Review of SGA 3 (2011 edition), J.S. Milne

The study of algebraic groups, regarded as groups of matrices, is almost as old as group theory itself — the classical algebraic groups (special linear, orthogonal, symplectic) over the finite prime fields were introduced by Jordan in the 1860s. However, it was not until the work of Maurer, Kolchin, and Chevalley that the study of algebraic groups became a subject in its own right. While Kolchin's interest in algebraic groups was as preparation for his study of differential algebraic groups, Chevalley's viewed them as a link between Lie groups and finite groups.

A central problem in the subject is the classification of the simple algebraic groups. The similar problem for Lie groups was solved by Killing and Cartan: the classification of simple complex Lie groups is the same as that of simple complex Lie algebras, and Killing and Cartan showed that, in addition to the classical simple Lie algebras, there are only five exceptional algebras E_6, E_7, E_8, F_4, G_2. As all semisimple complex Lie groups are algebraic, the classification of simple algebraic groups over $\mathbb{C}$ is the same as that of the simple Lie algebras. This solves the classification problem over $\mathbb{C}$. About 1955, Borel proved his fixed point theorem and thereby obtained his important results on the solvable subgroups of algebraic groups. This enabled Chevalley (in 1956) to extend some of his earlier work and prove that the classification of simple algebraic groups over an algebraically closed field is *independent of the field*. For fields of nonzero characteristic, this was surprising because the similar statement for Lie algebras is false. Chevalley went further, and showed that for split groups, i.e., those containing a split maximal torus, the classification is independent of the base field, algebraically closed or not, and even applies over $\mathbb{Z}$. In his 1965 thesis, Grothendieck's student Demazure showed that Chevalley's classification theory extends in an entirely satisfactory way to split reductive group schemes over arbitrary base schemes. Thus, in a single remarkable decade, the subject of algebraic groups had gone from one in which many of its main results were known only for algebraic groups over $\mathbb{C}$ to one that had achieved a certain maturity as the study of group schemes over arbitrary bases.

Most of this work is documented in the published notes of seminars in the Paris region. The first of these is Séminaire "Sophus Lie" (1954–56), organized by Cartier, which developed (in improved form) the Killing-Cartan theory of real and complex Lie algebras. The second is Séminaire Chevalley (1956-58), organized by Chevalley, which explained Borel's work on solvable subgroups and his own work on the classification of simple algebraic groups over algebraically closed fields. Chevalley sketched the extension of his theory to split groups over arbitrary field (and even $\mathbb{Z}$) in a 1961 Bourbaki seminar. Finally, in 1962–64 Grothendieck and Demazure organized the seminar on group schemes that is now referred to as SGA 3. The first two-thirds of the seminar develops the theory of group schemes over an arbitrary base scheme, and the final third is a detailed exposition by Demazure of his results on reductive group schemes over an arbitrary base scheme. Many of the participants of these seminars were also involved in the writing of Bourbaki's "Groupes et Algebres de Lie".

Grothendieck envisaged that his seminars would provide only a first exposition of a topic which, would soon be superseded by the "canonical" exposition in EGA.[1] However, they have proved much more durable than expected — for example, SGA 1 is still widely read, and has recently been reprinted in a corrected TEXed version, even though its material has been incorporated into EGA. SGA 3 was never incorporated into EGA — it was to have been Chapter VII — and has been the standard reference on group schemes since the notes

[1] Eléments de Géométrie Algébrique (Grothendieck, Dieudonné).

for it first became available almost fifty years ago. It has remained the only comprehensive treatment of group schemes over an arbitrary base scheme.

The notes for SGA 3 were originally distributed by IHES[2] in typed mimeographed form under the title SGAD (the reviewer's copy occupies 21cm on his bookcase). In 1970, they were reprinted as three volumes in Springer's series, Lecture Notes in Mathematics. That version is identical to the original except that some misprints were fixed, part of Exposé VI_B was rewritten, and indexes and tables of contents added. The present version has been thoroughly revised by the editors Philippe Gille and Patrick Polo, with the support of the mathematical community, especially Demazure, Gabber, and Raynaud. Only the first and third volumes are currently available — the remaining volume is expected to become available in 2013.

In the new version, the editors have retained the structure and the numbering of the original, but there have been a large number of improvements, which we now list.

1. The typed originals have been TEXed. This has allowed the editors to improve the typography by replacing underlined letters with calligraphic letters and doubly underlined letters with boldface letters. The residue field at a point x of a scheme is now denoted $\kappa(x)$, and the roots of a semisimple group are denoted $\alpha, \beta, \gamma, \ldots$ rather than $r, s, t, \ldots$. Terminology has been modernised: for example "prescheme/scheme" has been replaced by "scheme/separated scheme".
2. There are numerous small improvements to the text, all carefully footnoted — in Volume 1 there are over 800 of these, and in Volume 3 there are about 300. Some of these correct errors in the original and some add comments and references, but most expand the original exposition. For example, the editors may add lemmas to make explicit what was only implicit, add arguments omitted in the original, or add necessary background material. At the end of each section they have added a bibliography of the additional references cited.
3. In Exposé I, the editors added an eight-page section (Section 6) on G-equivariant objects and morphisms in a category, where G is a group object in the category of contravariant set-valued functors on the category.
4. In Exposé III, the editors greatly expanded the zeroth section, which mainly reviews parts of SGA 1 (now in EGA IV). They add several pages (pp. 148–152) on complete intersections, which enables them to give a complete proof (instead of a sketch) that a subgroup H of a smooth group scheme G is a local complete intersection in G if it is flat and locally of finite presentation over the base scheme (Prop. 4.15).
5. In Exposé V, the editors have re-ordered the pages, which were shuffled between SGAD and SGA 3(!). The main results in this section concern the construction of quotients by groupoid schemes. The editors have added an eight-page section in which they work out in detail the consequences of these results for quotients by group schemes. They state without proof the more recent results on quotients (Keel, Mori, Kollar).
6. In Exposé VI_A, the editors have added two sections (2.6, 6) explaining results from the 1975 thesis of D. Perrin, which extend some statements for affine group schemes to quasi-compact group schemes. For example, every quasi-compact group scheme over a field is a projective limit of its algebraic quotients.
7. In Exposé VI_B, the main theorem of Section 5 states that a group scheme and its homogeneous spaces are separated under certain hypotheses. Counterexamples of

[2] Institut des Hautes Études Scientifiques

Gabber show that the orginal hypotheses are inadequate. The editors have corrected the statement of the theorem, and rewritten the section to include a correct proof. Section 11, on affine group schemes and the affine envelope of a general group scheme, have been significantly rewritten, and the final two sections (12,13), have been added by the editors.

8. In Exposé VII_B, several subsections have been enlarged, and one has been added.
9. To Volume 3, the editors have added the published version of Demazure's thesis, which summarizes the material in the volume, and serves an introduction to it.

The volumes have been handsomely printed. The reviewer can attest that this version is much more pleasant to read, and work with, than the earlier versions. The editors and the Société Mathématique de France are to be congratulated for their efforts in rejuvenating this classic work. Everyone with an interest in group schemes will wish for a copy.

March 31, 2012; April 7, 2012.