M F \simeq \bigoplus\nolimits_v K_n^M F_v/2K_n^M F_v, \quad n \geq 3,$$

which implies that $K_n^M F/2K_n^M F \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1}$ $(n \geq 3)$ where r_1 is the number of real primes of F. Bass and Tate (1973a) improve this statement by showing that

$$K_n^M F \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1} \text{ for } n \geq 3.$$

The proof makes essential use of the "transfer maps"

$$\text{Tr}: K_* E \to K_* F,$$

defined whenever E is a finite field extension of F. Since these had only been defined for $n \leq 2$, a major part of the article is taken up with proving results on $K_* F$ for a general field, including the existence of the transfer maps.

The theorem of Bass and Tate completes the determination of the Milnor K-groups of a global field, except for K_1 and K_2.

7.4 Other results on K_2F

Let F be a field containing a primitive mth root ζ of 1 for some $m > 1$. Tate (1976) showed that, when F is a global field, every element of $K_2 F$ killed by m can be represented as $\{\zeta, a\}$ for some $a \in F^\times$. Tate (1977b) examines whether this holds for other fields and obtains a number of positive results.

For a finite extension of fields $E \supset F$, there is a transfer (or trace) map $\text{Tr}_{E/F}: K_2 E \to K_2 F$. As $K_2 E$ is generated by symbols $\{a, b\}$, in order to describe $\text{Tr}_{E/F}$ it suffices to describe its action on each symbol. This Rosset and Tate (1983c) do.

8 The Stark conjectures

In a series of four papers, Stark examined the behaviour of Artin L-series near $s = 0$ (equivalently, $s = 1$), and stated his now famous conjectures.[140] Tate gave a seminar at Harvard on Stark's conjectures in the spring of 1978, after Stark had given some talks on the subject in the number theory seminar in the fall of 1977. In 1980/81 Tate gave a course at the Université de Paris-Sud (Orsay) in which he clarified and extended Stark's work in important ways. The notes of Tate's course, when published in 1984, included most of the results known at that date, and became the basic reference for the Stark conjectures.

Let $\zeta_k(s)$ be the zeta function of a number field k. A celebrated theorem of Dedekind shows that

$$\zeta_k(s) \sim -\frac{R}{(e/h)}\, s^{r_1+r_2-1} \quad \text{as } s \to 0, \tag{35}$$

where h is the class number of k, R is its regulator, $e = |\mu(k)|$, and $r_1 + r_2 - 1$ is the rank of the group of units in k.

[140] Stark, H. M., Values of L-functions at $s = 1$. I. L-functions for quadratic forms. Advances in Math. 7 1971 301–343 (1971); II. Artin L-functions with rational characters. ibid. 17 (1975), no. 1, 60–92; III. Totally real fields and Hilbert's twelfth problem. ibid. 22 (1976), no. 1, 64–84; IV. First derivatives at $s = 0$. ibid. 35 (1980), no. 3, 197–235.

Let K be a finite Galois extension of k, with Galois group $G = G(K/k)$. Stark's insight was that the decomposition of $\zeta_K(s)$ into a product of Artin L-series indexed by the irreducible characters of G should induce an interesting decomposition of (35).

STARK'S MAIN CONJECTURE

Let $\chi: G \to \mathbb{C}$ be the character of a finite-dimensional complex representation $\rho: G \to \mathrm{GL}(V)$ of G. For a finite set S of primes of k containing the infinite primes, let

$$L(s,\chi) = \prod_{\mathfrak{p}\notin S} \frac{1}{\det(1-\rho(\sigma_\mathfrak{p})N\mathfrak{p}^{-s}\,|\,V^{I_\mathfrak{P}})}$$

(Artin L-function relative to S; cf. 1.1). Let S_K be the set of primes of K lying over a prime in S, let Y be the free $\mathbb{Z}$-module on S_K, and let X be the submodule of Y of elements $\sum n_w w$ such that $\sum n_w = 0$. Then $L(s,\chi)$ has a zero of multiplicity $r(\chi)$ at $s=0$, where

$$r(\chi) = \dim_{\mathbb{C}} \mathrm{Hom}_G(V^\vee, X_{\mathbb{C}}).$$

Let U be the group of S_K-units in K. The unit theorem provides us with an isomorphism

$$\lambda: U_{\mathbb{R}} \to X_{\mathbb{R}}, \quad u \mapsto \sum\nolimits_{w\in S_K} \log|u|_w\, w.$$

For each choice of an isomorphism of G-modules $f: X_{\mathbb{Q}} \to U_{\mathbb{Q}}$, Tate (1984, p.26) defines the Stark regulator, $R(\chi,f)$, to be the determinant of the endomorphism of $\mathrm{Hom}_G(V^\vee, X_{\mathbb{C}})$ induced by $\lambda_{\mathbb{C}} \circ f_{\mathbb{C}}$. Then

$$L(s,\chi) \quad\sim\quad \frac{R(\chi,f)}{A(\chi,f)}\, s^{r(\chi)} \qquad \text{as } s \to 0$$

for a complex number $A(\chi,f)$. The main conjecture of Stark, as formulated by Tate (1984, p.27), says that

$$A(\chi,f)^\alpha = A(\chi^\alpha, f) \text{ for all automorphisms } \alpha \text{ of } \mathbb{C},$$

where $\chi^\alpha = \alpha \circ \chi$. In particular, $A(\chi,f)$ is an algebraic number, lying in the cyclotomic field $\mathbb{Q}(\chi)$. Tate proves that the validity of the conjecture is independent of both f and S, and that it suffices to prove it for irreducible characters χ of dimension 1 (application of Brauer's theorem p.59).

CHARACTERS WITH VALUES IN $\mathbb{Q}$

When the character χ takes its values in $\mathbb{Q}$, Stark's conjecture predicts that $A(\chi,f) \in \mathbb{Q}$ for all f. If, in addition, χ is a $\mathbb{Z}$-linear combination of characters induced from trivial characters, then the proof of the conjecture comes down to the case of a trivial character, where it follows from (35). Some multiple of χ has this form, and so this shows that some power of $A(\chi,f)$ is in $\mathbb{Q}$ (Stark 1975). Tate proves (1984, Chapter II) that $A(\chi,f)$ itself lies in $\mathbb{Q}$. His proof makes heavy use of the cohomology of number fields, including the theorems in 1.3.

THE CASE THAT $L(s,\chi)$ IS NONZERO AT $s=0$

When $r(\chi)=0$, the Stark regulator $R(\chi,f)=1$, and Stark's conjecture becomes the statement:

$$L(0,\chi)^\alpha = L(0,\chi^\alpha) \text{ for all automorphisms } \alpha \text{ of } \mathbb{C}.$$

This special case of Stark's conjecture is also a special case of Deligne's conjecture on the critical values of motives (Deligne 1979, §6).[141] Using a refinement of Brauer's theorem (cf. p.59), Tate writes $L(s,\chi)$ as a sum of partial zeta functions:

$$L(s,\chi) = \sum_{\sigma\in G(K/k)} \chi(\sigma)\cdot\zeta(s,\sigma), \quad \zeta(s,\sigma) = \sum_{(\mathfrak{a},K/k)=\sigma} N\mathfrak{a}^{-s}$$

(Tate 1984, III 1). According to an important theorem of Siegel,[142] $\zeta(0,\sigma)\in\mathbb{Q}$, which proves Stark's conjecture in this case.

THE CASE THAT $L(s,\chi)$ HAS A FIRST ORDER ZERO AT $s=0$

By contrast, when $r(\chi)=1$, the conjecture is still unknown, but it has remarkable consequences. Let $\mathbb{C}[G]$ be the group algebra of G, and let

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma^{-1})\cdot\sigma$$

be the idempotent in $\mathbb{C}[G]$ that projects every representation of G onto its χ-component. For an $a\in\mathbb{Q}(\chi)$ and a character χ with $r(\chi)=1$, let

$$\pi(a,\chi) = \sum_{\alpha\in G(\mathbb{Q}(\chi)/\mathbb{Q})} a^\alpha\cdot L'(0,\chi^\alpha)\cdot e_{\overline{\chi}^\alpha} \in \mathbb{C}[G].$$

The character χ is realized on a $\mathbb{Q}[G]$-submodule U_W of $U_\mathbb{Q}$, and Stark's conjecture is true for χ if and only if

$$\pi(a,\chi)X_\mathbb{Q} = \lambda(U_W) \qquad \text{(inside } X_\mathbb{C}\text{)}.$$

(Tate 1984, III 2.1). More explicitly, let Ψ be a set of irreducible characters $\chi\neq 1$ of G such that $r(\chi)=1$, and assume that Ψ is stable under $\mathrm{Aut}(\mathbb{C})$. Let $(a_\chi)_{\chi\in\Psi}$ be a family of complex numbers such that $a_{\chi^\alpha}=(a_\chi)^\alpha$ for all $\alpha\in\mathrm{Aut}(\mathbb{C})$. If Stark's conjecture holds for the $\chi\in\Psi$, then for each prime v of S and extension of v to a prime w of K, there exists an integer $m>0$ and an S-unit ε of K such that

$$\lambda(\varepsilon) = m\sum_{\chi\in\Psi} a_\chi\cdot L'(0,\chi)\cdot e_{\overline{\chi}}\cdot w; \tag{36}$$

once m has been fixed, ε is unique up to a root of 1 in K (ibid. III §3). The units ε arising (conjecturally) in this way are called Stark units. They are analogous to the cyclotomic units in cyclotomic fields.

[141] Deligne, P. Valeurs de fonctions L et périodes d'intégrales. Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 313–346, Amer. Math. Soc., Providence, R.I., 1979. [142] Siegel, Carl Ludwig. Über die Fourierschen Koeffizienten von Modulformen. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1970 15–56.

FINER CONJECTURES WHEN K/k IS ABELIAN

When K/k is abelian, (36) can be made into a more precise form of Stark's conjecture, which Tate denotes $\mathrm{St}(K/k,S)$ (Stark 1980; Tate 1984, IV 2). For a real prime w of K and certain hypotheses on S, $\mathrm{St}(K/k,S)$ predicts the existence of a unit $\varepsilon(K,S,w)\in U$ such that

$$\varepsilon(K,S,w)^{\sigma}=\exp(-2\zeta'(0,\sigma)),\quad \text{all } \sigma\in G.$$

When we use w to embed K in $\mathbb{R}$, the $\varepsilon(K,S,w)$ lie in the abelian closure of k in $\mathbb{R}$. In the case that k is totally real, Tate (1984, 3.8) determines the subfield they generate; for example, when $[K:\mathbb{Q}]=2$, they generate the abelian closure of k in $\mathbb{R}$. This has implications for Hilbert's 12th problem. To paraphrase Tate (ibid. p.95):

> If the conjecture $\mathrm{St}(K/k,S)$ is true in this situation, then the formula
>
> $$\varepsilon=\exp(-2\zeta'(0,1))$$
>
> gives generators of abelian extensions of k that are special values of transcendental functions. Finding generators of class fields of this shape is the vague form of Hilbert's 12th problem, and the Stark conjecture represents an important contribution to this problem. However, it is a totally unexpected contribution: Hilbert asked that we discover the functions that play, for an arbitrary number field, the same role as the exponential function for $\mathbb{Q}$ and the elliptic modular functions for a quadratic imaginary field. In contrast, Stark's conjecture, by using L-functions directly, bypasses the transcendental functions that Hilbert asked for. Perhaps a knowledge of these last functions will be necessary for the proof of Stark's conjecture.

Remarkably, $\mathrm{St}(K/k,S)$ is useful for the explicit computation of class fields, and has even been incorporated into the computer algebra system PARI/GP.

For an abelian extension K/k, Tate introduced another conjecture, combining ideas of Brumer and Stark, and which he calls the Brumer-Stark conjecture. Let S be a set of primes of k including a finite prime $\mathfrak{p}$ that splits completely in K, and let $T=S\smallsetminus\{\mathfrak{p}\}$. Assume that T contains the infinite primes and the primes that ramify in K. Let

$$\theta_T(0)=\sum_{\chi\text{ irreducible}} L(0,\chi)e_{\overline{\chi}}\in\mathbb{C}[G].$$

Brumer conjectured that, for every ideal $\mathfrak{A}$ of K, $\mathfrak{A}^{e\theta_T(0)}$ is principal; the Brumer-Stark conjecture $\mathrm{BS}(K/k,T)$ says that $\mathfrak{A}^{e\theta_T(0)}=(\alpha)$ for an α satisfying certain conditions on the absolute values $|\alpha|_w$ ($w\in T$) and that $K(\alpha^{\frac{1}{e}})$ is an abelian extension of k (Tate 1984, 6.2).[143] Tate proved this conjecture for $k=\mathbb{Q}$ (ibid. 6.7) and for quadratic extensions K/k (Tate 1981b).

FUNCTION FIELDS

All of the conjectures make sense for a global field k of characteristic $p\neq 0$. In this case, the Artin L-series are rational functions in q^{-s}, where q is the order of the field of constants, and Stark's main conjecture follows easily from the known properties of these functions.

[143] Recall that $e=|\mu(k)|$.

However, as Mazur pointed out, the Brumer-Stark conjecture is far from trivial for function fields. Tate gave a seminar in Paris in early fall 1980 in which he discussed the conjecture and some partial results he had obtained. Deligne attended the seminar, and later gave a proof of the conjecture using his one-motives. This proof is included in Chapter V of Tate 1984.

p-ADIC ANALOGUES

Tate's reformulation of Stark's conjecture helped inspire two p-adic analogues of his main conjecture, one for $s = 0$ (Gross) and one for $s = 1$ (Serre) — the absence of a functional equation for the p-adic L-series makes these distinct conjectures. In a 1997 letter, Tate proposed a refinement of Gross's conjecture. This letter was published, with additional comments, as Tate 2004.

There is much numerical evidence for the Stark conjectures, found by Stark and others. As Tate (1981a, p.977) notes: "Taken all together, the evidence for the conjectures seems to me to be overwhelming".

9 Noncommutative ring theory

The Tate conjecture for divisors on a variety is related to the finiteness of the Brauer group of the variety, which is defined to be the set of the similarity classes of sheaves of (noncommutative) Azumaya algebras on the variety. This connection led M. Artin to an interest in noncommutative rings, which soon broadened beyond Azumaya algebras. Tate wrote a number of articles on noncommutative rings in collaboration with Artin and others.

9.1 Regular algebras

A ring A is said to have *finite global dimension* if there exists an integer d such that every A-module has a projective resolution of length at most d. The smallest such d is then called the *global dimension* of A. Serre[144] showed that a *commutative* ring is noetherian of finite global dimension if and only if it is regular.

Let k be a field. Artin and Schelter (1987)[145] defined a finitely generated k-algebra to be *regular* if it is of the form

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots, \tag{37}$$

and

(a) A has finite global dimension (defined in terms of *graded* A-modules),
(b) A has polynomial growth (i.e., $\dim A_n$ is bounded by a polynomial function in n), and
(c) A is Gorenstein (i.e., the k-vector space $\mathrm{Ext}^i_A(k, A)$ has dimension 1 when i is the global dimension of A, and is zero otherwise).

The only commutative graded k-algebras satisfying these conditions and generated in degree 1 are the polynomial rings. It is expected that the regular algebras have many of the good properties of polynomial rings. For example, Artin and Schelter conjecture that they are noetherian domains. The dimension of a regular algebra is its global dimension.

[144] Serre, Jean-Pierre, Algèbre locale. Multiplicités. Seconde édition, 1965. Lecture Notes in Mathematics, 11 Springer-Verlag, Berlin-New York 1965 [145] Artin, Michael; Schelter, William F.; Graded algebras of global dimension 3. Adv. in Math. 66 (1987), no. 2, 171–216.

In collaboration with Artin and others, Tate studied regular algebras, especially the classification of those of low dimension.

From now on, we require regular k-algebras to be generated in degree 1. Such an algebra is a quotient of a tensor algebra by a homogeneous ideal.

A regular k-algebra of dimension one is a polynomial ring, and one of dimension two is the quotient of the free associative algebra $k\langle X, Y\rangle$ by a single quadratic relation, which can be taken to be $XY - cYX$ ($c \neq 0$) or $XY - YX - Y^2$. Thus, the first interesting dimension is three. Artin and Schelter (ibid.) showed that a regular k-algebra of dimension three either has three generators and three relations of degree two, or two generators and two relations of degree three. Moreover, they showed that the algebras fall into thirteen families. While the generic members of each family are regular, they were unable to show that all the algebras in the families are regular. Artin, Tate, Van den Bergh (1990a) overcame this problem, and consequently gave a complete classification of these algebras. Having found an explicit description of all the algebras, they were able to show that they are all noetherian.

These two articles introduced new geometric techniques into noncommutative ring theory. They showed that the regular algebras of dimension 3 correspond to certain triples $(E, \mathcal{L}, \sigma)$ where E is a one-dimensional scheme of arithmetic genus 1 which is embedded either as a cubic divisor in $\mathbb{P}^2$ or as a divisor of bidegree $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}_E(1)$ is an invertible sheaf on E, and σ is an automorphism of E. The scheme E parametrizes the point modules for A, i.e., the graded cyclic right A-modules, generated in degree zero, such that $\dim_k(M_n) = 1$ for all $n \geq 0$. The geometry of (E, σ) is reflected in the structure of the point modules, and Artin, Tate, van den Bergh (1991a) exploit this relation to prove that the 3-dimensional regular algebra corresponding to a triple $(E, \mathcal{L}, \sigma)$ is finite over its centre if and only if the automorphism σ has finite order. They also show that noetherian regular k-algebras of dimension ≤ 4 are domains.

9.2 *Quantum groups*

A bialgebra A over a field k is a k-module equipped with compatible structures of an associative algebra with identity and of a coassociative coalgebra with coidentity. A bialgebra is called a Hopf algebra if it admits an antipodal map (linear map $S: A \to A$ such that certain diagrams commute).

A bialgebra is said to be commutative if it is commutative as a k-algebra. The commutative bialgebras (resp. Hopf algebras) over k are exactly the coordinate rings of affine monoid schemes (resp. affine group schemes) over k.

Certain Hopf algebras (not necessarily commutative) are called quantum groups. For example, there is a standard one-parameter family $\mathcal{O}(\mathrm{GL}_n(q))$, $q \in k^\times$, of Hopf algebras that takes the value $\mathcal{O}(\mathrm{GL}_n)$ for $q = 1$. This can be regarded as a one-parameter deformation of $\mathcal{O}(\mathrm{GL}_n)$ by Hopf algebras, or of GL_n by quantum groups.

Artin, Schelter, and Tate (1991b) construct a family of deformations of $\mathcal{O}(\mathrm{GL}_n)$, depending on $1 + \binom{n}{2}$ parameters, which includes the family $\mathcal{O}(\mathrm{GL}_n(q))$. The algebras in the family are all twists of $\mathcal{O}(\mathrm{GL}_n(q))$ by 2-cocycles. They first construct a family of deformations of $\mathcal{O}(M_n)$ by bialgebras that are graded algebras generated in degree 1, have the same Hilbert series as the polynomial ring in n^2 variables, and are noetherian domains. The family of deformations of $\mathcal{O}(\mathrm{GL}_n)$ is then obtained by inverting the quantum determinant. The algebras in the family of deformations of $\mathcal{O}(M_n)$ are regular in the sense of (9.1), and so this gives a large class of regular algebras with the expected good properties.

9.3 *Sklyanin algebras*

As noted in (9.1), regular algebras of degree 3 over a field k correspond to certain triples $(E,\mathcal{L},\sigma)$ with E a curve, $\mathcal{L}$ an invertible sheaf of degree 3 on E, and σ an automorphism of E. When E is a nonsingular elliptic curve and σ is translation by a point P in $E(k)$, the algebra $A(E,\mathcal{L},\sigma)$ is called a *Sklyanin algebra*. Let $U=\Gamma(E,\mathcal{L})$. This is a 3-dimensional k-vector space, and we can identify $U\otimes U$ with $\Gamma(E,\mathcal{L}\boxtimes\mathcal{L})$. The algebra $A(E,\mathcal{L},\sigma)$ is the quotient of the tensor algebra $T(U)$ of U by $\{f\in U\otimes U \mid f(x,x+P)=0\}$. It is essentially independent of $\mathcal{L}$, because any two invertible sheaves of degree 3 differ by a translation. More generally, there is Sklyanin algebra $A(E,\mathcal{L},\sigma)$ for every triple consisting of a nonsingular elliptic curve, an invertible sheaf $\mathcal{L}$ of degree d on E, and a translation by a point in $E(k)$. The algebra $A(E,\mathcal{L},\sigma)$ has dimension d.

Artin, Schelter, and Tate (1994c) give a precise description of the centres of Sklyanin algebras of dimension three, and Smith and Tate (1994d) extend the description to those of dimension four.

Tate and van den Bergh (1996) prove that every d-dimensional Sklyanin algebra $A(E,\mathcal{L},\sigma)$ is a noetherian domain, is Koszul, has the same Hilbert series as a polynomial ring in d variables, and is regular in the sense of (9.1); moreover, if σ has finite order, then $A(E,\mathcal{L},\sigma)$ is finite over its centre.

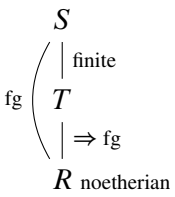
10 Miscellaneous articles

1951a TATE, JOHN. ON THE RELATION BETWEEN EXTREMAL POINTS OF CONVEX SETS AND HOMOMORPHISMS OF ALGEBRAS. COMM. PURE APPL. MATH. 4, (1951). 31–32.

Tate considers a convex set K of linear functionals on a commutative algebra A over $\mathbb{R}$. Under certain hypotheses on A and K, he proves that the extremal points of K are exactly the homomorphisms from A into $\mathbb{R}$.

1951b ARTIN, EMIL; TATE, JOHN T. A NOTE ON FINITE RING EXTENSIONS. J. MATH. SOC. JAPAN 3, (1951). 74–77.

Artin and Tate prove that if S is a commutative finitely generated algebra over a noetherian ring R, and T is a subalgebra of S such that S is finitely generated as a T-module, then T is also finitely generated over R. This statement generalizes a lemma of Zariski, and is now known as the Artin-Tate lemma. There are various generalizations of it to noncommutative rings.



1952a TATE, JOHN. GENUS CHANGE IN INSEPARABLE EXTENSIONS OF FUNCTION FIELDS. PROC. AMER. MATH. SOC. 3, (1952). 400–406.

Let C be a complete normal geometrically integral curve over a field k of characteristic p, and let C' be the curve obtained from C by an extension of the base field $k\to k'$. If k' is inseparable over k, then C' need not be normal, and its normalization $\tilde{C}'$ may have genus $g(\tilde{C}')$ less than the genus $g(C)$ of C. However, Tate proves that

$$\frac{p-1}{2} \text{ divides } g(C)-g(\tilde{C}'). \tag{38}$$

In particular, the genus of C can't change if $g(C) < (p-1)/2$ (which implies that C is smooth in this case).

Statement (38) is widely used. Tate derives it from a "Riemann-Hurwitz formula" for purely inseparable coverings, which he proves using the methods of the day (function fields and repartitions). A modern proof has been given of (38)[146], but not, as far as I know, of the more general formula.

1952b LANG, SERGE; TATE, JOHN. ON CHEVALLEY'S PROOF OF LUROTH'S THEOREM. PROC. AMER. MATH. SOC. 3, (1952). 621–624.

Chevalley (1951, p. 106)[147] proved Lüroth's theorem in the following form: let k_0 be a field, and let $k = k_0(X)$ be the field of rational functions in the symbol X (i.e., k is the field of fractions of the polynomial ring $k_0[X]$); then every intermediate field k', $k_0 \subsetneqq k' \subset k$, is of the form $k_0(f)$ for some $f \in k$.

A classical proof of Lüroth's theorem uses the Riemann-Hurwitz formula. Let k be a function field in one variable over a field k_0, and let k' be an intermediate field; the Riemann-Hurwitz formula shows that, if k/k' is separable, then

$$g(k') \leq g(k).$$

Therefore, if k has genus zero, so also does k'; if, in addition, k has a prime of degree 1, so also does k', and so k' is a rational field (by a well-known criterion).

However, if k/k' is not separable, it may happen that $g(k') > g(k)$. Chevalley proved Lüroth's theorem in nonzero characteristic by showing directly that, when $k = k_0(X)$, every intermediate field k has genus zero. Lang and Tate generalized Chevalley's argument to prove:

> Let k be a function field in one variable over a field k_0, and let k' be an intermediate field; if k is separably generated over k_0, then $g(k') \leq g(k)$.

In other words, they showed that Chevalley's argument doesn't require that $k = k_0(X)$ but only that it be separably generated over k_0. They also prove a converse statement:

> A field of genus zero that is not separably generated over its field of constants contains subfields of arbitrarily high genus.

Finally, to complete their results, they exhibit a field of genus zero, not separably generated over its field of constants.

1955b BRAUER, RICHARD; TATE, JOHN. ON THE CHARACTERS OF FINITE GROUPS. ANN. OF MATH. (2) 62, (1955). 1–7.

Recall (p.5) that Brauer's theorem says that every character χ of a finite group G can be expressed in the form

$$\chi = \sum\nolimits_i n_i \operatorname{Ind} \chi_i, \quad n_i \in \mathbb{Z},$$

[146] Schröer, Stefan, On genus change in algebraic curves over imperfect fields. Proc. Amer. Math. Soc. 137 (2009), no. 4, 1239–1243. [147] Chevalley, Claude. Introduction to the Theory of Algebraic Functions of One Variable. Mathematical Surveys, No. VI. American Mathematical Society, New York, N. Y., 1951.

with the χ_i *one-dimensional* characters on subgroups of G (as conjectured by Artin). Brauer and Tate found what is probably the simplest known proof of Brauer's theorem. Recall that a group is said to be elementary if it can be expressed as the product of a cyclic group with a p-group for some prime p. An elementary group is nilpotent, and so every irreducible character of it is induced from a one-dimensional character on a subgroup. Let G be a finite group, and let $\mathcal{H}$ be a set of subgroups of G. Consider the following three $\mathbb{Z}$-submodules of the space of complex-valued functions on G:

$$\begin{aligned} X(G) &= \text{span}\{\text{irreducible characters of } G\} \qquad \text{(module of virtual characters)} \\ Y &= \text{span}\{\text{characters of } G \text{ induced from an irreducible character of an } H \text{ in } \mathcal{H}\} \\ U &= \{\text{class functions } \chi \text{ on } G \text{ such that } \chi|H \in X(H) \text{ for all } H \text{ in } \mathcal{H}\}. \end{aligned}$$

Brauer and Tate show that

$$U \supset X(G) \supset Y,$$

that U is a ring, and that Y is an ideal of U. Using this, they show that if $\mathcal{H}$ consists of the elementary subgroups of G, then $U = Y$, thereby elegantly proving not only Artin's conjecture (the equality $X(G) = Y$), but also the main theorem of Brauer 1953[148] (the equality $U = X(G)$).

1957 TATE, JOHN. HOMOLOGY OF NOETHERIAN RINGS AND LOCAL RINGS. ILLINOIS J. MATH. 1 (1957), 14–27.

Tate makes systematic use of the skew-commutative graded differential algebras over a noetherian commutative ring R to obtain results concerning R and its quotient rings. The differential of such an R-algebra allows it to be regarded as a complex, and Tate proves that every quotient $R/\mathfrak{a}$ of R has a free resolution that is an R-algebra (in the above sense). Such resolutions are now called Tate resolutions.

Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$. The Betti series of R is defined to be the formal power series $\mathcal{R} = \sum_{r\geq 0} b_r Z^r$ with b_r equal to the length of $\operatorname{Tor}_r^R(R/\mathfrak{m}, R/\mathfrak{m})$. Serre (1956)[149] showed that R is regular if and only if $\mathcal{R}$ is a polynomial, in which case $\mathcal{R} = (1+Z)^d$ with $d = \dim(R)$. Tate showed that $\mathcal{R} = (1+Z)^d/(1-Z^2)^{b_1-d}$ if R is a complete intersection. In general, he showed that the natural homomorphism

$$\bigwedge\nolimits^* \operatorname{Tor}_1^R(R/\mathfrak{m}, R/\mathfrak{m}) \to \operatorname{Tor}^R(R/\mathfrak{m}, R/\mathfrak{m})$$

is injective and realizes $\operatorname{Tor}^R(R/\mathfrak{m}, R/\mathfrak{m})$ as a free module over $\bigwedge^* \operatorname{Tor}_1^R(R/\mathfrak{m}, R/\mathfrak{m})$ with a homogeneous basis. If R is regular, then the homomorphism is an isomorphism; conversely, if the homomorphism is an isomorphism on one homogeneous component of degree ≥ 2, then R is regular.

1962a FRÖHLICH, A.; SERRE, J.-P.; TATE, J. A DIFFERENT WITH AN ODD CLASS. J. REINE ANGEW. MATH. 209 1962 6–7.

Let A be a Dedekind domain with field of fractions K, and let B be the integral closure of A in a finite separable extension of K. The different $\mathfrak{D}$ of B/A is an ideal in B, and its norm $\mathfrak{d}$

[148] Brauer, Richard, A characterization of the characters of groups of finite order. Ann. of Math. (2) 57, (1953). 357–377. [149] Serre, Jean-Pierre, Sur la dimension homologique des anneaux et des modules noethériens. Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 175–189. Science Council of Japan, Tokyo, 1956.

is the discriminant ideal of B/A. The ideal class of $\mathfrak{d}$ is always a square, and Hecke (1954, §63)[150] proved that the ideal class of $\mathfrak{D}$ is a square when K is a number field, but the authors show that it need not be a square otherwise. Specifically, they construct examples of affine curves over perfect fields whose coordinate rings A have extensions B for which the ideal class of the different is not a square.[151]

This is not a major result. However, Martin Taylor[152] writes:

> [This article and Fröhlich's earlier work on discriminants] seems to have marked the start of [his] interest in parity questions. He would go on to be interested in whether conductors of real-valued characters were squares; this in turn led to questions about the signs of Artin root numbers — an issue that lay right at the heart of his work on Galois modules.

Fröhlich's work on Artin root numbers and Galois module structures was his most important.

1963 SEN, SHANKAR; TATE, JOHN. RAMIFICATION GROUPS OF LOCAL FIELDS. J. INDIAN MATH. SOC. (N.S.) 27 1963 197–202 (1964).

Let F be a field, complete with respect to a discrete valuation, and let K be a finite Galois extension of F. Assume initially that the residue field is finite, and let W be the Weil group of K/F (extension of $G(K/F)$ by $K^\times$ determined by the fundamental class of K/F). Shafarevich showed that there is a homomorphism s making the following diagram commute

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times & \longrightarrow & W & \longrightarrow & G(K/F) & \longrightarrow & 1 \\ & & \downarrow r & & \downarrow s & & \| & & \\ 1 & \longrightarrow & G(K^{\mathrm{ab}}/K) & \longrightarrow & G(K^{\mathrm{ab}}/F) & \longrightarrow & G(K/F) & \longrightarrow & 1, \end{array}$$

where r is the reciprocity map. For a real $t > 0$, let $G(K^{\mathrm{ab}}/K)^t$ denote the tth ramification subgroup of $G(K^{\mathrm{ab}}/K)$. Then

$$r^{-1}(G(K^{\mathrm{ab}}/K)^t) = U_K^t \overset{\text{def}}{=} \begin{cases} \{u \in K^\times \mid \mathrm{ord}_K(u) = 0\} \text{ if } t = 0 \\ \{u \in K^\times \mid \mathrm{ord}_K(u-1) \geq t\} \text{ if } t > 0. \end{cases} \tag{39}$$

Artin and Tate (1961) proved the existence of the Shafarevich map s for a general class formation. When the residue field of F is algebraically closed, the groups $\pi_1(U_K)$ (fundamental group of U_K regarded as a pro-algebraic group) form a class formation, and so the above diagram exists with $K^\times$ replaced by $\pi_1(U_K)$. In this case,

$$r^{-1}(G(K^{\mathrm{ab}}/K)^t) = \pi_1(U_K^t). \tag{40}$$

In both cases, Sen and Tate give a description of the subgroups $s^{-1}(G(K^{\mathrm{ab}}/F)^t)$ of W generalizing those in (39) and (40), which can be considered the case $K = F$. Specifically, let $G(K/F)_x$, $x \geq 0$, denote with ramification groups of K/F with the lower numbering, and let

$$\varphi(x) = \int_0^x \frac{du}{(G(K/F)_0 : G(K/F)_u)} \text{ for } x \geq 0.$$

[150] Hecke, Erich, Vorlesungen über die Theorie der algebraischen Zahlen. 2te Aufl. (German) Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1954. [151] Hecke's theorem can be proved for global fields of characteristic $p \neq 0$ by methods similar to those of Hecke (Armitage, J. V., On a theorem of Hecke in number fields and function fields. Invent. Math. 2 1967 238–246). [152] Taylor, M. J. Obituary: Albrecht Fröhlich, 1916–2001. Bull. London Math. Soc. 38 (2006), no. 2, 329–350.

For $u \in W$, let $m(u) > 0$ be the smallest integer such that $u^{m(u)} \in K^\times$ (resp. $\pi_1(U_K)$). Then

$$W^{\varphi(x)} = \left\{ u \in W \,\middle|\, u^{m(u)} \in U_K^{m(u)\cdot x} \text{ (resp. } \pi_1(U_K^{m(u)\cdot x})) \right\}.$$

1964c TATE, JOHN. NILPOTENT QUOTIENT GROUPS. TOPOLOGY 3 1964 SUPPL. 1 109–111.

For a finite group G, subgroup S, and positive integer p, there are restriction maps r and transfer maps t,

$$H^i(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{r^i} H^i(S, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{t^i} H^i(G, \mathbb{Z}/p\mathbb{Z}), \quad i \geq 0,$$

whose composite is multiplication by $(G\colon S)$.

Let S be Sylow p-subgroup of G (so p is prime). If S has a normal p-complement in G, then the restriction maps are isomorphisms, and Atiyah asked whether the converse is true. Thompson pointed out that the answer is yes, and that results of his and Huppert show that one need only require that r^1 is an isomorphism. Tate gives a very short cohomological proof of a somewhat stronger result.

Specifically, for a finite group G, define a descending sequence of normal subgroups of G as follows:

$$G_0 = G, \quad G_{n+1} = (G_n)^p\,[G, G_n] \text{ for } n \geq 0, \quad G_\infty = \bigcap\nolimits_{n=0}^{\infty} G_n$$

(p not necessarily prime). Thus, G/G_1 (resp. G/G_∞) is the largest quotient group of G that is abelian of exponent p (resp. nilpotent and p-primary). Let S be a subgroup of G of index prime to p. The following three conditions are (obviously) equivalent,

- ◇ the restriction map $r^1\colon H^1(G, \mathbb{Z}/p\mathbb{Z}) \to H^1(S, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism,
- ◇ the map $S/S_1 \to G/G_1$ is an isomorphism,
- ◇ $S \cap G_1 = S_1$,

and Tate proves that they imply

- ◇ $S \cap G_n = S_n$ for all $1 \leq n \leq \infty$.

When S is a Sylow p-subgroup of G, $S \cap G_\infty = 1$, and so the conditions imply that G_∞ is a normal p-complement of S in G (thereby recovering the Huppert-Thompson theorem).

1968b TATE, JOHN. RESIDUES OF DIFFERENTIALS ON CURVES. ANN. SCI. ÉCOLE NORM. SUP. (4) 1 1968 149–159.

Tate defines the residues of differentials on curves as the traces of certain "finite potent" linear maps. From his definition, all the standard theorems on residues follow naturally and easily. In particular, the residue formula

$$\sum\nolimits_{P \in C} \operatorname{res}_P(\omega) = 0 \qquad (C \text{ a complete curve})$$

follows directly, without computation, from the finite dimensionality of the cohomology groups $H^0(C, \mathcal{O}_C)$ and $H^1(C, \mathcal{O}_C)$ "almost as though one had an abstract Stokes's Theorem available".

A linear map $\theta\colon V \to V$ is finite potent if $\theta^n V$ is finite dimensional for some n. The trace $\operatorname{Tr}_V(\theta)$ of such a map can be defined to be its trace on any finite dimensional subspace W of V such that $\theta W \subset W$ and $\theta^n V \subset W$ for some n. Many of the properties of the usual

trace continue to hold, but not all. For example, linearity fails even for two finite potent operators on an infinite-dimensional vector space.[153] Tate defines the residue of a differential $f\,dg$ at a closed point p of a curve C to be the trace of the commutator $[f_p, g_p]$, where f_p, g_p are representatives of f, g in a certain subspace of $\mathrm{End}(k(C)_p)$.

Tate's approach to residues has found its way into the text books (e.g., Iwasawa 1993[154]). Elzein[155] used Tate's ideas to give a definition of the residue that recaptures both Leray's in the case of a complex algebraic variety and Grothendieck's in the case of a smooth integral morphisms of relative dimension n.

Others have adapted his proof of the residue formula to other situations; for example, Arbarello et al (1989)[156] use it to prove an "abstract reciprocity law" for tame symbols on a curve over an algebraically closed field, and Beilinson et al (2002)[157] use it to prove a product formula for ε-factors in the de Rham setting.

In reading Tate's article, Beilinson recognized that a certain linear algebra construction there can be reformulated as the construction of a canonical central extension of Lie algebras. This led to the notion of a Tate extension in various settings; see Beilinson and Schechtman 1988,[158] and Beilinson and Drinfeld 2004, 2.7.[159]

1978a CARTIER, P.; TATE, J. A SIMPLE PROOF OF THE MAIN THEOREM OF ELIMINATION THEORY IN ALGEBRAIC GEOMETRY. ENSEIGN. MATH. (2) 24 (1978), NO. 3-4, 311–317.

The authors give an elementary one-page proof of the homogeneous form of Hilbert's theorem of zeros:

> let $\mathfrak{a}$ be a graded ideal in a polynomial ring $k[X_0, \dots, X_n]$ over a field k; either the radical of $\mathfrak{a}$ contains the ideal $(X_0, \dots, X_n)$, or $\mathfrak{a}$ has a nontrivial zero in an algebraic closure of k.

From this, they quickly deduce the main theorem of elimination theory, both in its classical form and in its modern form:

> let $A = \bigoplus_{d \geq 0} A_d$ be a graded commutative algebra such that A is generated as an A_0-algebra by A_1 and each A_0-module A_d is finitely generated; then the map of topological spaces $\mathrm{proj}(A) \to \mathrm{spec}(A_0)$ is closed.

[153] Julia Ramos Gonzalez and Fernando Pablos Romo, A Negative Answer to the Question Of the Linearity of Tate's Trace for the Sum of Two Endomorphisms (preprint 2012). [154] Iwasawa, Kenkichi Algebraic functions. Translated from the 1973 Japanese edition by Goro Kato. Translations of Mathematical Monographs, 118. American Mathematical Society, Providence, RI, 1993. [155] Elzein, Fouad, Résidus en géométrie algébrique. C. R. Acad. Sci. Paris Sér. A-B 272 (1971), A878–A881. [156] Arbarello, E.; De Concini, C.; Kac, V. G. The infinite wedge representation and the reciprocity law for algebraic curves. Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), 171–190, Proc. Sympos. Pure Math., 49, Part 1, Amer. Math. Soc., Providence, RI, 1989. [157] Beilinson, Alexander; Bloch, Spencer; Esnault, Hélène, ε-factors for Gauss-Manin determinants. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. Mosc. Math. J. 2 (2002), no. 3, 477–532. [158] Beĭlinson, A. A.; Schechtman, V. V. Determinant bundles and Virasoro algebras. Comm. Math. Phys. 118 (1988), no. 4, 651–701. [159] Beilinson, Alexander; Drinfeld, Vladimir. Chiral algebras. American Mathematical Society Colloquium Publications, 51. American Mathematical Society, Providence, RI, 2004.

1989 GROSS, B.; TATE, J. COMMENTARY ON ALGEBRA. A CENTURY OF MATHEMATICS IN AMERICA, PART II, 335–336, HIST. MATH., 2, AMER. MATH. SOC., PROVIDENCE, RI, 1989.

For the bicentenary of Princeton University in 1946, there was a three-day conference in which various distinguished mathematicians discussed Problems in Mathematics. Artin, Brauer, and others contributed to the discussion on algebra, and in 1989 Gross and Tate wrote a commentary on their remarks. For example:

> Artin's belief that "whatever can be said about non-Abelian class field theory follows from what we know now," and that "our difficulty is not in the proofs, but in learning what to prove," seems overly optimistic.

1994b TATE, JOHN. THE NON-EXISTENCE OF CERTAIN GALOIS EXTENSIONS OF $\mathbb{Q}$ UNRAMIFIED OUTSIDE 2. ARITHMETIC GEOMETRY (TEMPE, AZ, 1993), 153–156, CONTEMP. MATH., 174, AMER. MATH. SOC., PROVIDENCE, RI, 1994.

In a 1973 letter to Tate, Serre suggested that certain two-dimensional mod p representations of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ should be modular. In response, Tate verified this for $p = 2$ by showing that every two-dimensional mod 2 representation unramified outside 2 has zero trace. The article is based on his letter.

Serre's suggestion became Serre's conjecture on the modularity of two-dimensional mod p representations, which attracted much attention because of its relation to the modularity conjecture for elliptic curves over $\mathbb{Q}$ and Fermat's last theorem. Serre's conjecture was recently proved by an inductive argument that uses Tate's result as one of the base cases.[160]

1996a TATE, JOHN; VOLOCH, JOSÉ FELIPE . LINEAR FORMS IN p-ADIC ROOTS OF UNITY. INTERNAT. MATH. RES. NOTICES 1996, NO. 12, 589–601.

The authors make the following conjecture: for a semi-abelian variety A over $\mathbb{C}_p$ and a closed subvariety X, there exists a lower bound $c > 0$ for the p-adic distance of torsion points of A, not in X, to X. Here, as usual, $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. They prove the conjecture for the torus

$$A = \mathrm{Spec}\,\mathbb{C}_p[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}].$$

This comes down to proving the following explicit statement: for every hyperplane

$$a_1 T_1 + \cdots + a_n T_n = 0$$

in $\mathbb{C}_p^n$, there exists a constant $c > 0$, depending on $(a_1, \ldots, a_n)$, such that, for any n-tuple $\zeta_1, \ldots, \zeta_n$ of roots of 1 in $\mathbb{C}_p$, either $a_1\zeta_1 + \cdots + a_1\zeta_n = 0$ or $|a_1\zeta_1 + \cdots + a_1\zeta_n| \geq c$.

2002a TATE, JOHN. ON A CONJECTURE OF FINOTTI. BULL. BRAZ. MATH. SOC. (N.S.) 33 (2002), NO. 2, 225–229.

In his study of the Teichmüller points in canonical lifts of elliptic curves, Finotti was led to a conjecture on remainders of division by polynomials.[161] He checked it by computer for all primes $p \leq 877$, and Tate proved it in general. The statement is:

[160] Khare, Chandrashekhar; Wintenberger, Jean-Pierre. Serre's modularity conjecture. I. Invent. Math. 178 (2009), no. 3, 485–504. II. Ibid. 505–586. [161] L. R. A. Finotti, Canonical and minimal degree liftings of curves, J. Math. Sci. Univ. Tokyo 11 (2004), no. 1, 1–47 (Ph.D. thesis, Univ. Texas, 2001).

Let k be a field of characteristic $p = 2m + 1 \geq 5$. Let $F \in k[X]$ be a monic cubic polynomial, and let A be the coefficient of X^{p-1} in F^m. Let $G \in k[X]$ be a polynomial of degree $3m + 1$ such that $G' = F^m - AX^{p-1}$. Then the remainder in the division of G^2 by $X^p F^{m+1}$ has degree $\leq 5m + 2 = \frac{5p-1}{2}$.

Acknowledgements

I thank B. Gross for help with dates, J-P. Serre for correcting a misstatement, and J. Tate for answering my queries and pointing out some mistakes.

Added September 2012

I should have mentioned the work of Tate on liftings of Galois representations, as included in Part II of: Serre, J.-P. Modular forms of weight one and Galois representations. Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 193–268. Academic Press, London, 1977. See also: Variations on a theorem of Tate. Stefan Patrikis. arXiv:1207.6724.

Also, "An oft cited (1979) letter from Tate to Serre on computing local heights on elliptic curves." was posted on the arXiv by Silverman (arXiv:1207.5765).

The collected works of Tate, which will include other unpublished letters, is in preparation.

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[162] Lang calls his Chapter VII an "unpublished article of Tate", but gives no date. In his MR review, Shatz writes that "It appears here in almost the same form the reviewer remembers from the original seminar of Tate in 1958. "

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[163] The original notes don't give a date or a publisher. I copied this information from the footnote p. 162 of 1967a. The volume was prepared by the staff of the Institute of Advanced Study, but it was distributed by the Harvard University Mathematics Department.

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