A Primer of Commutative Algebra

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Abstract

These notes prove the basic theorems in commutative algebra required for algebraic geometry and algebraic groups. They assume only a knowledge of the algebra usually taught in advanced undergraduate or first-year graduate courses.

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Contents

NOTATIONS AND CONVENTIONS

Our convention is that rings have identity elements,[1] and homomorphisms of rings respect the identity elements. A ***unit*** of a ring is an element admitting an inverse. The units of a ring A form a group, which we denote[2] $A^\times$. Throughout "ring" means "commutative ring".

[1] An element e of a ring A is an ***identity element*** if $ea = a = ae$ for all elements a of the ring. It is usually denoted 1_A or just 1. Some authors call this a unit element, but then an element can be a unit without being a unit element. Worse, a unit need not be the unit.

[2] This notation differs from that of Bourbaki, who writes $A^\times$ for the multiplicative monoid $A \smallsetminus \{0\}$ and A^* for the group of units. We shall rarely need the former, and * is overused.

Following Bourbaki, we let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For a field k, k^{al} denotes an algebraic closure of k.

$X \subset Y$	X is a subset of Y (not necessarily proper).
$X \stackrel{\text{def}}{=} Y$	X is defined to be Y, or equals Y by definition.
$X \approx Y$	X is isomorphic to Y.
$X \simeq Y$	X and Y are canonically isomorphic (or there is a given or unique isomorphism).

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1 Rings and algebras

Let A be a ring. A ***subring*** of A is a subset that contains 1_A and is closed under addition, multiplication, and the formation of negatives. An A-***algebra*** is a ring B together with a homomorphism $i_B: A \to B$. A ***homomorphism*** of A-algebras $B \to C$ is a homomorphism of rings $\varphi: B \to C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$.

Elements $x_1, \ldots, x_n$ of an A-algebra B are said to ***generate*** it if every element of B can be expressed as a polynomial in the x_i with coefficients in $i_B(A)$, i.e., if the homomorphism of A-algebras $A[X_1, \ldots, X_n] \to B$ acting as i_B on A and sending X_i to x_i is surjective. We then write $B = (i_B A)[x_1, \ldots, x_n]$.

A ring homomorphism $A \to B$ is of ***finite type***, and B is a ***finitely generated*** A-algebra, if B is generated by a finite set of elements as an A-algebra.

A ring homomorphism $A \to B$ is ***finite***, and B is a ***finite***[3] A-algebra, if B is finitely generated as an A-module. If $A \to B$ and $B \to C$ are finite ring homomorphisms, then so also is their composite $A \to C$.

Let k be a field, and let A be a k-algebra. When $1_A \neq 0$, the map $k \to A$ is injective, and we can identify k with its image, i.e., we can regard k as a subring of A. When $1_A = 0$, the ring A is the zero ring $\{0\}$.

Let $A[X]$ be the ring of polynomials in the symbol X with coefficients in A. If A is an integral domain, then $\deg(fg) = \deg(f) + \deg(g)$, and so $A[X]$ is also an integral domain; moreover, $A[X]^\times = A^\times$.

Let A be an algebra over a field k. If A is an integral domain and finite as a k-algebra, then it is a field because, for each nonzero $a \in A$, the k-linear map $x \mapsto ax: A \to A$ is injective, and hence is surjective, which shows that a has an inverse. If A is an integral domain and each element of A is algebraic over k, then for each $a \in A$, $k[a]$ is an integral domain finite over k, and hence contains an inverse of a; again A is a field.

PRODUCTS AND IDEMPOTENTS

An element e of a ring A is ***idempotent*** if $e^2 = e$. For example, 0 and 1 are both idempotents — they are called the ***trivial idempotents***. Idempotents $e_1, \ldots, e_n$ are ***orthogonal*** if $e_i e_j = 0 = e_j e_i$ for $i \neq j$. Any sum of orthogonal idempotents is again idempotent. A set

[3] This is Bourbaki's terminology. Finite homomorphisms of rings correspond to finite maps of varieties and schemes. Some other authors say "module-finite".

$\{e_1,\dots,e_n\}$ of orthogonal idempotents is ***complete*** if $e_1+\cdots+e_n=1$. Any set of orthogonal idempotents $\{e_1,\dots,e_n\}$ can be made into a complete set of orthogonal idempotents by adding the idempotent $e=1-(e_1+\cdots+e_n)$.

If $A=A_1\times\cdots\times A_n$ (direct product of rings), then the elements

$$e_i=(0,\dots,\overset{i}{1},\dots,0),\quad 1\le i\le n,$$

form a complete set of orthogonal idempotents in A. Conversely, if $\{e_1,\dots,e_n\}$ is a complete set of orthogonal idempotents in A, then Ae_i becomes a ring[4] with the addition and multiplication induced by that of A, and $A\simeq Ae_1\times\cdots\times Ae_n$.

2 Ideals

Let A be a ring. An ***ideal*** $\mathfrak{a}$ in A is a subset such that

- ◇ $\mathfrak{a}$ is a subgroup of A regarded as a group under addition;
- ◇ $a\in\mathfrak{a}$, $r\in A\Rightarrow ra\in\mathfrak{a}$.

The ***ideal generated by a subset*** S of A is the intersection of all ideals $\mathfrak{a}$ containing S — it is easy to verify that this is in fact an ideal, and that it consists of all finite sums of the form $\sum r_is_i$ with $r_i\in A$, $s_i\in S$. The ideal generated by the empty set is the zero ideal $\{0\}$. When $S=\{s_1,s_2,\dots\}$, we write $(s_1,s_2,\dots)$ for the ideal it generates.

An ideal is ***principal*** if it is generated by a single element. Such an ideal (a) is proper if and only a is not a unit. Thus a ring A is a field if and only if $1_A\neq 0$ and A contains no nonzero proper ideals.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in A. The set $\{a+b\mid a\in\mathfrak{a},b\in\mathfrak{b}\}$ is an ideal, denoted $\mathfrak{a}+\mathfrak{b}$. The ideal generated by $\{ab\mid a\in\mathfrak{a},b\in\mathfrak{b}\}$ is denoted by $\mathfrak{ab}$. Clearly $\mathfrak{ab}$ consists of all finite sums $\sum a_ib_i$ with $a_i\in\mathfrak{a}$ and $b_i\in\mathfrak{b}$, and if $\mathfrak{a}=(a_1,\dots,a_m)$ and $\mathfrak{b}=(b_1,\dots,b_n)$, then $\mathfrak{ab}=(a_1b_1,\dots,a_ib_j,\dots,a_mb_n)$. Note that $\mathfrak{ab}\subset\mathfrak{a}A=\mathfrak{a}$ and $\mathfrak{ab}\subset A\mathfrak{b}=\mathfrak{b}$, and so

$$\mathfrak{ab}\subset\mathfrak{a}\cap\mathfrak{b}. \tag{1}$$

The kernel of a homomorphism $A\to B$ is an ideal in A. Conversely, for any ideal $\mathfrak{a}$ in a ring A, the set of cosets of $\mathfrak{a}$ in A forms a ring $A/\mathfrak{a}$, and $a\mapsto a+\mathfrak{a}$ is a homomorphism $\varphi\colon A\to A/\mathfrak{a}$ whose kernel is $\mathfrak{a}$. There is a one-to-one correspondence

$$\{\text{ideals of } A \text{ containing } \mathfrak{a}\}\underset{\varphi^{-1}(\mathfrak{b})\hookleftarrow\mathfrak{b}}{\overset{\mathfrak{b}\mapsto\varphi(\mathfrak{b})}{\longleftrightarrow}}\{\text{ideals of } A/\mathfrak{a}\}. \tag{2}$$

For any ideal $\mathfrak{b}$ of A, $\varphi^{-1}\varphi(\mathfrak{b})=\mathfrak{a}+\mathfrak{b}$.

The ideals of $A\times B$ are all of the form $\mathfrak{a}\times\mathfrak{b}$ with $\mathfrak{a}$ and $\mathfrak{b}$ ideals in A and B. To see this, note that if $\mathfrak{c}$ is an ideal in $A\times B$ and $(a,b)\in\mathfrak{c}$, then $(a,0)=(1,0)(a,b)\in\mathfrak{c}$ and $(0,b)=(0,1)(a,b)\in\mathfrak{c}$. Therefore, $\mathfrak{c}=\mathfrak{a}\times\mathfrak{b}$ with

$$\mathfrak{a}=\{a\mid(a,0)\in\mathfrak{c}\},\quad \mathfrak{b}=\{b\mid(0,b)\in\mathfrak{c}\}.$$

An ideal $\mathfrak{p}$ in A is ***prime*** if $\mathfrak{p}\neq A$ and $ab\in\mathfrak{p}\Rightarrow a\in\mathfrak{p}$ or $b\in\mathfrak{p}$. Thus $\mathfrak{p}$ is prime if and only if the quotient ring $A/\mathfrak{p}$ is nonzero and has the property that

$$ab=0,\quad b\neq 0\Rightarrow a=0,$$

[4]But Ae_i is not a subring of A if $n\neq 1$ because its identity element is $e_i\neq 1_A$. However, the map $a\mapsto ae_i\colon A\to Ae_i$ realizes Ae_i as a quotient of A.

i.e., $A/\mathfrak{p}$ is an integral domain. Note that if $\mathfrak{p}$ is prime and $a_1 \cdots a_n \in \mathfrak{p}$, then either $a_1 \in \mathfrak{p}$ or $a_2 \cdots a_n \in \mathfrak{p}$; if the latter, then either $a_2 \in \mathfrak{p}$ or $a_3 \cdots a_n \in \mathfrak{p}$; continuing in this fashion, we find that at least one of the $a_i \in \mathfrak{p}$.

An ideal $\mathfrak{m}$ in A is ***maximal*** if it is a maximal element of the set of proper ideals in A. Therefore an ideal $\mathfrak{m}$ is maximal if and only if the quotient ring $A/\mathfrak{m}$ is nonzero and has no proper nonzero ideals (by (2)), and so is a field. Note that

$$\mathfrak{m} \text{ maximal} \implies \mathfrak{m} \text{ prime}.$$

A ***multiplicative subset*** of a ring A is a subset S with the property:

$$1 \in S, \quad a, b \in S \implies ab \in S.$$

For example, the following are multiplicative subsets:
the multiplicative subset $\{1, f, \dots, f^r, \dots\}$ generated by an element f of A;
the complement of a prime ideal (or of a union of prime ideals);
$1 + \mathfrak{a} \stackrel{\text{def}}{=} \{1 + a \mid a \in \mathfrak{a}\}$ for any ideal $\mathfrak{a}$ of A.

PROPOSITION 2.1. *Let S be a subset of a ring A, and let $\mathfrak{a}$ be an ideal disjoint from S. The set of ideals in A containing $\mathfrak{a}$ and disjoint from S contains maximal elements (i.e., an element not properly contained in any other ideal in the set). If S is multiplicative, then any such maximal element is prime.*

PROOF. The set Σ of ideals containing $\mathfrak{a}$ and disjoint from S is nonempty (it contains $\mathfrak{a}$). If A is noetherian (see §3 below), Σ automatically contains maximal elements. Otherwise, we apply Zorn's lemma. Let $\mathfrak{b}_1 \subset \mathfrak{b}_2 \subset \cdots$ be a chain of ideals in Σ, and let $\mathfrak{b} = \bigcup \mathfrak{b}_i$. Then $\mathfrak{b} \in \Sigma$, because otherwise some element of S lies in $\mathfrak{b}$, and hence in some $\mathfrak{b}_i$, which contradicts the definition of Σ. Therefore $\mathfrak{b}$ is an upper bound for the chain. As every chain in Σ has an upper bound, Zorn's lemma implies that Σ has a maximal element.

Assume that S is a multiplicative subset of A, and let $\mathfrak{c}$ be maximal in Σ. Let $bb' \in \mathfrak{c}$. If b is not in $\mathfrak{c}$, then $\mathfrak{c} + (b)$ properly contains $\mathfrak{c}$, and so it is not in Σ. Therefore there exist an $f \in S \cap (\mathfrak{c} + (b))$, say, $f = c + ab$ with $c \in \mathfrak{c}$. Similarly, if b' is not in $\mathfrak{c}$, then there exists an $f' \in S$ such that $f = c' + a'b'$ with $c' \in \mathfrak{c}$. Now

$$ff' = cc' + abc' + a'b'c + aa'bb' \in \mathfrak{c},$$

which contradicts

$$ff' \in S.$$

□

COROLLARY 2.2. *Every proper ideal in a ring is contained in a maximal ideal.*

PROOF. For a proper ideal $\mathfrak{a}$ of A, apply the proposition with $S = \{1\}$. □

EXERCISE 2.3. A multiplicative subset S of a ring A is said to be ***saturated*** if

$$ab \in S \implies a, b \in S.$$

Show the saturated multiplicative subsets are exactly the complements of unions of prime ideals (hence every multiplicative subset S is contained in a smallest saturated multiplicative subset, namely, $A \smallsetminus \bigcup\{\mathfrak{p} \mid \mathfrak{p} \text{ prime}, \mathfrak{p} \cap S = \emptyset\}$).

REMARK 2.4. The proof of (2.1) is one of many in commutative algebra in which an ideal, maximal with respect to some property, is shown to be prime. For a general examination of this phenomenon, see Lam and Reyes 2008.

The ***radical*** $\mathrm{rad}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ is

$$\{f \in A \mid f^r \in \mathfrak{a}, \text{ some } r \in \mathbb{N}, r > 0\}.$$

An ideal $\mathfrak{a}$ is said to be ***radical*** if it equals its radical. Thus $\mathfrak{a}$ is radical if and only if the quotient ring $A/\mathfrak{a}$ is ***reduced***, i.e., without nonzero ***nilpotent*** elements (elements some power of which is zero). Since integral domains are reduced, prime ideals (*a fortiori* maximal ideals) are radical. The radical of (0) consists of the nilpotent elements of A — it is called the ***nilradical*** of A.

If $\mathfrak{b} \leftrightarrow \mathfrak{b}'$ under the one-to-one correspondence (2), then $A/\mathfrak{b} \simeq (A/\mathfrak{a})/\mathfrak{b}'$, and so $\mathfrak{b}$ is prime (resp. maximal, radical) if and only if $\mathfrak{b}'$ is prime (resp. maximal, radical).

PROPOSITION 2.5. *Let $\mathfrak{a}$ be an ideal in a ring A.*
(a) The radical of $\mathfrak{a}$ is an ideal.
(b) $\mathrm{rad}(\mathrm{rad}(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$.

PROOF. (a) If $a \in \mathrm{rad}(\mathfrak{a})$, then clearly $fa \in \mathrm{rad}(\mathfrak{a})$ for all $f \in A$. Suppose $a, b \in \mathrm{rad}(\mathfrak{a})$, with say $a^r \in \mathfrak{a}$ and $b^s \in \mathfrak{a}$. When we expand $(a+b)^{r+s}$ using the binomial theorem, we find that every term has a factor a^r or b^s, and so lies in $\mathfrak{a}$.

(b) If $a^r \in \mathrm{rad}(\mathfrak{a})$, then $a^{rs} = (a^r)^s \in \mathfrak{a}$ for some $s > 0$. □

Note that (b) of the proposition shows that $\mathrm{rad}(\mathfrak{a})$ is radical, and so is the smallest radical ideal containing $\mathfrak{a}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are radical, then $\mathfrak{a} \cap \mathfrak{b}$ is radical, but $\mathfrak{a} + \mathfrak{b}$ need not be: consider, for example, $\mathfrak{a} = (X^2 - Y)$ and $\mathfrak{b} = (X^2 + Y)$; they are both prime ideals in $k[X, Y]$ (by 4.7 below), but $\mathfrak{a} + \mathfrak{b} = (X^2, Y)$, which contains X^2 but not X.

PROPOSITION 2.6. *The radical of an ideal is equal to the intersection of the prime ideals containing it. In particular, the nilradical of a ring A is equal to the intersection of the prime ideals of A.*

PROOF. If $\mathfrak{a} = A$, then the set of prime ideals containing it is empty, and so the intersection is A. Thus we may suppose that $\mathfrak{a}$ is a proper ideal of A. Then $\mathrm{rad}(\mathfrak{a}) \subset \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$ because prime ideals are radical and $\mathrm{rad}(\mathfrak{a})$ is the smallest radical ideal containing $\mathfrak{a}$.

Conversely, suppose that $f \notin \mathrm{rad}(\mathfrak{a})$. According to Proposition 2.1, there exists a prime ideal containing $\mathfrak{a}$ and disjoint the multiplicative subset $\{1, f, \ldots\}$. Therefore $f \notin \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$. □

DEFINITION 2.7. The ***Jacobson radical*** $\mathfrak{J}$ of a ring is the intersection of the maximal ideals of the ring:

$$\mathfrak{J}(A) = \bigcap \{\mathfrak{m} \mid \mathfrak{m} \text{ maximal in } A\}.$$

A ring A is ***local*** if it has exactly one maximal ideal. For such a ring, the Jacobson radical is $\mathfrak{m}$.

PROPOSITION 2.8. *An element c of A is in the Jacobson radical of A if and only if $1 - ac$ is a unit for all $a \in A$.*

PROOF. We prove the contrapositive: there exists a maximal ideal $\mathfrak{m}$ such that $c \notin \mathfrak{m}$ if and only if there exists an $a \in A$ such that $1-ac$ is not a unit.

$\Leftarrow$: As $1-ac$ is not a unit, it lies in some maximal ideal $\mathfrak{m}$ of A (by 2.4a). Then $c \notin \mathfrak{m}$, because otherwise $1 = (1-ac)+ac \in \mathfrak{m}$.

$\Rightarrow$: Suppose that c is not in the maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}+(c) = A$, and so $1 = m+ac$ for some $m \in \mathfrak{m}$ and $a \in A$. Now $1-ac \in \mathfrak{m}$, and so it is not a unit. □

PROPOSITION 2.9. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, $r \geq 1$, be ideals in A with $\mathfrak{p}_2, \dots, \mathfrak{p}_r$ prime, and let $\mathfrak{a}$ be an ideal in A. Then*

$$\mathfrak{a} \subset \bigcup\nolimits_{1 \leq i \leq r} \mathfrak{p}_i \implies \mathfrak{a} \subset \mathfrak{p}_i \text{ for some } i.$$

PROOF. We prove the contrapositive:

> if the ideal $\mathfrak{a}$ in not contained in any of the ideals $\mathfrak{p}_i$, then it is not contained in their union.

For $r = 1$, there is nothing to prove, and so we may assume that $r > 1$ and (inductively) that the statement is true for $r-1$. As $\mathfrak{a}$ is not contained in any of the ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, for each i, there exists an a_i in $\mathfrak{a}$ not in the union of the ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_{i-1}, \mathfrak{p}_{i+1}, \dots, \mathfrak{p}_r$. If there exists an i such that a_i does not lie in $\mathfrak{p}_i$, then that $a_i \in \mathfrak{a} \smallsetminus \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$, and the proof is complete. Thus suppose that every $a_i \in \mathfrak{p}_i$, and consider

$$a = a_1 \cdots a_{r-1} + a_r.$$

Because $\mathfrak{p}_r$ is prime and none of the elements $a_1, \dots, a_{r-1}$ lies in $\mathfrak{p}_r$, their product does not lie in $\mathfrak{p}_r$; however, $a_r \in \mathfrak{p}_r$, and so $a \notin \mathfrak{p}_r$. Next consider a prime $\mathfrak{p}_i$ with $i \leq r-1$. In this case $a_1 \cdots a_{r-1} \in \mathfrak{p}_i$ because the product involves a_i, but $a_r \notin \mathfrak{p}_i$, and so again $a \notin \mathfrak{p}_i$. Now $a \in \mathfrak{a} \smallsetminus \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r$, and so $\mathfrak{a}$ is not contained in the union of the $\mathfrak{p}_i$. □

EXTENSION AND CONTRACTION OF IDEALS

Let $\varphi\colon A \to B$ be a homomorphism of rings.

NOTATION 2.10. For an ideal $\mathfrak{b}$ of B, $\varphi^{-1}(\mathfrak{b})$ is an ideal in A, called the ***contraction*** of $\mathfrak{b}$ to A, which is often denoted $\mathfrak{b}^c$. For an ideal $\mathfrak{a}$ of A, the ideal in B generated by $\varphi(\mathfrak{a})$ is called the ***extension*** of $\mathfrak{a}$ to B, and is often denoted $\mathfrak{a}^e$.

When φ is surjective, $\varphi(\mathfrak{a})$ is already an ideal, and when A is a subring of B, $\mathfrak{b}^c = \mathfrak{b} \cap A$.

2.11. There are the following equalities ($\mathfrak{a}, \mathfrak{a}'$ ideals in A; $\mathfrak{b}, \mathfrak{b}'$ ideals in B):

$$(\mathfrak{a}+\mathfrak{a}')^e = \mathfrak{a}^e + \mathfrak{a}'^e, \quad (\mathfrak{a}\mathfrak{a}')^e = \mathfrak{a}^e\mathfrak{a}'^e, \quad (\mathfrak{b} \cap \mathfrak{b}')^c = \mathfrak{b}^c \cap \mathfrak{b}'^c, \quad \operatorname{rad}(\mathfrak{b})^c = \operatorname{rad}(\mathfrak{b}^c).$$

2.12. Obviously (i) $\mathfrak{a} \subset \mathfrak{a}^{ec}$ and (ii) $\mathfrak{b}^{ce} \subset \mathfrak{b}$ ($\mathfrak{a}$ an ideal of A; $\mathfrak{b}$ an ideal of B). On applying e to (i), we find that $\mathfrak{a}^e \subset \mathfrak{a}^{ece}$, and (ii) with $\mathfrak{b}$ replaced by $\mathfrak{a}^e$ shows that $\mathfrak{a}^{ece} \subset \mathfrak{a}^e$; therefore $\mathfrak{a}^e = \mathfrak{a}^{ece}$. Similarly, $\mathfrak{b}^{cec} = \mathfrak{b}^c$. It follows that extension and contraction define inverse bijections between the set of contracted ideals in A and the set of extended ideals in B:

$$\{\mathfrak{b}^c \subset A \mid \mathfrak{b} \text{ an ideal in } B\} \underset{c}{\overset{e}{\rightleftharpoons}} \{\mathfrak{a}^e \subset B \mid \mathfrak{a} \text{ an ideal in } A\}$$

Note that, for any ideal $\mathfrak{b}$ in B, the map $A/\mathfrak{b}^c \to B/\mathfrak{b}$ is injective, and so $\mathfrak{b}^c$ is prime (resp. radical) if $\mathfrak{b}$ is prime (resp. radical).

THE CHINESE REMAINDER THEOREM

Recall the classical form of the theorem: let $d_1, ..., d_n$ be integers, relatively prime in pairs; then for any integers $x_1, ..., x_n$, the congruences

$$x \equiv x_i \bmod d_i$$

have a simultaneous solution $x \in \mathbb{Z}$; moreover, if x is one solution, then the other solutions are the integers of the form $x + md$ with $m \in \mathbb{Z}$ and $d = \prod d_i$.

We want to translate this in terms of ideals. Integers m and n are relatively prime if and only if $(m,n) = \mathbb{Z}$, i.e., if and only if $(m) + (n) = \mathbb{Z}$. This suggests defining ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a ring A to be ***relatively prime*** (or ***coprime***) if $\mathfrak{a} + \mathfrak{b} = A$.

If $m_1, ..., m_k$ are integers, then $\bigcap(m_i) = (m)$ where m is the least common multiple of the m_i. Thus $\bigcap(m_i) \supset (\prod m_i)$, which equals $\prod(m_i)$. If the m_i are relatively prime in pairs, then $m = \prod m_i$, and so we have $\bigcap(m_i) = \prod(m_i)$. Note that in general,

$$\mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdots \mathfrak{a}_n \subset \mathfrak{a}_1 \cap \mathfrak{a}_2 \cap ... \cap \mathfrak{a}_n,$$

but the two ideals need not be equal.

These remarks suggest the following statement.

THEOREM 2.13 (CHINESE REMAINDER THEOREM). *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals in a ring A. If $\mathfrak{a}_i$ is relatively prime to $\mathfrak{a}_j$ whenever $i \neq j$, then the map*

$$a \mapsto (\dots, a + \mathfrak{a}_i, \dots): A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n \tag{3}$$

is surjective with kernel $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$.

PROOF. Suppose first that $n = 2$. As $\mathfrak{a}_1 + \mathfrak{a}_2 = A$, there exist $a_i \in \mathfrak{a}_i$ such that $a_1 + a_2 = 1$. Then $a_1 x_2 + a_2 x_1$ maps to $(x_1 \bmod \mathfrak{a}_1, x_2 \bmod \mathfrak{a}_2)$, which shows that (3) is surjective.

For each i, there exist elements $a_i \in \mathfrak{a}_1$ and $b_i \in \mathfrak{a}_i$ such that

$$a_i + b_i = 1, \text{ all } i \geq 2.$$

The product $\prod_{i \geq 2}(a_i + b_i) = 1$, and lies in $\mathfrak{a}_1 + \prod_{i \geq 2} \mathfrak{a}_i$, and so

$$\mathfrak{a}_1 + \prod_{i \geq 2} \mathfrak{a}_i = A.$$

We can now apply the theorem in the case $n = 2$ to obtain an element y_1 of A such that

$$y_1 \equiv 1 \bmod \mathfrak{a}_1, \quad y_1 \equiv 0 \bmod \prod_{i \geq 2} \mathfrak{a}_i.$$

These conditions imply

$$y_1 \equiv 1 \bmod \mathfrak{a}_1, \quad y_1 \equiv 0 \bmod \mathfrak{a}_j, \text{ all } j > 1.$$

Similarly, there exist elements $y_2, ..., y_n$ such that

$$y_i \equiv 1 \bmod \mathfrak{a}_i, \quad y_i \equiv 0 \bmod \mathfrak{a}_j \text{ for } j \neq i.$$

The element $x = \sum x_i y_i$ maps to $(x_1 \bmod \mathfrak{a}_1, \dots, x_n \bmod \mathfrak{a}_n)$, which shows that (3) is surjective.

It remains to prove that $\bigcap \mathfrak{a}_i = \prod \mathfrak{a}_i$. Obviously $\prod \mathfrak{a}_i \subset \bigcap \mathfrak{a}_i$. Suppose first that $n = 2$, and let $a_1 + a_2 = 1$, as before. For $c \in \mathfrak{a}_1 \cap \mathfrak{a}_2$, we have

$$c = a_1 c + a_2 c \in \mathfrak{a}_1 \cdot \mathfrak{a}_2$$

which proves that $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \mathfrak{a}_1 \mathfrak{a}_2$. We complete the proof by induction. This allows us to assume that $\prod_{i \geq 2} \mathfrak{a}_i = \bigcap_{i \geq 2} \mathfrak{a}_i$. We showed above that $\mathfrak{a}_1$ and $\prod_{i \geq 2} \mathfrak{a}_i$ are relatively prime, and so

$$\mathfrak{a}_1 \cdot (\prod_{i \geq 2} \mathfrak{a}_i) = \mathfrak{a}_1 \cap (\prod_{i \geq 2} \mathfrak{a}_i)$$

by the $n = 2$ case. Now $\mathfrak{a}_1 \cdot (\prod_{i \geq 2} \mathfrak{a}_i) = \prod_{i \geq 1} \mathfrak{a}_i$ and $\mathfrak{a}_1 \cap (\prod_{i \geq 2} \mathfrak{a}_i) = \mathfrak{a}_1 \cap (\bigcap_{i \geq 2} \mathfrak{a}_i) = \bigcap_{i \geq 1} \mathfrak{a}_i$, which completes the proof. □

3 Noetherian rings

PROPOSITION 3.1. *The following three conditions on a ring A are equivalent:*

(a) *every ideal in A is finitely generated;*
(b) *every ascending chain of ideals $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ eventually becomes constant, i.e., for some m, $\mathfrak{a}_m = \mathfrak{a}_{m+1} = \cdots$.*
(c) *every nonempty set of ideals in A has a maximal element.*

PROOF. (a) $\Rightarrow$ (b): If $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ is an ascending chain, then $\mathfrak{a} = \bigcup \mathfrak{a}_i$ is an ideal, and hence has a finite set $\{a_1, \ldots, a_n\}$ of generators. For some m, all the a_i belong $\mathfrak{a}_m$, and then

$$\mathfrak{a}_m = \mathfrak{a}_{m+1} = \cdots = \mathfrak{a}.$$

(b) $\Rightarrow$ (c): Let Σ be a nonempty set of ideals in A. Then (b) certainly implies that every ascending chain of ideals in Σ has an upper bound in Σ, and so Zorn's lemma shows that Σ has a maximal element.

(c) $\Rightarrow$ (a): Let $\mathfrak{a}$ be an ideal, and let Σ be the set of finitely generated ideals contained in $\mathfrak{a}$. Then S is nonempty because it contains the zero ideal, and so it contains a maximal element $\mathfrak{c} = (a_1, \ldots, a_r)$. If $\mathfrak{c} \neq \mathfrak{a}$, then there exists an element $a \in \mathfrak{a} \setminus \mathfrak{c}$, and $(a_1, \ldots, a_r, a)$ will be a finitely generated ideal in $\mathfrak{a}$ properly containing $\mathfrak{c}$. This contradicts the definition of $\mathfrak{c}$. □

A ring A is ***noetherian*** if it satisfies the equivalent conditions of the proposition. For example, fields and principal ideal domains are noetherian. On applying (c) to the set of all proper ideals containing a fixed proper ideal, we see that every proper ideal in a noetherian ring is contained in a maximal ideal. We saw in (2.4) that this is, in fact, true for any ring, but the proof for non-noetherian rings requires Zorn's lemma.

A quotient $A/\mathfrak{a}$ of a noetherian ring A is noetherian, because the ideals in $A/\mathfrak{a}$ are all of the form $\mathfrak{b}/\mathfrak{a}$ with $\mathfrak{b}$ an ideal in A, and any set of generators for $\mathfrak{b}$ generates $\mathfrak{b}/\mathfrak{a}$.

PROPOSITION 3.2. *Let A be a ring. The following conditions on an A-module M are equivalent:*

(a) *every submodule of M is finitely generated (in particular, M is finitely generated);*
(b) *every ascending chain of submodules $M_1 \subset M_2 \subset \cdots$ eventually becomes constant.*
(c) *every nonempty set of submodules of M has a maximal element.*

PROOF. Essentially the same as that of (3.1). □

An A-module M is ***noetherian*** if it satisfies the equivalent conditions of the proposition. Let ${}_AA$ denote A regarded as a left A-module. Then the submodules of ${}_AA$ are exactly the ideals in A, and so ${}_AA$ is noetherian (as an A-module) if and only if A is noetherian (as a ring).

PROPOSITION 3.3. *Let M be an A-module, and let N be a submodule of M. The module M is noetherian if and only both N and M/N are noetherian.*

PROOF. $\Rightarrow$: An ascending chain of submodules in N or in M/N gives rise to an ascending chain in M, and therefore becomes constant.

$\Leftarrow$: I claim that if $M' \subset M''$ are submodules of M such that $M' \cap N = M'' \cap N$ and M' and M'' have the same image in M/N, then $M' = M''$. To see this, let $x \in M''$; the second condition implies that there exists a $y \in M'$ with the same image as x in M/N, i.e., such that $x - y \in N$. Then $x - y \in M'' \cap N \subset M'$, and so $x \in M'$.

Now consider an ascending chain of submodules of M. If M/N is Noetherian, the image of the chain in M/N becomes stationary, and if N is Noetherian, the intersection of the chain with N becomes stationary. Now the claim shows that the chain itself becomes stationary. □

More generally, consider an exact sequence

$$0 \to M' \xrightarrow{i} M \xrightarrow{q} M'' \to 0$$

of A-modules. The module M is noetherian if and only if M' and M'' are both noetherian. For example, a direct sum

$$M = M_1 \oplus M_2$$

of A-modules is noetherian if and only if M_1 and M_2 are both noetherian (because $0 \to M_1 \to M \to M_2 \to 0$ is exact).

PROPOSITION 3.4. *Let A be a noetherian ring. Then every finitely generated A-module is noetherian.*

PROOF. If M is generated by a single element, then $M \approx A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ in A, and the statement is obvious. We argue by induction on the minimum number n of generators of M. Since M contains a submodule N generated by $n-1$ elements such that the quotient M/N is generated by a single element, the statement follows from (3.3). □

PROPOSITION 3.5. *Every finitely generated module M over a noetherian ring A contains a finite chain of submodules $M \supset M_r \supset \cdots \supset M_1 \supset 0$ such that each quotient M_i/M_{i-1} is isomorphic to $A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$.*

PROOF. The annihilator $\operatorname{ann}(x)$ of an element x of M is $\{a \in A \mid ax = 0\}$. It is an ideal in A, which is proper if $x \neq 0$. I claim that any ideal $\mathfrak{a}$ that is maximal among the annihilators of nonzero elements of A is prime. Suppose $\mathfrak{a} = \operatorname{ann}(x)$, and let $ab \in \mathfrak{a}$, so that $abx = 0$. Then $\mathfrak{a} \subset (a) + \mathfrak{a} \subset \operatorname{ann}(bx)$. If $b \notin \mathfrak{a}$, then $bx \neq 0$, and so $\mathfrak{a} = \operatorname{ann}(bx)$ by maximality, which implies that $a \in \mathfrak{a}$.

We now prove the proposition. Note that, for any $x \in M$, the submodule Ax of M is isomorphic to $A/\operatorname{ann}(x)$. Therefore, if M is nonzero, then it contains a submodule M_1

isomorphic to $A/\mathfrak{p}_1$ for some prime ideal $\mathfrak{p}_1$. Similarly, M/M_1 contains a submodule M_2/M_1 isomorphic $A/\mathfrak{p}_2$ for some prime ideal $\mathfrak{p}_2$, and so on. The chain $0 \subset M_1 \subset M_2 \subset \cdots$ terminates because M is noetherian (by 3.4). □

THEOREM 3.6 (HILBERT BASIS THEOREM). *Every finitely generated algebra over a noetherian ring is noetherian.*

PROOF. Let A be noetherian. Since every finitely generated A-algebra is a quotient of a polynomial algebra, it suffices to prove the theorem for $A[X_1,\dots,X_n]$. Note that

$$A[X_1,\dots,X_n] = A[X_1,\dots,X_{n-1}][X_n]. \tag{4}$$

This simply says that every polynomial f in n symbols $X_1,\dots,X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1,\dots,X_{n-1}]$,

$$f(X_1,\dots,X_n) = a_0(X_1,\dots,X_{n-1})X_n^r + \cdots + a_r(X_1,\dots,X_{n-1}).$$

Thus an induction argument shows that it suffices to prove the theorem for $A[X]$.

Recall that for a polynomial

$$f(X) = c_0X^r + c_1X^{r-1} + \cdots + c_r, \quad c_i \in A, \quad c_0 \neq 0,$$

c_0 is the ***leading coefficient*** of f.

Let $\mathfrak{a}$ be an ideal in $A[X]$, and let $\mathfrak{c}_i$ be the set of elements of A that occur as the leading coefficient of a polynomial in $\mathfrak{a}$ of degree i (we also include 0). Then $\mathfrak{c}_i$ is an ideal in A, and $\mathfrak{c}_{i-1} \subset \mathfrak{c}_i$, because if $cX^{i-1} + \cdots \in \mathfrak{a}$, then so also does $X(cX^{i-1} + \cdots) = cX^i + \cdots$. As A is noetherian, the sequence of ideals

$$\mathfrak{c}_1 \subset \mathfrak{c}_2 \subset \cdots \subset \mathfrak{c}_i \subset \cdots$$

eventually becomes constant, say, $\mathfrak{c}_d = \mathfrak{c}_{d+1} = \dots$ (and then $\mathfrak{c}_d$ contains the leading coefficients of *all* polynomials in $\mathfrak{a}$).

For each $i \le d$, choose a finite generating set $\{c_{i1}, c_{i2}, \dots\}$ for $\mathfrak{c}_i$, and for each (i,j), choose a polynomial $f_{ij} \in \mathfrak{a}$ of degree i with leading coefficient c_{ij}. We shall show that the f_{ij}'s generate $\mathfrak{a}$.

Let $f \in \mathfrak{a}$; we have to show that $f \in (f_{ij})$. Suppose first that f has degree $s \ge d$. Then $f = cX^s + \cdots$ with $c \in \mathfrak{c}_d$, and so

$$c = \sum\nolimits_j a_j c_{dj}, \quad \text{some } a_j \in A.$$

Now

$$f - \sum\nolimits_j a_j f_{dj} X^{s-d}$$

is either zero and $f \in (f_{ij})$, or it has degree $< \deg(f)$. In the second case, we repeat the argument, until we obtain a polynomial f with degree $s < d$ that differs from the original polynomial by an element of (f_{ij}). By a similar argument, we then construct elements $a_j \in A$ such that

$$f - \sum\nolimits_j a_j f_{sj}$$

is either zero or has degree $< \deg(f)$. In the second case, we repeat the argument, until we obtain zero. □

PROPOSITION 3.7 (NAKAYAMA'S LEMMA). *Let $\mathfrak{a}$ be an ideal in a ring A contained in all maximal ideals of A, and let M be a finitely generated A-module.*

(a) *If $M = \mathfrak{a}M$, then $M = 0$.*

(b) *If N is a submodule of M such that $M = N + \mathfrak{a}M$, then $M = N$.*

PROOF. (a) Suppose $M \neq 0$. Choose a minimal set of generators $\{e_1, \dots, e_n\}$ for M, $n \geq 1$, and write

$$e_1 = a_1e_1 + \cdots + a_ne_n, \quad a_i \in \mathfrak{a}.$$

Then

$$(1-a_1)e_1 = a_2e_2 + \cdots + a_ne_n$$

and, as $1-a_1$ is a unit (see 2.8), $e_2, \dots, e_n$ generate M. This contradicts the minimality of the set.

(b) The hypothesis implies that $M/N = \mathfrak{a}(M/N)$, and so $M/N = 0$. □

Recall that the Jacobson radical $\mathfrak{J}$ of A is the intersection of the maximal ideals of A, and so the condition on $\mathfrak{a}$ is that $\mathfrak{a} \subset \mathfrak{J}$. For example, if A is local, then the proposition holds for every proper ideal.

COROLLARY 3.8. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k \stackrel{\text{def}}{=} A/\mathfrak{m}$, and let M be a finitely generated module over A. The action of A on $M/\mathfrak{m}M$ factors through k, and elements $a_1, \dots, a_n$ of M generate it as an A-module if and only if $a_1 + \mathfrak{m}M, \dots, a_n + \mathfrak{m}M$ span $M/\mathfrak{m}M$ as k-vector space.*

PROOF. If $a_1, \dots, a_n$ generate M, then it is obvious that their images generate the vector space $M/\mathfrak{m}M$. Conversely, suppose that $a_1 + \mathfrak{m}M, \dots, a_n + \mathfrak{m}M$ span $M/\mathfrak{m}M$, and let N be the submodule of M generated by $a_1, \dots, a_n$. The composite $N \to M \to M/\mathfrak{m}M$ is surjective, and so $M = N + \mathfrak{m}M$. Now Nakayama'a lemma shows that $M = N$. □

COROLLARY 3.9. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. Elements $a_1, \dots, a_n$ of $\mathfrak{m}$ generate $\mathfrak{m}$ as an ideal if and only if $a_1 + \mathfrak{m}^2, \dots, a_n + \mathfrak{m}^2$ span $\mathfrak{m}/\mathfrak{m}^2$ as a vector space over $k \stackrel{\text{def}}{=} A/\mathfrak{m}$. In particular, the minimum number of generators for the maximal ideal is equal to the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$.*

PROOF. Because A is noetherian, $\mathfrak{m}$ is finitely generated, and we can apply the preceding corollary with $M = \mathfrak{m}$. □

DEFINITION 3.10. Let A be a noetherian ring.

(a) The ***height*** $\operatorname{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ in A is the greatest length d of a chain of distinct prime ideals

$$\mathfrak{p} = \mathfrak{p}_d \supset \mathfrak{p}_{d-1} \supset \cdots \supset \mathfrak{p}_0. \tag{5}$$

(b) The ***(Krull) dimension*** of A is $\sup\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \subset A, \quad \mathfrak{p} \text{ prime}\}$.

Thus, the Krull dimension of a ring A is the supremum of the lengths of chains of prime ideals in A (the length of a chain is the number of gaps, so the length of (5) is d). For example, a field has Krull dimension 0, and conversely an integral domain of Krull dimension 0 is a field. The height of every nonzero prime ideal in a principal ideal domain is 1, and so such a ring has Krull dimension 1 (provided it is not a field). It is convenient to define the Krull dimension of the zero ring to be -1.

We shall see in §15 that the height of any prime ideal in a noetherian ring is finite. However, the Krull dimension of the ring may be infinite, because it may contain a sequence $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \dots$ of prime ideals such that $\operatorname{ht}(\mathfrak{p}_i)$ tends to infinity (see Krull 1938 or Nagata 1962, p.203, for examples).

LEMMA 3.11. *In a noetherian ring, every set of generators for an ideal contains a finite generating set.*

PROOF. Let $\mathfrak{a}$ be an ideal in a noetherian ring A, and let S be a set of generators for $\mathfrak{a}$. An ideal maximal in the set of ideals generated by finite subsets of S must contain every element of S (otherwise it wouldn't be maximal), and so equals $\mathfrak{a}$. □

THEOREM 3.12 (KRULL INTERSECTION THEOREM). *Let $\mathfrak{a}$ be an ideal in a noetherian ring A. If $\mathfrak{a}$ is contained in all maximal ideals of A, then $\bigcap_{n\geq 1}\mathfrak{a}^n = \{0\}$.*

PROOF. We shall show that, for *any* ideal $\mathfrak{a}$ in a noetherian ring,

$$\bigcap\nolimits_{n\geq 1}\mathfrak{a}^n = \mathfrak{a}\cdot\bigcap\nolimits_{n\geq 1}\mathfrak{a}^n. \tag{6}$$

When $\mathfrak{a}$ is contained in all maximal ideals of A, Nakayama's lemma shows that $\bigcap_{n\geq 1}\mathfrak{a}^n$ is zero.

Let $a_1,\dots,a_r$ generate $\mathfrak{a}$. Then $\mathfrak{a}^n$ consists of finite sums

$$\sum_{i_1+\cdots+i_r=n} c_{i_1\cdots i_r}a_1^{i_1}\cdots a_r^{i_r}, \quad c_{i_1\cdots i_r}\in A.$$

In other words, $\mathfrak{a}^n$ consists of the elements of A of the form $g(a_1,\dots,a_r)$ for some homogeneous polynomial $g(X_1,\dots,X_r)\in A[X_1,\dots,X_r]$ of degree n. Let S_m be the set of homogeneous polynomials f of degree m such that $f(a_1,\dots,a_r)\in\bigcap_{n\geq 1}\mathfrak{a}^n$, and let $\mathfrak{c}$ be the ideal in $A[X_1,\dots,X_r]$ generated by all the S_m. According to the lemma, there exists a finite set $\{f_1,\dots,f_s\}$ of elements of $\bigcup_m S_m$ that generates $\mathfrak{c}$. Let $d_i=\deg f_i$, and let $d=\max d_i$. Let $b\in\bigcap_{n\geq 1}\mathfrak{a}^n$; then $b\in\mathfrak{a}^{d+1}$, and so $b=f(a_1,\dots,a_r)$ for some homogeneous polynomial f of degree $d+1$. By definition, $f\in S_{d+1}\subset\mathfrak{a}$, and so

$$f = g_1f_1+\cdots+g_sf_s$$

for some $g_i\in A[X_1,\dots,X_n]$. As f and the f_i are homogeneous, we can omit from each g_i all terms not of degree $\deg f-\deg f_i$, since these terms cancel out. Thus, we may choose the g_i to be homogeneous of degree $\deg f-\deg f_i = d+1-d_i>0$. Then $g_i(a_1,\dots,a_r)\in\mathfrak{a}$, and so

$$b=f(a_1,\dots,a_r)=\textstyle\sum_i g_i(a_1,\dots,a_r)\cdot f_i(a_1,\dots,a_r)\in\mathfrak{a}\cdot\bigcap_n\mathfrak{a}^n,$$

which completes the proof of (6). □

The equality (6) can also be proved using primary decompositions — see (13.15).

PROPOSITION 3.13. *In a noetherian ring, every ideal contains a power of its radical; in particular, some power of the nilradical of the ring is zero.*

PROOF. Let $a_1,\dots,a_n$ generate $\operatorname{rad}(\mathfrak{a})$. For each i, some power of a_i, say $a_i^{r_i}$, lies in $\mathfrak{a}$. Then every term of the expansion of

$$(c_1a_1+\cdots+c_na_n)^{r_1+\cdots+r_n}, \quad c_i\in A,$$

has a factor of the form $a_i^{r_i}$ for some i, and so lies in $\mathfrak{a}$. □

4 Unique factorization

Let A be an integral domain, and let a be an element of A that is neither zero nor a unit. Then is said to be ***irreducible*** if it admits only trivial factorizations, i.e.,

$$a = bc \implies b \text{ or } c \text{ is a unit.}$$

The element a is said to be ***prime*** if (a) is a prime ideal, i.e.,

$$a|bc \implies a|b \text{ or } a|c.$$

An integral domain A is called a ***unique factorization domain*** if every nonzero nonunit in A can be written as a finite product of irreducible elements in exactly one way up to units and the order of the factors. Every principal ideal domain, for example, the polynomial ring $k[X]$ over a field k, is a unique factorization domain (proved in most algebra courses).

PROPOSITION 4.1. *Let A be an integral domain, and let a be an element of A that is neither zero nor a unit. If a is prime, then a is irreducible, and the converse holds when A is a unique factorization domain.*

PROOF. Assume that a is prime. If $a = bc$, then a divides bc and so a divides b or c. Suppose the first, and write $b = aq$. Now $a = bc = aqc$, which implies that $qc = 1$, and that c is a unit. Therefore a is irreducible.

For the converse, assume that a is irreducible. If $a|bc$, then

$$bc = aq, \text{ some } q \in A.$$

On writing each of b, c, and q as a product of irreducible elements, and using the uniqueness of factorizations, we see that a differs from one of the irreducible factors of b or c by a unit. Therefore a divides b or c. □

PROPOSITION 4.2 (GAUSS'S LEMMA). *Let A be a unique factorization domain with field of fractions F. If $f(X) \in A[X]$ factors into the product of two nonconstant polynomials in $F[X]$, then it factors into the product of two nonconstant polynomials in $A[X]$.*

PROOF. Let $f = gh$ in $F[X]$. For suitable $c, d \in A$, the polynomials $g_1 = cg$ and $h_1 = dh$ have coefficients in A, and so we have a factorization

$$cdf = g_1 h_1 \text{ in } A[X].$$

If an irreducible element p of A divides cd, then, looking modulo (p), we see that

$$0 = \overline{g_1} \cdot \overline{h_1} \text{ in } (A/(p))[X].$$

According to Proposition 4.1, the ideal (p) is prime, and so $(A/(p))[X]$ is an integral domain. Therefore, p divides all the coefficients of at least one of the polynomials g_1, h_1, say g_1, so that $g_1 = pg_2$ for some $g_2 \in A[X]$. Thus, we have a factorization

$$(cd/p)f = g_2 h_1 \text{ in } A[X].$$

Continuing in this fashion, we can remove all the irreducible factors of cd, and so obtain a factorization of f in $A[X]$. □

Let A be a unique factorization domain. A nonzero polynomial

$$f = a_0 + a_1X + \cdots + a_mX^m$$

in $A[X]$ is said to be ***primitive*** if the coefficients a_i have no common factor other than units. Every polynomial f in $A[X]$ can be written $f = c(f) \cdot f_1$ with $c(f) \in A$ and f_1 primitive. The element $c(f)$, well-defined up to multiplication by a unit, is called the ***content*** of f.

LEMMA 4.3. *The product of two primitive polynomials is primitive.*

PROOF. Let

$$f = a_0 + a_1X + \cdots + a_mX^m$$
$$g = b_0 + b_1X + \cdots + b_nX^n,$$

be primitive polynomials, and let p be an irreducible element of A. Let a_{i_0} be the first coefficient of f not divisible by p and b_{j_0} the first coefficient of g not divisible by p. Then all the terms in $\sum_{i+j=i_0+j_0} a_ib_j$ are divisible by p, except $a_{i_0}b_{j_0}$, which is not divisible by p. Therefore, p doesn't divide the $(i_0 + j_0)$th-coefficient of fg. We have shown that no irreducible element of A divides all the coefficients of fg, which must therefore be primitive. □

LEMMA 4.4. *For polynomials $f, g \in A[X]$, $c(fg) = c(f) \cdot c(g)$; hence every factor in $A[X]$ of a primitive polynomial is primitive.*

PROOF. Let $f = c(f)f_1$ and $g = c(g)g_1$ with f_1 and g_1 primitive. Then $fg = c(f)c(g)f_1g_1$ with f_1g_1 primitive, and so $c(fg) = c(f)c(g)$. □

PROPOSITION 4.5. *If A is a unique factorization domain, then so also is $A[X]$.*

PROOF. From the factorization $f = c(f)f_1$, we see that the irreducible elements of $A[X]$ are to be found among the constant polynomials and the primitive polynomials, but a constant polynomial a is irreducible if and only if a is an irreducible element of A (obvious) and a primitive polynomial is irreducible if and only if it has no primitive factor of lower degree (by 4.4). From this it is clear that every nonzero nonunit f in $A[X]$ is a product of irreducible elements.

Let

$$f = c_1 \cdots c_m f_1 \cdots f_n = d_1 \cdots d_r g_1 \cdots g_s$$

be two factorizations of an element f of $A[X]$ into irreducible elements with the c_i, d_j constants and the f_i, g_j primitive polynomials. Then

$$c(f) = c_1 \cdots c_m = d_1 \cdots d_r \text{ (up to units in } A),$$

and, on using that A is a unique factorization domain, we see that $m = r$ and the c_i's differ from the d_i's only by units and ordering. Hence,

$$f_1 \cdots f_n = g_1 \cdots g_s \text{ (up to units in } A).$$

Gauss's lemma shows that the f_i, g_j are irreducible polynomials in $F[X]$ and, on using that $F[X]$ is a unique factorization domain, we see that $n = s$ and that the f_i's differ from the g_i's only by units in F and by their ordering. But if $f_i = \frac{a}{b}g_j$ with a and b nonzero elements of A, then $bf_i = ag_j$. As f_i and g_j are primitive, this implies that $b = a$ (up to a unit in A), and hence that $\frac{a}{b}$ is a unit in A. □

Let k be a field. A ***monomial*** in $X_1,\dots,X_n$ is an expression of the form

$$X_1^{a_1}\cdots X_n^{a_n},\quad a_j\in\mathbb{N}.$$

The ***total degree*** of the monomial is $\sum a_i$. The ***degree***, $\deg(f)$, of a nonzero polynomial $f(X_1,\dots,X_n)$ is the largest total degree of a monomial occurring in f with nonzero coefficient. Since

$$\deg(fg)=\deg(f)+\deg(g),$$

$k[X_1,\dots,X_n]$ is an integral domain and $k[X_1,\dots,X_n]^\times=k^\times$. Therefore, an element f of $k[X_1,\dots,X_n]$ is irreducible if it is nonconstant and $f=gh\implies g$ or h is constant.

THEOREM 4.6. *The ring $k[X_1,\dots,X_n]$ is a unique factorization domain.*

PROOF. This is trivially true when $n=0$, and an induction argument using (4), p.10, proves it for all n. □

COROLLARY 4.7. *A nonzero proper principal ideal (f) in $k[X_1,\dots,X_n]$ is prime if and only f is irreducible.*

PROOF. Special case of (4.1). □

5 Integrality

Let A be a subring of a ring B. An element α of B is said to be ***integral*** over A if it is a root of a monic[5] polynomial with coefficients in A, i.e., if it satisfies an equation

$$\alpha^n+a_1\alpha^{n-1}+\cdots+a_n=0,\quad a_i\in A.$$

If every element of B is integral over A, then B is said to be ***integral*** over A.

In the next proof, we shall need to apply Cramer's formula. As usually stated in linear algebra courses, this says that, if $x_1,\dots,x_m$ is a solution to the system of linear equations

$$\sum_{j=1}^{m}c_{ij}x_j=d_i,\quad i=1,\dots,m,$$

then

$$x_j=\frac{\det(C_j)}{\det(C)},\quad \text{where } C=(c_{ij}) \text{ and}$$

$$C_j=\begin{pmatrix} c_{11} & \cdots & c_{1,j-1} & d_1 & c_{1,j+1} & \cdots & c_{1m}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ c_{m1} & \cdots & c_{m,j-1} & d_m & c_{m,j+1} & \cdots & c_{mm}\end{pmatrix}.$$

When one restates the formula as

$$\det(C)\cdot x_j=\det(C_j)$$

[5] A polynomial is ***monic*** if its leading coefficient is 1, i.e., $f(X)=X^n+$ terms of degree less than n.

it becomes true over any ring (whether or not $\det(C)$ is a unit). The proof is elementary—expand out the right hand side of

$$\det C_j = \det \begin{pmatrix} c_{11} & \dots & \sum c_{1j}x_j & \dots & c_{1m} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & \sum c_{mj}x_j & \dots & c_{mm} \end{pmatrix}$$

using standard properties of determinants.

PROPOSITION 5.1. *Let A be a subring of a ring B. An element α of B is integral over A if and only if there exists a faithful*[6] *finitely generated A-submodule M of B such that $\alpha M \subset M$.*

PROOF. $\Rightarrow$: Suppose

$$\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0, \quad a_i \in A.$$

Then the A-submodule M of B generated by $1, \alpha, ..., \alpha^{n-1}$ has the property that $\alpha M \subset M$, and it is faithful because it contains 1.

$\Leftarrow$: Let M be a nonzero A-module in B such that $\alpha M \subset M$, and let $e_1, \dots, e_n$ be a finite set of generators for M. Then, for each i,

$$\alpha e_i = \sum a_{ij}e_j, \text{ some } a_{ij} \in A.$$

We can rewrite this system of equations as

$$\begin{aligned} (\alpha - a_{11})e_1 - a_{12}e_2 - a_{13}e_3 - \cdots &= 0 \\ -a_{21}e_1 + (\alpha - a_{22})e_2 - a_{23}e_3 - \cdots &= 0 \\ \cdots &= 0. \end{aligned}$$

Let C be the matrix of coefficients on the left-hand side. Then Cramer's formula tells us that $\det(C) \cdot e_i = 0$ for all i. As the e_i generate M and M is faithful, this implies that $\det(C) = 0$. On expanding out the determinant, we obtain an equation

$$\alpha^n + c_1\alpha^{n-1} + c_2\alpha^{n-2} + \cdots + c_n = 0, \quad c_i \in A. \qquad \square$$

PROPOSITION 5.2. *An A-algebra B is finite if and only if it is finitely generated and integral over A.*

PROOF. $\Leftarrow$: Suppose $B = A[\alpha_1, \dots, \alpha_m]$ and that

$$\alpha_i^{n_i} + a_{i1}\alpha_i^{n_i-1} + \cdots + a_{in_i} = 0, \quad a_{ij} \in A, \quad i = 1, \dots, m.$$

Any monomial in the α_i's divisible by $\alpha_i^{n_i}$ is equal (in B) to a linear combination of monomials of lower degree. Therefore, B is generated as an A-module by the monomials $\alpha_1^{r_1} \cdots \alpha_m^{r_m}$, $1 \le r_i < n_i$.

$\Rightarrow$: As an A-module, B is faithful (because $a \cdot 1_B = a$), and so (5.1) implies that every element of B is integral over A. As B is finitely generated as an A-module, it is certainly finitely generated as an A-algebra. $\square$

[6] An A-module M is ***faithful*** if $aM = 0$, $a \in A$, implies $a = 0$.

THEOREM 5.3. *Let A be a subring of the ring B. The elements of B integral over A form a subring of B.*

PROOF. Let α and β be two elements of B integral over A. The argument in the proof of (5.2) shows that $A[\alpha,\beta]$ is finitely generated as an A-module. It is obviously a faithful A-module, and it is stable under multiplication by $\alpha\pm\beta$ and $\alpha\beta$. Therefore (5.1) shows that $\alpha\pm\beta$ and $\alpha\beta$ are integral over A. □

DEFINITION 5.4. Let A be a subring of the ring B. The ***integral closure*** of A in B is the subring of B consisting of the elements integral over A .

PROPOSITION 5.5. *Let A be an integral domain with field of fractions F, and let L be a field containing F. If $\alpha\in L$ is algebraic over F, then there exists a $d\in A$ such that $d\alpha$ is integral over A.*

PROOF. By assumption, α satisfies an equation

$$\alpha^m+a_1\alpha^{m-1}+\cdots+a_m=0,\quad a_i\in F.$$

Let d be a common denominator for the a_i, so that $da_i\in A$ for all i, and multiply through the equation by d^m:

$$d^m\alpha^m+a_1d^m\alpha^{m-1}+\cdots+a_md^m=0.$$

We can rewrite this as

$$(d\alpha)^m+a_1d(d\alpha)^{m-1}+\cdots+a_md^m=0.$$

As $a_1d,\dots,a_md^m\in A$, this shows that $d\alpha$ is integral over A. □

COROLLARY 5.6. *Let A be an integral domain and let L be an algebraic extension of the field of fractions of A. Then L is the field of fractions of the integral closure of A in L.*

PROOF. In fact, the proposition shows that every element of L is a quotient β/d with β integral over A and $d\in A$. □

DEFINITION 5.7. An integral domain A is ***integrally closed*** if it is equal to its integral closure in its field of fractions F, i.e., if

$$\alpha\in F,\quad \alpha \text{ integral over } A \implies \alpha\in A.$$

PROPOSITION 5.8. *Every unique factorization domain is integrally closed.*

PROOF. An element of the field of fractions of A not in A can be written a/b with $a,b\in A$ and b divisible by some irreducible element p not dividing a. If a/b is integral over A, then it satisfies an equation

$$(a/b)^n+a_1(a/b)^{n-1}+\cdots+a_n=0,\quad a_i\in A.$$

On multiplying through by b^n, we obtain the equation

$$a^n+a_1a^{n-1}b+\cdots+a_nb^n=0.$$

The element p then divides every term on the left except a^n, and hence must divide a^n. Since it doesn't divide a, this is a contradiction. □

PROPOSITION 5.9. *Let A be an integrally closed integral domain, and let L be a finite extension of the field of fractions F of A. An element of L is integral over A if and only if its minimum polynomial[7] over F has coefficients in A.*

PROOF. Let α be integral over A, so that

$$\alpha^m + a_1\alpha^{m-1} + \cdots + a_m = 0, \quad \text{some } a_i \in A, \quad m > 0.$$

Let α' be a conjugate of α, i.e., a root of the minimum polynomial $f(X)$ of α over F in some field containing L. Then there is an F-isomorphism[8]

$$\sigma\colon F[\alpha] \to F[\alpha'], \quad \sigma(\alpha) = \alpha'$$

On applying σ to the above equation we obtain the equation

$$\alpha'^m + a_1\alpha'^{m-1} + \cdots + a_m = 0,$$

which shows that α' is integral over A. Hence all the conjugates of α are integral over A, and it follows from (5.3) that the coefficients of $f(X)$ are integral over A. They lie in F, and A is integrally closed, and so they lie in A. This proves the "only if" part of the statement, and the "if" part is obvious. □

COROLLARY 5.10. *Let A be an integrally closed integral domain with field of fractions F, and let $f(X)$ be a monic polynomial in $A[X]$. Then every monic factor of $f(X)$ in $F[X]$ has coefficients in A.*

PROOF. It suffices to prove this for an irreducible monic factor g of f in $F[X]$. Let α be a root of g in some extension field of F. Then g is the minimum polynomial α, which, being also a root of f, is integral. Therefore g has coefficients in A. □

THEOREM 5.11 (NOETHER NORMALIZATION THEOREM). *Every finitely generated algebra A over a field k contains a polynomial algebra R such that A is a finite R-algebra.*

In other words, there exist elements $y_1, \ldots, y_r$ of A such that A is a finite $k[y_1, \ldots, y_r]$-algebra and $y_1, \ldots, y_r$ are algebraically independent[9] over k.

PROOF. We use induction on the minimum number n of generators of A as a k-algebra. If $n = 0$, there is nothing to prove, and so we may suppose that $n \geq 1$ and that the statement is true for k-algebras generated by $n-1$ (or fewer) elements.

Let $A = k[x_1, \ldots, x_n]$. If the x_i are algebraically independent, then there is nothing to prove, and so we may suppose that there exists a nonconstant polynomial $f(T_1, \ldots, T_n)$ such that $f(x_1, \ldots, x_n) = 0$. Some T_i occurs in f, say T_1, and we can write

$$f = c_0T_1^N + c_1T_1^{N-1} + \cdots + c_N, \quad c_i \in k[T_2, \ldots, T_n], \quad c_0 \neq 0.$$

[7]Most authors write "minimal polynomial" but the polynomial in question is in fact minimum (smallest element in the set of monic polynomials having α as a root).

[8]Recall that the homomorphism $X \mapsto \alpha\colon F[X] \to F[\alpha]$ defines an isomorphism $F[X]/(f) \to F[\alpha]$.

[9]Recall that this means that the homomorphism of k-algebras $k[X_1, \ldots, X_n] \to k[y_1, \ldots, y_n]$ sending X_i to y_i is an isomorphism, or, equivalently, that if

$$P(y_1, \ldots, y_n) = 0, \quad P(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n],$$

then $P = 0$.

If $c_0 \in k$, then the equation

$$c_0 x_1^N + c_1(x_2,\dots,x_n)x_1^{N-1} + \cdots + c_N(x_2,\dots,x_n) = 0$$

shows that x_1 is integral over $k[x_2,\dots,x_n]$. By induction, there exist algebraically independent elements $y_1,\dots,y_r$ such that $k[x_2,\dots,x_n]$ is finite over $k[y_1,\dots,y_r]$. It follows that A is finite over $k[y_1,\dots,y_r]$.

If $c_0 \notin k$, then we choose different generators for A. Fix an integer $m > 0$, and let

$$y_1 = x_1, y_2 = x_2 - x_1^{m^2}, \dots, y_r = x_r - x_1^{m^r}.$$

Then

$$k[y_1,\dots,y_n] = k[x_1,\dots,x_n] = A$$

and

$$f(y_1, y_2 + y_1^{m^2}, \dots, y_r + y_1^{m^r}) = 0.$$

In other words, $g(y_1,\dots,y_n) = 0$ where $g(T_1,\dots,T_n) = f(T_1, T_2 + T_1^{m^2},\dots,T_r + T_1^{m^r})$. When m is chosen sufficiently large,

$$g(T_1,\dots,T_n) = c_0' T_1^N + c_1' T_1^{N-1} + \cdots + c_N', \quad c_i' \in k[T_2,\dots,T_r], \quad c_0' \neq 0$$

with $c_0' \in k$.[10] Therefore, the previous argument applies. □

REMARK 5.12. When k is infinite, there is a simpler proof of a somewhat stronger result: let $A = k[x_1,\dots,x_n]$; then there exist algebraically independent elements $f_1,\dots,f_r$ that are *linear combinations* of the x_i such that A is finite over $k[f_1,\dots,f_r]$ (see 8.13 of my algebraic geometry notes).

6 Rings of fractions

Recall that a multiplicative subset of a ring is a nonempty subset closed under the formation of finite products.

Let S be a multiplicative subset of A, and define an equivalence relation on $A \times S$ by

$$(a,s) \sim (b,t) \iff u(at - bs) = 0 \text{ for some } u \in S.$$

Write $\frac{a}{s}$ for the equivalence class containing (a,s), and define addition and multiplication of equivalence classes in the way suggested by the notation:

$$\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}, \quad \frac{a}{s}\frac{b}{t} = \frac{ab}{st}.$$

[10] Let

$$f(T_1,\dots,T_r) = \sum c_{j_1\cdots j_r} T_1^{j_1} \cdots T_r^{j_r}.$$

If m is chosen so large that the numbers

$$j_1 + m^2 j_2 + \cdots + m^r j_r,$$

with $j_1,\dots,j_r$ running over the r-tuples such that $c_{j_1,\dots,j_r} \neq 0$, are distinct, say with largest value N, then

$$f(T_1, T_2 + T_1^{m^2},\dots,T_r + T_1^{m^r}) = cT_1^N + c_1 T_1^{N-1} + \cdots$$

with $c \in k \smallsetminus \{0\}$.

It is easy to show that these do not depend on the choices of representatives for the equivalence classes, and that we obtain in this way a ring

$$S^{-1}A = \{\tfrac{a}{s} \mid a \in A, s \in S\}$$

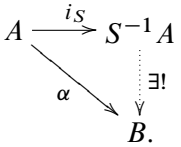
and a ring homomorphism $a \mapsto \frac{a}{1}: A \xrightarrow{i_S} S^{-1}A$ whose kernel is

$$\{a \in A \mid sa = 0 \text{ for some } s \in S\}.$$

If S contains no zero-divisors, for example, if A is an integral domain and $0 \notin S$, then $i_S: A \to S^{-1}A$ is injective. At the opposite extreme, if $0 \in S$, then $S^{-1}A$ is the zero ring.

PROPOSITION 6.1. *The pair $(S^{-1}A, i_S)$ has the following universal property:*

every element of S maps to a unit in $S^{-1}A$, and any other ring homomorphism $\alpha: A \to B$ with this property factors uniquely through i_S



PROOF. Let $\alpha: A \to B$ be a homomorphism, and let $\beta: S^{-1}A \to B$ be a homomorphism such that $\beta \circ i_S = \alpha$. Then

$$\tfrac{s}{1}\tfrac{a}{s} = \tfrac{a}{1} \implies \beta(\tfrac{s}{1})\beta(\tfrac{a}{s}) = \beta(\tfrac{a}{1}),$$

and so

$$\beta(\tfrac{a}{s}) = \alpha(a)\alpha(s)^{-1}. \tag{7}$$

This shows that there can be at most one β such that $\beta \circ i_S = \alpha$. When α maps the elements of S to units in B, we define β by the formula (7). Then

$$\tfrac{a}{s} = \tfrac{b}{t} \implies u(at - bs) = 0 \text{ some } u \in S \overset{\alpha(u) \in B^\times}{\implies} \alpha(a)\alpha(t) - \alpha(b)\alpha(s) = 0,$$

which shows that β is well-defined, and it is easy to check that it is a homomorphism. □

As usual, this universal property determines the pair $(S^{-1}A, i_S)$ uniquely up to a unique isomorphism.[11]

When A is an integral domain and $S = A \setminus \{0\}$, the ring $S^{-1}A$ is the field of fractions F of A. In this case, for any other multiplicative subset T of A not containing 0, the ring $T^{-1}A$ can be identified with the subring of F consisting of the fractions $\frac{a}{t}$ with $a \in A$ and $t \in T$.

EXAMPLE 6.2. Let $h \in A$. Then $S_h = \{1, h, h^2, \dots\}$ is a multiplicative subset of A, and we let $A_h = S_h^{-1}A$. Thus every element of A_h can be written in the form a/h^m, $a \in A$, and

$$\tfrac{a}{h^m} = \tfrac{b}{h^n} \iff h^N(ah^n - bh^m) = 0, \quad \text{some } N.$$

If h is nilpotent, then $A_h = 0$, and if A is an integral domain with field of fractions F and $h \neq 0$, then A_h is the subring of F of elements of the form a/h^m, $a \in A$, $m \in \mathbb{N}$.

[11] Recall the proof: let (A_1, i_1) and (A_2, i_2) have the universal property in the proposition; because every element of S maps to a unit in A_2, there exists a unique homomorphism $\alpha: A_1 \to A_2$ such that $\alpha \circ i_1 = i_2$ (universal property of A_1, i_1); similarly, there exists a unique homomorphism $\alpha': A_2 \to A_1$ such that $\alpha' \circ i_2 = i_1$; now

$$\alpha' \circ \alpha \circ i_1 = \alpha' \circ i_2 = i_1 = \mathrm{id}_{A_1} \circ i_1,$$

and so $\alpha' \circ \alpha = \mathrm{id}_{A_1}$ (universal property of A_1, i_1); similarly, $\alpha \circ \alpha' = \mathrm{id}_{A_2}$, and so α and α' are inverse isomorphisms (and they are uniquely determined by the conditions $\alpha \circ i_1 = i_2$ and $\alpha' \circ i_2 = i_1$).

PROPOSITION 6.3. *For any ring A and $h \in A$, the map $\sum a_i X^i \mapsto \sum \frac{a_i}{h^i}$ defines an isomorphism*

$$A[X]/(1-hX) \to A_h.$$

PROOF. If $h = 0$, both rings are zero, and so we may assume $h \neq 0$. In the ring $A[x] = A[X]/(1-hX)$, $1 = hx$, and so h is a unit. Let $\alpha: A \to B$ be a homomorphism of rings such that $\alpha(h)$ is a unit in B. The homomorphism $\sum a_i X^i \mapsto \sum \alpha(a_i)\alpha(h)^{-i}: A[X] \to B$ factors through $A[x]$ because $1-hX \mapsto 1-\alpha(h)\alpha(h)^{-1} = 0$, and this is the unique extension of α to $A[x]$. Therefore $A[x]$ has the same universal property as A_h, and so the two are (uniquely) isomorphic by an A-algebra isomorphism that makes h^{-1} correspond to x. □

Let S be a multiplicative subset of a ring A, and let $S^{-1}A$ be the corresponding ring of fractions. For any ideal $\mathfrak{a}$ in A, the ideal generated by the image of $\mathfrak{a}$ in $S^{-1}A$ is

$$S^{-1}\mathfrak{a} = \{\tfrac{a}{s} \mid a \in \mathfrak{a}, \quad s \in S\}.$$

If $\mathfrak{a}$ contains an element of S, then $S^{-1}\mathfrak{a}$ contains 1, and so is the whole ring. Thus some of the ideal structure of A is lost in the passage to $S^{-1}A$, but, as the next lemma shows, some is retained.

PROPOSITION 6.4. *Let S be a multiplicative subset of the ring A, and consider extension $\mathfrak{a} \mapsto \mathfrak{a}^e = S^{-1}\mathfrak{a}$ and contraction $\mathfrak{a} \mapsto \mathfrak{a}^c = \{a \in A \mid \frac{a}{1} \in \mathfrak{a}\}$ of ideals with respect to the homomorphism $A \to S^{-1}A$. Then*

$$\mathfrak{a}^{ce} = \mathfrak{a} \qquad \text{for all ideals of } S^{-1}A$$
$$\mathfrak{a}^{ec} = \mathfrak{a} \qquad \text{if } \mathfrak{a} \text{ is a prime ideal of } A \text{ disjoint from } S.$$

Moreover, the $\mathfrak{p} \mapsto \mathfrak{p}^e$ is a bijection from the set of prime ideals of A disjoint from S onto the set of all prime ideals of $S^{-1}A$; the inverse map is $\mathfrak{p} \mapsto \mathfrak{p}^c$.

PROOF. Let $\mathfrak{a}$ be an ideal in $S^{-1}A$. Certainly $\mathfrak{a}^{ce} \subset \mathfrak{a}$. For the reverse inclusion, let $b \in \mathfrak{a}$. We can write $b = \frac{a}{s}$ with $a \in A$, $s \in S$. Then $\frac{a}{1} = s(\frac{a}{s}) \in \mathfrak{a}$, and so $a \in \mathfrak{a}^c$. Thus $b = \frac{a}{s} \in \mathfrak{a}^{ce}$, and so $\mathfrak{a} \subset \mathfrak{a}^{ce}$.

Let $\mathfrak{p}$ be a prime ideal of A disjoint from S. Clearly $\mathfrak{p}^{ec} \supset \mathfrak{p}$. For the reverse inclusion, let $a \in \mathfrak{p}^{ec}$ so that $\frac{a}{1} = \frac{a'}{s}$ for some $a' \in \mathfrak{p}$, $s \in S$. Then $t(as - a') = 0$ for some $t \in S$, and so $ast \in \mathfrak{p}$. Because $st \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, this implies that $a \in \mathfrak{p}$, and so $\mathfrak{p}^{ec} \subset \mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of A disjoint from S, and let $\bar{S}$ be the image of S in $A/\mathfrak{p}$. Then $(S^{-1}A)/\mathfrak{p}^e \simeq \bar{S}^{-1}(A/\mathfrak{p})$ because $S^{-1}A/\mathfrak{p}^e$ has the correct universal property, and $\bar{S}^{-1}(A/\mathfrak{p})$ is an integral domain because $A/\mathfrak{p}$ is an integral domain and $\bar{S}$ doesn't contain 0. Therefore $\mathfrak{p}^e$ is prime. From §2 we know that $\mathfrak{p}^c$ is prime if $\mathfrak{p}$ is, and so $\mathfrak{p} \mapsto \mathfrak{p}^e$ and $\mathfrak{p} \mapsto \mathfrak{p}^c$ are inverse bijections on the two sets. □

COROLLARY 6.5. *If A is noetherian, then so also is $S^{-1}A$ for any multiplicative set S.*

PROOF. As $\mathfrak{b}^c$ is finitely generated, so also is $(\mathfrak{b}^c)^e = \mathfrak{b}$. □

EXAMPLE 6.6. Let $\mathfrak{p}$ be a prime ideal in A. Then $S_{\mathfrak{p}} = A \smallsetminus \mathfrak{p}$ is a multiplicative subset of A, and we let $A_{\mathfrak{p}} = S_{\mathfrak{p}}^{-1}A$. Thus each element of $A_{\mathfrak{p}}$ can be written in the form $\frac{a}{c}$, $c \notin \mathfrak{p}$, and

$$\tfrac{a}{c} = \tfrac{b}{d} \iff s(ad - bc) = 0, \text{ some } s \notin \mathfrak{p}.$$

According to (6.4b), the prime ideals of $A_{\mathfrak{p}}$ correspond to the prime ideals of A disjoint from $A \smallsetminus \mathfrak{p}$, i.e., contained in $\mathfrak{p}$. Therefore, $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{p}^e = \{\frac{a}{s} \mid a \in \mathfrak{p}, s \notin \mathfrak{p}\}$.

PROPOSITION 6.7. *Let $\mathfrak{m}$ be a maximal ideal of a noetherian ring A, and let $\mathfrak{n} = \mathfrak{m}A_{\mathfrak{m}}$ be the maximal ideal of $A_{\mathfrak{m}}$. For all n, the map*

$$a + \mathfrak{m}^n \mapsto a + \mathfrak{n}^n : A/\mathfrak{m}^n \to A_{\mathfrak{m}}/\mathfrak{n}^n$$

is an isomorphism. Moreover, it induces isomorphisms

$$\mathfrak{m}^r/\mathfrak{m}^n \to \mathfrak{n}^r/\mathfrak{n}^n$$

for all pairs (r,n) with $r < n$.

PROOF. The second statement follows from the first, because of the exact commutative diagram ($r < n$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}^r/\mathfrak{m}^n & \longrightarrow & A/\mathfrak{m}^n & \longrightarrow & A/\mathfrak{m}^r & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\simeq & & \downarrow\simeq & & \\ 0 & \longrightarrow & \mathfrak{n}^r/\mathfrak{n}^n & \longrightarrow & A_{\mathfrak{m}}/\mathfrak{n}^n & \longrightarrow & A_{\mathfrak{m}}/\mathfrak{n}^r & \longrightarrow & 0. \end{array}$$

We consider extension and contraction with respect to $a \mapsto \frac{a}{1} : A \to A_{\mathfrak{m}}$. In order to show that the map $A/\mathfrak{m}^n \to A_{\mathfrak{m}}/\mathfrak{n}^n$ is injective, we have to show that $(\mathfrak{m}^n)^{ec} = \mathfrak{m}^n$. If $a \in (\mathfrak{m}^n)^{ec}$, then $\frac{a}{1} = \frac{b}{s}$ with $b \in \mathfrak{m}^n$ and $s \in S$. Then $s'sa \in \mathfrak{m}^n$ for some $s' \in S$, and so $s'sa = 0$ in $A/\mathfrak{m}^n$. The only maximal ideal containing $\mathfrak{m}^n$ is $\mathfrak{m}$, and so the only maximal ideal in $A/\mathfrak{m}^n$ is $\mathfrak{m}/\mathfrak{m}^n$. As $s's$ is not in $\mathfrak{m}/\mathfrak{m}^n$, it must be a unit in $A/\mathfrak{m}^n$, and so $a = 0$ in $A/\mathfrak{m}^n$, i.e., $a \in \mathfrak{m}^n$. We have shown that $(\mathfrak{m}^n)^{ec} \subset \mathfrak{m}$, and the reverse inclusion is always true.

We now prove that $A/\mathfrak{m}^n \to A_{\mathfrak{m}}/\mathfrak{n}^n$ is surjective. Let $\frac{a}{s} \in A_{\mathfrak{m}}$, $a \in A$, $s \in A \setminus \mathfrak{m}$. The only maximal ideal of A containing $\mathfrak{m}^n$ is $\mathfrak{m}$, and so no maximal ideal contains both s and $\mathfrak{m}^n$; it follows that $(s) + \mathfrak{m}^n = A$. Therefore, there exist $b \in A$ and $q \in \mathfrak{m}^n$ such that $sb + q = 1$. Because s is invertible in $A_{\mathfrak{m}}/\mathfrak{n}^n$, $\frac{a}{s}$ is the *unique* element of this ring such that $s\frac{a}{s} = a$. As $s(ba) = a(1-q)$, the image of ba in $A_{\mathfrak{m}}$ also has this property and therefore equals $\frac{a}{s}$. □

PROPOSITION 6.8. *In a noetherian ring, only 0 lies in all powers of all maximal ideals.*

PROOF. Let a be an element of a noetherian ring A. If $a \neq 0$, then $\{b \mid ba = 0\}$ is a proper ideal, and so it is contained in some maximal ideal $\mathfrak{m}$. Then $\frac{a}{1}$ is nonzero in $A_{\mathfrak{m}}$, and so $\frac{a}{1} \notin (\mathfrak{m}A_{\mathfrak{m}})^n$ for some n (by the Krull intersection theorem 3.12), which implies that $a \notin \mathfrak{m}^n$ (by 6.7). □

MODULES OF FRACTIONS

Let S be a multiplicative subset of the ring A, and let M be an A-module. Define an equivalence relation on $M \times S$ by

$$(m,s) \sim (n,t) \iff u(mt - ns) = 0 \text{ for some } u \in S.$$

Write $\frac{m}{s}$ for the equivalence class containing (m,s), and define addition and multiplication of equivalence classes by the formulas:

$$\frac{m}{s} + \frac{n}{t} = \frac{mt+ns}{st}, \quad \frac{a}{s}\frac{m}{t} = \frac{am}{st}, \quad m,n \in M, \quad s,t \in S, \quad a \in A.$$

It is easy to show that these definitions do not depend on the choices of representatives for the equivalence classes, and that we obtain in this way an $S^{-1}A$-module

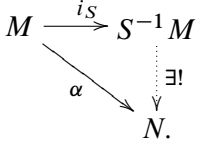
$$S^{-1}M = \{\tfrac{m}{s} \mid m \in M, s \in S\}$$

and a homomorphism $m \mapsto \frac{m}{1}: M \xrightarrow{i_S} S^{-1}M$ of A-modules whose kernel is

$$\{a \in M \mid sa = 0 \text{ for some } s \in S\}.$$

PROPOSITION 6.9. *The pair $(S^{-1}M, i_S)$ has the following universal property:*

every element of S acts invertibly on $S^{-1}M$, and any other homomorphism $\alpha: M \to N$ of A-modules such that the elements of S act invertibly on N factors uniquely through i_S



PROOF. Similar to that of Proposition 6.1. □

EXAMPLE 6.10. Let M be an A-module. For $h \in A$, let $M_h = S_h^{-1}M$ where $S_h = \{1, h, h^2, \ldots\}$. Then every element of M_h can be written in the form $\frac{m}{h^r}$, $m \in M$, $r \in \mathbb{N}$, and $\frac{m}{h^r} = \frac{m'}{h^{r'}}$ if and only if $h^N(h^{r'}m - h^r m') = 0$ for some $N \in \mathbb{N}$.

PROPOSITION 6.11. *The functor $M \rightsquigarrow S^{-1}M$ is exact. In other words, if the sequence of A-modules*

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

is exact, then so also is the sequence of $S^{-1}A$-modules

$$S^{-1}M' \xrightarrow{S^{-1}\alpha} S^{-1}M \xrightarrow{S^{-1}\beta} S^{-1}M''.$$

PROOF. Because $\beta \circ \alpha = 0$, we have $0 = S^{-1}(\beta \circ \alpha) = S^{-1}\beta \circ S^{-1}\alpha$. Therefore $\mathrm{Im}(S^{-1}\alpha) \subset \mathrm{Ker}(S^{-1}\beta)$. For the reverse inclusion, let $\frac{m}{s} \in \mathrm{Ker}(S^{-1}\beta)$ where $m \in M$ and $s \in S$. Then $\frac{\beta(m)}{s} = 0$ and so, for some $t \in S$, we have $t\beta(m) = 0$. Then $\beta(tm) = 0$, and so $tm = \alpha(m')$ for some $m' \in M'$. Now

$$\frac{m}{s} = \frac{tm}{ts} = \frac{\alpha(m')}{ts} \in \mathrm{Im}(S^{-1}\alpha).$$ □

EXERCISE 6.12. Let A be an integral domain. A multiplicative subset S of A is said to be *saturated* if

$$ab \in S \Rightarrow a \text{ and } b \in S.$$

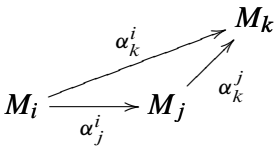
(a) Show that S is saturated $\iff$ its complement is a union of prime ideals.
(b) Show that given a multiplicative system S, there is a unique smallest saturated multiplicative system S' containing S, and that $S' = A \setminus \bigcup \mathfrak{p}$, where $\mathfrak{p}$ runs over the prime ideals disjoint from S. Show that $S'^{-1}A = S^{-1}A$. Deduce that $S^{-1}A$ is characterized by the set of prime ideals of A that remain prime in $S^{-1}A$.

7 Direct limits

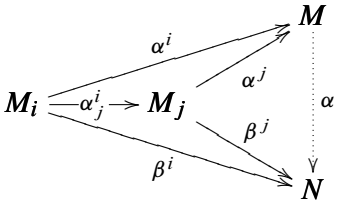
DEFINITION 7.1. A partial ordering $\leq$ on a set I is said to be ***directed***, and the pair $(I, \leq)$ is called a ***directed set***, if for all $i, j \in I$ there exists a $k \in I$ such that $i, j \leq k$.

DEFINITION 7.2. Let $(I, \leq)$ be a directed set, and let A be a ring.

A ***direct system*** of A-modules indexed by $(I, \leq)$ is a family $(M_i)_{i \in I}$ of A-modules together with a family $(\alpha^i_j : M_i \to M_j)_{i \leq j}$ of A-linear maps such that $\alpha^i_i = \mathrm{id}_{M_i}$ and $\alpha^j_k \circ \alpha^i_j = \alpha^i_k$ all $i \leq j \leq k$.



An A-module M together with a family $(\alpha^i : M_i \to M)_{i \in I}$ of A-linear maps satisfying $\alpha^i = \alpha^j \circ \alpha^i_j$ all $i \leq j$ is said to be a ***direct limit*** of the system $((M_i), (\alpha^i_j))$ if it has the following universal property: for any other A-module N and family $(\beta^i : M_i \to N)$ of A-linear maps such that $\beta^i = \beta^j \circ \alpha^i_j$ all $i \leq j$, there exists a unique morphism $\alpha : M \to N$ such that $\alpha \circ \alpha^i = \beta^i$ for all i.



As usual, the universal property determines the direct limit (if it exists) uniquely up to a unique isomorphism. We denote it $\varinjlim(M_i, \alpha^j_i)$, or just $\varinjlim M_i$.

CRITERION

An A-module M together with A-linear maps $\alpha^i : M_i \to M$ is the direct limit of a system (M_i, α^j_i) if and only if

(a) $M = \bigcup_{i \in I} \alpha^i(M_i)$, and

(b) $m_i \in M_i$ maps to zero in M if and only if it maps to zero in M_j for some $j \geq i$.

CONSTRUCTION

Let

$$M = \bigoplus_{i \in I} M_i / M'$$

where M' is the A-submodule generated by the elements

$$m_i - \alpha^i_j(m_i) \quad \text{all } i < j, m_i \in M_i.$$

Let $\alpha^i(m_i) = m_i + M'$. Then certainly $\alpha^i = \alpha^j \circ \alpha^i_j$ for all $i \leq j$. For any A-module N and A-linear maps $\beta^j : M_j \to N$, there is a unique map

$$\bigoplus_{i \in I} M_i \to N,$$

namely, $\sum m_i \mapsto \sum \beta^i(m_i)$, sending m_i to $\beta^i(m_i)$, and this map factors through M and is the unique A-linear map with the required properties.

Direct limits of A-algebras, etc., are defined similarly.

AN EXAMPLE

PROPOSITION 7.3. *For any multiplicative subset S of a ring A, $S^{-1}A \simeq \varinjlim A_h$, where h runs over the elements of S (partially ordered by division).*

PROOF. When $h|h'$, say, $h' = hg$, there is a unique homomorphism $A_h \to A_{h'}$ respecting the maps $A \to A_h$ and $A \to A_{h'}$, namely, $\frac{a}{h} \mapsto \frac{ag}{h'}$, and so the rings A_h form a direct system indexed by the set S. When $h \in S$, the homomorphism $A \to S^{-1}A$ extends uniquely to a homomorphism $\frac{a}{h} \mapsto \frac{a}{h}: A_h \to S^{-1}A$ (see 6.1), and these homomorphisms are compatible with the maps in the direct system. Now apply the criterion p.24 to see that $S^{-1}A$ is the direct limit of the A_h. □

8 Tensor Products

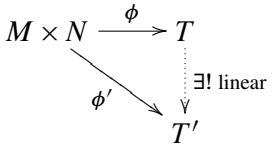
TENSOR PRODUCTS OF MODULES

Let A be a ring, and let M, N, and P be A-modules. A map $\phi: M \times N \to P$ of A-modules is said to be A-***bilinear*** if

$$\begin{aligned}
\phi(x+x',y) &= \phi(x,y)+\phi(x',y), && x,x' \in M, \quad y \in N\\
\phi(x,y+y') &= \phi(x,y)+\phi(x,y'), && x \in M, \quad y,y' \in N\\
\phi(ax,y) &= a\phi(x,y), && a \in A, \quad x \in M, \quad y \in N\\
\phi(x,ay) &= a\phi(x,y), && a \in A, \quad x \in M, \quad y \in N,
\end{aligned}$$

i.e., if ϕ is A-linear in each variable.

An A-module T together with an A-bilinear map $\phi: M \times N \to T$ is called the ***tensor product*** of M and N over A if it has the following universal property: every A-bilinear map $\phi': M \times N \to T'$ factors uniquely through ϕ.



As usual, the universal property determines the tensor product uniquely up to a unique isomorphism. We write it $M \otimes_A N$. Note that

$$\mathrm{Hom}_{A\text{-bilinear}}(M \times N, T) \simeq \mathrm{Hom}_{A\text{-linear}}(M \otimes_A N, T).$$

Construction

Let M and N be A-modules, and let $A^{(M\times N)}$ be the free A-module with basis $M \times N$. Thus each element $A^{(M\times N)}$ can be expressed uniquely as a finite sum

$$\sum a_i(x_i, y_i), \quad a_i \in A, \quad x_i \in M, \quad y_i \in N.$$

Let P be the submodule of $A^{(M\times N)}$ generated by the following elements

$$\begin{aligned}
(x+x',y)-(x,y)-(x',y), &\quad x,x' \in M, \quad y \in N\\
(x,y+y')-(x,y)-(x,y'), &\quad x \in M, \quad y,y' \in N\\
(ax,y)-a(x,y), &\quad a \in A, \quad x \in M, \quad y \in N\\
(x,ay)-a(x,y), &\quad a \in A, \quad x \in M, \quad y \in N,
\end{aligned}$$

and define

$$M \otimes_A N = A^{(M\times N)}/P.$$

Write $x \otimes y$ for the class of (x, y) in $M \otimes_A N$. Then

$$(x, y) \mapsto x \otimes y : M \times N \to M \otimes_A N$$

is A-bilinear — we have imposed the fewest relations necessary to ensure this. Every element of $M \otimes_A N$ can be written as a finite sum

$$\sum a_i(x_i \otimes y_i), \quad a_i \in A, \quad x_i \in M, \quad y_i \in N,$$

and all relations among these symbols are generated by the following relations

$$\begin{aligned}(x + x') \otimes y &= x \otimes y + x' \otimes y\\ x \otimes (y + y') &= x \otimes y + x \otimes y'\\ a(x \otimes y) &= (ax) \otimes y = x \otimes ay.\end{aligned}$$

The pair $(M \otimes_A N, (x, y) \mapsto x \otimes y)$ has the correct universal property because any bilinear map $\phi' : M \times N \to T'$ defines an A-linear map $A^{(M\times N)} \to T'$, which factors through $A^{(M\times N)}/K$, and gives a commutative triangle.

Extension of scalars

Let A be a commutative ring and let B be an A-algebra (not necessarily commutative) such that the image of $A \to B$ lies in the centre of B. Then $M \rightsquigarrow B \otimes_A M$ is a functor from left A-modules to left B-modules, which has the following universal property:

$$\mathrm{Hom}_{A\text{-linear}}(M, N) \simeq \mathrm{Hom}_{B\text{-linear}}(B \otimes_A M, N), \quad N \text{ a } B\text{-module}. \tag{8}$$

If $(e_\alpha)_{\alpha\in I}$ is a family of generators (resp. basis) for M as an A-module, then $(1 \otimes e_\alpha)_{\alpha\in I}$ is a family of generators (resp. basis) for $B \otimes_A M$ as a B-module.

Behaviour with respect to direct limits

PROPOSITION 8.1. *Direct limits commute with tensor products:*

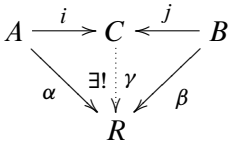
$$\varinjlim_{i\in I} M_i \otimes_A \varinjlim_{j\in J} N_j \simeq \varinjlim_{(i,j)\in I\times J} M_i \otimes_A N_j.$$

PROOF. Using the universal properties of direct limits and tensor products, one sees easily that $\varinjlim(M_i \otimes_A N_j)$ has the universal property to be the tensor product of $\varinjlim M_i$ and $\varinjlim N_j$. □

TENSOR PRODUCTS OF ALGEBRAS

Let k be a ring, and let A and B be k-algebras. A k-algebra C together with homomorphisms $i : A \to C$ and $j : B \to C$ is called the ***tensor product*** of A and B if it has the following universal property:

for every pair of homomorphisms (of k-algebras) $\alpha : A \to R$ and $\beta : B \to R$, there exists a unique homomorphism $\gamma : C \to R$ such that $\gamma \circ i = \alpha$ and $\gamma \circ j = \beta$,



If it exists, the tensor product, is uniquely determined up to a unique isomorphism by this property. We write it $A \otimes_k B$. Note that the universal property says that

$$\mathrm{Hom}_{k\text{-algebra}}(A \otimes_k B, R) \simeq \mathrm{Hom}_{k\text{-algebra}}(A, R) \times \mathrm{Hom}_{k\text{-algebra}}(B, R). \tag{9}$$

Construction

Regard A and B as k-modules, and form the tensor product $A \otimes_k B$. There is a multiplication map $A \otimes_k B \times A \otimes_k B \to A \otimes_k B$ for which

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb', \quad \text{all } a, a' \in A, \quad b, b' \in B.$$

This makes $A \otimes_k B$ into a ring, and the homomorphism

$$c \mapsto c(1 \otimes 1) = c \otimes 1 = 1 \otimes c$$

makes it into a k-algebra. The maps

$$a \mapsto a \otimes 1 \colon A \to A \otimes_k B \text{ and } b \mapsto 1 \otimes b \colon B \to A \otimes_k B$$

are homomorphisms, and they make $A \otimes_k B$ into the tensor product of A and B in the above sense.

EXAMPLE 8.2. The algebra A, together with the given map $k \to A$ and the identity map $A \to A$, has the universal property characterizing $k \otimes_k A$. In terms of the constructive definition of tensor products, the map $c \otimes A \mapsto cA \colon k \otimes_k A \to A$ is an isomorphism.

EXAMPLE 8.3. The ring $k[X_1, \dots, X_m, X_{m+1}, \dots, X_{m+n}]$, together with the obvious inclusions

$$k[X_1, \dots, X_m] \hookrightarrow k[X_1, \dots, X_{m+n}] \hookleftarrow k[X_{m+1}, \dots, X_{m+n}]$$

is the tensor product of $k[X_1, \dots, X_m]$ and $k[X_{m+1}, \dots, X_{m+n}]$. To verify this we only have to check that, for every k-algebra R, the map

$$\mathrm{Hom}_{k\text{-alg}}(k[X_1, \dots, X_{m+n}], R) \to \mathrm{Hom}_{k\text{-alg}}(k[X_1, \dots], R) \times \mathrm{Hom}_{k\text{-alg}}(k[X_{m+1}, \dots], R)$$

induced by the inclusions is a bijection. But this map can be identified with the bijection

$$R^{m+n} \to R^m \times R^n.$$

In terms of the constructive definition of tensor products, the map

$$k[X_1, \dots, X_m] \otimes_k k[X_{m+1}, \dots, X_{m+n}] \to k[X_1, \dots, X_{m+n}]$$

sending $f \otimes g$ to fg is an isomorphism.

REMARK 8.4. (a) Let $k \hookrightarrow k'$ be a homomorphism of rings. Then

$$k' \otimes_k k[X_1, \dots, X_n] \simeq k'[1 \otimes X_1, \dots, 1 \otimes X_n] \simeq k'[X_1, \dots, X_n].$$

If $A = k[X_1, \dots, X_n]/(g_1, \dots, g_m)$, then

$$k' \otimes_k A \simeq k'[X_1, \dots, X_n]/(g_1, \dots, g_m).$$

(b) If A and B are algebras of k-valued functions on sets S and T respectively, then definition

$$(f \otimes g)(x, y) = f(x)g(y), \quad f \in A, g \in B, x \in S, y \in T,$$

realizes $A \otimes_k B$ as an algebra of k-valued functions on $S \times T$.

THE TENSOR ALGEBRA OF A MODULE

Let M be a module over a ring A. For each $A \geq 0$, set

$$T^r M = M \otimes_A \cdots \otimes_A M \qquad (r \text{ factors}),$$

so that $T^0 M = A$ and $T^1 M = M$, and define

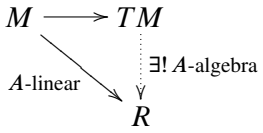
$$TM = \bigoplus\nolimits_{r \geq 0} T^r M.$$

This can be made into a noncommutative A-algebra, called the ***tensor algebra*** of M, by requiring that the multiplication map

$$T^r M \times T^s M \to T^{r+s} M$$

send $(m_1 \otimes \cdots \otimes m_r, m_{r+1} \otimes \cdots \otimes m_{r+s})$ to $m_1 \otimes \cdots \otimes m_{r+s}$.

The pair $(TM, M \to TM)$ has the following universal property: any A-linear map from M to an A-algebra R (not necessarily commutative) extends uniquely to an A-algebra homomorphism $TM \to R$.



If M is a free A-module with basis $x_1, \ldots, x_n$, then TM is the (noncommutative) polynomial ring over A in the noncommuting symbols x_i (because this A-algebra has the same universal property as TM).

THE SYMMETRIC ALGEBRA OF A MODULE

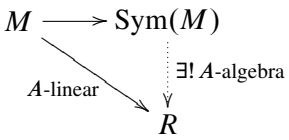
The ***symmetric algebra*** $\mathrm{Sym}(M)$ of an A-module M is the quotient of TM by the ideal generated by all elements of $T^2 M$ of the form

$$m \otimes n - n \otimes m, \quad m, n \in M.$$

It is a graded algebra $\mathrm{Sym}(M) = \bigoplus_{r \geq 0} \mathrm{Sym}^r(M)$ with $\mathrm{Sym}^r(M)$ equal to the quotient of $M^{\otimes r}$ by the A-submodule generated by all elements of the form

$$m_1 \otimes \cdots \otimes m_r - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(r)}, \quad m_i \in M, \quad \sigma \in B_r \text{ (symmetric group)}.$$

The pair $(\mathrm{Sym}(M), M \to \mathrm{Sym}(M))$ has the following universal property: any A-linear map $M \to R$ from M to a commutative A-algebra R extends uniquely to an A-algebra homomorphism $\mathrm{Sym}(M) \to R$ (because it extends to an A-algebra homomorphism $TM \to R$, which factors through $\mathrm{Sym}(M)$ because R is commutative).



If M is a free A-module with basis $x_1, \ldots, x_n$, then $\mathrm{Sym}(M)$ is the polynomial ring over A in the (commuting) symbols x_i (because this A-algebra has the same universal property as TM).

9 Flatness

Let M be an A-module. If the sequence of A-modules

$$0 \to N' \to N \to N'' \to 0 \tag{10}$$

is exact, then the sequence

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is exact, but $M \otimes_A N' \to M \otimes_A N$ need not be injective. For example, when we tensor the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$$

with $\mathbb{Z}/m\mathbb{Z}$, we get the sequence

$$\mathbb{Z}/m\mathbb{Z} \xrightarrow{m=0} \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \to 0.$$

Moreover, $M \otimes_A N$ may be zero even when neither M nor N is nonzero. For example,

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$$

because it is killed by both 2 and 3.[12]

DEFINITION 9.1. An A-module M is ***flat*** if

$$N' \to N \text{ injective } \implies M \otimes_A N' \to M \otimes_A N \text{ injective.}$$

It is ***faithfully flat*** if, in addition,

$$M \otimes_A N = 0 \implies N = 0.$$

A homomorphism of rings $A \to B$ is said to be (***faithfully***) ***flat*** when B is (faithfully) flat as an A-module.

Thus, an A-module M is flat if and only if $M \otimes_A -$ is an exact functor, i.e.,

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0 \tag{11}$$

is exact whenever (10) is exact.

The functor $M \otimes -$ takes direct sums to direct sums, and therefore split-exact sequences to split-exact sequences. Therefore, all vector spaces over a field are flat, and nonzero vector spaces are faithfully flat.

PROPOSITION 9.2. *Let $i: A \to B$ be a homomorphism of rings. If i is faithfully flat, then a sequence of A-modules*

$$0 \to N' \to N \to N'' \to 0 \tag{12}$$

is exact if and only if

$$0 \to B \otimes_A N' \to B \otimes_A N \to B \otimes_A N'' \to 0 \tag{13}$$

is exact. Conversely, if

$$\text{(12) exact} \iff \text{(13) exact,}$$

then $i: A \to B$ is faithfully flat.

[12] It was once customary to require a ring to have an identity element $1 \neq 0$ (see, for example, Northcott 1953, p.3). However, the example shows that tensor products do not always exist in the category of such objects, .

PROOF. For the first statement, we have to show that (12) is exact if (13) is exact. Let N_0 be the kernel of $N' \to N$. Then, because $A \to B$ is flat, $B \otimes_A N_0$ is the kernel of $B \otimes_A N' \to B \otimes_A N$, which is zero by assumption. Because $A \to B$ is *faithfully* flat, this implies that $N_0 = 0$. This proves the exactness at N', and the proof of exactness elsewhere is similar.

For the converse statement, the condition implies that i is flat (this is the definition). Now let N be an A-module, and consider the sequence

$$0 \to 0 \to N \to 0 \to 0.$$

If $B \otimes_A N = 0$, then this sequence becomes exact when tensored with B, and so is itself exact, which implies that $N = 0$. This shows that i is faithfully flat. □

PROPOSITION 9.3. *Let $i: A \to B$ be a faithfully flat homomorphism. For any A-module M, the sequence*

$$0 \to M \xrightarrow{d_0} B \otimes_A M \xrightarrow{d_1} B \otimes_A B \otimes_A M \qquad (*)$$
$$\begin{cases} d_0(m) &= 1 \otimes m, \\ d_1(b \otimes m) &= 1 \otimes b \otimes m - b \otimes 1 \otimes m \end{cases}$$

is exact.

PROOF. Assume first that there exists an A-linear section to $A \to B$, i.e., an A-linear map $f: B \to A$ such that $f \circ i = \mathrm{id}_A$, and define

$$k_0: B \otimes_A M \to M, \qquad k_0(b \otimes m) = f(b)m$$
$$k_1: B \otimes_A B \otimes_A M \to B \otimes_A M, \qquad k_1(b \otimes b' \otimes m) = f(b)b' \otimes m.$$

Then $k_0 d_0 = \mathrm{id}_M$, which shows that d_0 is injective. Moreover,

$$k_1 \circ d_1 + d_0 \circ k_0 = \mathrm{id}_{B \otimes_A M}$$

which shows that, if $d_1(x) = 0$, then $x = d_0(k_0(x))$, as required.

We now consider the general case. Because $A \to B$ is faithfully flat, it suffices to prove that the sequence (*) becomes exact after tensoring in B. But the sequence obtained from (*) by tensoring with B is isomorphic to the sequence (*) for the homomorphism of rings $B \mapsto 1 \otimes B: B \to B \otimes_A B$ and the B-module $B \otimes_A M$, because, for example,

$$B \otimes_A (B \otimes_A M) \simeq (B \otimes_A B) \otimes_B (B \otimes_A M).$$

Now $B \to B \otimes_A B$ has an B-linear section, namely, $f(B \otimes B') = BB'$, and so we can apply the first part. □

COROLLARY 9.4. *If $A \to B$ is faithfully flat, then it is injective with image the set of elements on which the maps*

$$\begin{cases} b &\mapsto 1 \otimes b \\ b &\mapsto b \otimes 1 \end{cases} : B \to B \otimes_A B$$

agree.

PROOF. This is the special case $M = A$ of the Proposition. □

PROPOSITION 9.5. *Let $A \to A'$ be a homomorphism of rings. If $A \to B$ is flat (or faithfully flat), then so also is $A' \to B \otimes_A A'$.*

PROOF. For any A'-module M,

$$(B \otimes_A A') \otimes_{A'} M \simeq B \otimes_A (A' \otimes_{A'} M) \simeq B \otimes_A M,$$

from which the statement follows. □

PROPOSITION 9.6. *For any multiplicative subset S of a ring A and A-module M,*

$$S^{-1}A \otimes_A M \simeq S^{-1}M.$$

Therefore the homomorphism $a \mapsto \frac{a}{1}: A \to S^{-1}A$ is flat.

PROOF. To give an $S^{-1}A$-module is the same as giving an A-module on which the elements of S act invertibly. Therefore $S^{-1}A \otimes_A M$ and $S^{-1}M$ satisfy the same universal property (see §8, especially (8)), which proves the first statement. As $M \rightsquigarrow S^{-1}M$ is exact (6.11), so also is $M \rightsquigarrow S^{-1}A \otimes_A M$, which proves the second statement. □

PROPOSITION 9.7. *The following conditions on a flat homomorphism $\varphi: A \to B$ are equivalent:*

(a) *φ is faithfully flat;*
(b) *for every maximal ideal $\mathfrak{m}$ of A, the ideal $\varphi(\mathfrak{m})B \neq B$;*
(c) *every maximal ideal $\mathfrak{m}$ of A is of the form $\varphi^{-1}(\mathfrak{n})$ for some maximal ideal $\mathfrak{n}$ of B.*

PROOF. (a) $\Rightarrow$ (b): Let $\mathfrak{m}$ be a maximal ideal of A, and let $M = A/\mathfrak{m}$; then

$$B \otimes_A M \simeq B/\varphi(\mathfrak{m})B.$$

As $B \otimes_A M \neq 0$, we see that $\varphi(\mathfrak{m})B \neq B$.

(b) $\Rightarrow$ (c): If $\varphi(\mathfrak{m})B \neq B$, then $\varphi(\mathfrak{m})$ is contained in a maximal ideal $\mathfrak{n}$ of B. Now $\varphi^{-1}(\mathfrak{n})$ is a proper ideal in A containing $\mathfrak{m}$, and hence equals $\mathfrak{m}$.

(c) $\Rightarrow$ (a): Let M be a nonzero A-module. Let x be a nonzero element of M, and let $\mathfrak{a} = \{a \in A \mid ax = 0\}$. Then $\mathfrak{a}$ is an ideal in A, and $M' \stackrel{\text{def}}{=} Ax \simeq A/\mathfrak{a}$. Moreover, $B \otimes_A M' \simeq B/\varphi(\mathfrak{a}) \cdot B$ and, because $A \to B$ is flat, $B \otimes_A M'$ is a submodule of $B \otimes_A M$. Because $\mathfrak{a}$ is proper, it is contained in a maximal ideal $\mathfrak{m}$ of A, and therefore

$$\varphi(\mathfrak{a}) \subset \varphi(\mathfrak{m}) \subset \mathfrak{n}$$

for some maximal ideal $\mathfrak{n}$ of A. Hence $\varphi(\mathfrak{a}) \cdot B \subset \mathfrak{n} \neq B$, and so $B \otimes_A M \supset B \otimes_A M' \neq 0$. □

THEOREM 9.8 (GENERIC FLATNESS). *Let A an integral domain with field of fractions F, and let B be a finitely generated A-algebra such that $B \subset F \otimes_A B$. Then for some nonzero elements a of A and b of B, the homomorphism $A_a \to B_b$ is faithfully flat.*

PROOF. As $F \otimes_A B$ is a finitely generated F-algebra, the Noether normalization theorem (5.11) shows that there exist elements $x_1, \dots, x_m$ of $F \otimes_A B$ such that $F[x_1, \dots, x_m]$ is a polynomial ring over F and $F \otimes_A B$ is a finite $F[x_1, \dots, x_m]$-algebra. After multiplying each x_i by an element of A, we may suppose that it lies in B. Let $b_1, \dots, b_n$ generate B as an A-algebra. Each b_i satisfies a monic polynomial equation with coefficients in $F[x_1, \dots, x_m]$. Let $a \in A$ be a common denominator for the coefficients of these polynomials. Then each b_i is integral over A_a. As the b_i generate B_a as an A_a-algebra, this shows that B_a is a finite $A_a[x_1, \dots, x_m]$-algebra (by 5.2). Therefore, after replacing A with A_a and B with B_a, we may suppose that B is a finite $A[x_1, \dots, x_m]$-algebra.

$$\begin{array}{ccccc}
B & \xrightarrow{\text{injective}} & F \otimes_A B & \longrightarrow & E \otimes_{A[x_1,\dots,x_m]} B \\
\uparrow{\scriptstyle \text{finite}} & & \uparrow{\scriptstyle \text{finite}} & & \uparrow{\scriptstyle \text{finite}} \\
A[x_1, \dots, x_m] & \longrightarrow & F[x_1, \dots, x_m] & \longrightarrow & E \overset{\text{def}}{=} F(x_1, \dots, x_n) \\
\uparrow & & \uparrow & & \\
A & \longrightarrow & F & &
\end{array}$$

Let $E = F(x_1, \dots, x_m)$ be the field of fractions of $A[x_1, \dots, x_m]$, and let $b_1, \dots, b_r$ be elements of B that form a basis for $E \otimes_{A[x_1,\dots,x_m]} B$ as an E-vector space. Each element of B can be expressed a linear combination of the b_i with coefficients in E. Let q be a common denominator for the coefficients arising from a set of generators for B as an $A[x_1, \dots, x_m]$-module. Then $b_1, \dots, b_r$ generate B_q as an $A[x_1, \dots, x_m]_q$-module. In other words, the map

$$(c_1, \dots, c_r) \mapsto \textstyle\sum c_i b_i : A[x_1, \dots, x_m]_q^r \to B_q \tag{14}$$

is surjective. This map becomes an isomorphism when tensored with E over $A[x_1, \dots, x_m]_q$, which implies that each element of its kernel is killed by a nonzero element of $A[x_1, \dots, x_m]_q$ and so is zero (because $A[x_1, \dots, x_n]_q$ is an integral domain). Hence the map (14) is an isomorphism, and B_q is free of finite rank over $A[x_1, \dots, x_m]_q$. Let a be some nonzero coefficient of the polynomial q, and consider the maps

$$A_a \to A_a[x_1, \dots, x_m] \to A_a[x_1, \dots, x_m]_q \to B_{aq}.$$

The first and third arrows realize their targets as nonzero free modules over their sources, and so are faithfully flat. The middle arrow is flat by (9.6). Let $\mathfrak{m}$ be a maximal ideal in A_a. Then $\mathfrak{m}A_a[x_1, \dots, x_m]$ does not contain the polynomial q because the coefficient a of q is invertible in A_a. Hence $\mathfrak{m}A_a[x_1, \dots, x_m]_q$ is a proper ideal of $A_a[x_1, \dots, x_m]_q$, and so the map $A_a \to A_a[x_1, \dots, x_m]_q$ is faithfully flat (apply 9.7). This completes the proof. □

REMARK 9.9. The theorem holds for any finitely generated B-algebra, i.e., without the requirement that $B \subset F \otimes_A B$. To see this, note that $F \otimes_A B$ is the ring of fractions of B with respect to the multiplicative subset $A \setminus \{0\}$ (see 9.6), and so the kernel of $B \to F \otimes_A B$ is the ideal

$$\mathfrak{n} = \{b \in B \mid ab = 0 \text{ for some nonzero } a \in A\}.$$

This is finitely generated (Hilbert basis theorem 3.6), and so there exists a nonzero $c \in A$ such that $cb = 0$ for *all* $b \in \mathfrak{n}$. I claim that the homomorphism $B_c \to F \otimes_{A_c} B_c$ is injective. If $\frac{b}{c^r}$ lies in its kernel, then $\frac{a}{c^s}\frac{b}{c^r} = 0$ in B_c for some nonzero $\frac{a}{c^s} \in A_c$, and so $c^N ab = 0$ in B for some N; therefore $b \in \mathfrak{n}$, and so $cb = 0$, which implies that $\frac{b}{c^r} = 0$ already in B_c.

Therefore, after replacing A, B, and M with A_c, B_c, and M_c, we may suppose that the map $B \to F \otimes_A B$ is injective. On identifying B with its image, we arrive at the situation of the theorem.

10 The Hilbert Nullstellensatz

THEOREM 10.1 (ZARISKI'S LEMMA). *Let $k \subset K$ be fields. If K is finitely generated as a k-algebra, then it is algebraic over k (hence K is finite over k, and equals it if k is algebraically closed).*

PROOF. [13]We shall prove this by induction on r, the smallest number of elements required to generate K as a k-algebra. The case $r = 0$ being trivial, we may suppose that

$$K = k[x_1, \dots, x_r] \text{ with } r \geq 1.$$

If K is not algebraic over k, then at least one x_i, say x_1, is not algebraic over k. Then, $k[x_1]$ is a polynomial ring in one symbol over k, and its field of fractions $k(x_1)$ is a subfield of K. Clearly K is generated as a $k(x_1)$-algebra by $x_2, \dots, x_r$, and so the induction hypothesis implies that $x_2, \dots, x_r$ are algebraic over $k(x_1)$. Proposition 5.5 shows that there exists a $c \in k[x_1]$ such that $cx_2, \dots, cx_r$ are integral over $k[x_1]$. Let $f \in K$. For a sufficiently large N, $c^N f \in k[x_1, cx_2, \dots, cx_r]$, and so $c^N f$ is integral over $k[x_1]$ by 5.3. When we apply this statement to an element f of $k(x_1)$, it shows that $c^N f \in k[x_1]$ because $k[x_1]$ is integrally closed. Therefore, $k(x_1) = \bigcup_N c^{-N} k[x_1]$, but this is absurd, because $k[x_1]$ ($\simeq k[X]$) has infinitely many distinct monic irreducible polynomials[14] that can occur as denominators of elements of $k(x_1)$. □

Recall that k^{al} denotes an algebraic closure of the field k.

[13]The following alternative proof of (10.1) is a simplication by Swan of a proof of Munshi — see `www.math.uchicago.edu/~swan/`.

LEMMA: For an integral domain A, there does not exist an $f \in A[X]$ such that $A[X]_f$ is a field.

PROOF: Suppose, on the contrary, that $A[X]_f$ is a field. Then $\deg f > 0$, and so $f - 1 \notin R$. Write $(f-1)^{-1} = g/f^n$ with $g \in A[X]$ and $n \geq 1$. Then $(f-1)g = f^n = (1+(f-1))^n = 1 + (f-1)h$ with $h \in A[X]$. It follows that $f - 1$ is a unit, which is absurd.

LEMMA: Consider rings $A \subset B$. If B is integral over A, then $A \cap B^\times = A^\times$. In particular, if B is a field, then so also is A.

PROOF: Let a be an element of A that becomes a unit in B, say, $ab = 1$ with $b \in B$. Write $b^n + a_1 b^{n-1} + \cdots + a_n = 0$ with $a_1, \dots, a_n \in A$. On multiplying by a^{n-1}, we find that $b = -a_1 - \cdots - a_n a^{n-1} \in A$ and so $a \in A^\times$.

PROPOSITION: Let A be an integral domain, and let $\mathfrak{m}$ be a maximal ideal of $A[X_1, \dots, X_n]$. If $A \cap \mathfrak{m} = (0)$, then there exists a nonzero element a in A such that A_a is a field and $A[X_1, \dots, X_n]/\mathfrak{m}$ is a finite extension of A_a.

PROOF: We argue by induction on n. The statement being trivial for $n = 0$, we may suppose that $n \geq 1$. Regard $A[X_1, \dots, X_n]$ as a polynomial ring in $n-1$ symbols over $A[X_i]$. The induction hypothesis and the first lemma show that there exists a nonzero element $P_i(X_i) = a_i X_i^{n_i} + \cdots$ in $\mathfrak{m} \cap A[X_i]$. Let $a = a_1 \cdots a_n$, and let $K = A[X_1, \dots, X_n]/\mathfrak{m}$. The image x_i of X_i in K satisfies the monic equation $a_i^{-1} P_i = 0$, and so K is integral over A_a. By the second lemma, A_a is a field, and K is finite over it because it is integral (algebraic) and finitely generated.

COROLLARY: Let A be a finitely generated algebra over a field k. For any maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ is a finite extension of k.

PROOF: Take A in the proposition to be a field.

[14]When k is infinite, there are infinitely many polynomials $X - a$, and when k is finite, we can adapt Euclid's argument: if $p_1, \dots, p_r$ are monic irreducible polynomials in $k[X]$, then $p_1 \cdots p_r + 1$ is divisible by a monic irreducible polynomial distinct from $p_1, \dots, p_r$.

THEOREM 10.2 (NULLSTELLENSATZ). *Every proper ideal $\mathfrak{a}$ in $k[X_1,\dots,X_n]$ has a zero in $(k^{\mathrm{al}})^n \stackrel{\text{def}}{=} k^{\mathrm{al}}\times\cdots\times k^{\mathrm{al}}$, i.e., there exists a point $(a_1,\dots,a_n)\in(k^{\mathrm{al}})^n$ such that $f(a_1,\dots,a_n)=0$ for all $f\in\mathfrak{a}$.*

PROOF. We have to show that there exists a k-algebra homomorphism $k[X_1,\dots,X_n]\to k^{\mathrm{al}}$ containing $\mathfrak{a}$ in its kernel. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a}$. Then $k[X_1,\dots,X_n]/\mathfrak{m}$ is a field, which is algebraic over k by Zariski's lemma, and so there exists a k-algebra homomorphism $k[X_1,\dots,X_n]/\mathfrak{m}\to k^{\mathrm{al}}$. The composite of this with the quotient map $k[X_1,\dots,X_n]\to k[X_1,\dots,X_n]/\mathfrak{m}$ contains $\mathfrak{a}$ in its kernel. □

COROLLARY 10.3. *When k is algebraically closed, the maximal ideals in $k[X_1,\dots,X_n]$ are exactly the ideals $(X_1-a_1,\dots,X_n-a_n)$, $(a_1,\dots,a_n)\in k^n$.*

PROOF. Clearly, $k[X_1,\dots,X_n]/(X_1-a_1,\dots,X_n-a_n)\simeq k$, and so $(X_1-a_1,\dots,X_n-a_n)$ is maximal. Conversely, because k is algebraically closed, a proper ideal $\mathfrak{a}$ has a zero $(a_1,\dots,a_n)$ in k^n. Let $f\in k[X_1,\dots,X_n]$; when we write f as a polynomial in $X_1-a_1,\dots,X_n-a_n$, its constant term is $f(a_1,\dots,a_n)$. Therefore, if $f\in\mathfrak{a}$, then $f\in(X_1-a_1,\dots,X-a_n)$. □

THEOREM 10.4 (STRONG NULLSTELLENSATZ). *For an ideal $\mathfrak{a}$ in $k[X_1,\dots,X_n]$, let $Z(\mathfrak{a})$ be the set of zeros of $\mathfrak{a}$ in $(k^{\mathrm{al}})^n$. If a polynomial $h\in k[X_1,\dots,X_n]$ is zero on $Z(\mathfrak{a})$, then some power of h lies in $\mathfrak{a}$.*

PROOF. We may assume $h\neq 0$. Let $g_1,\dots,g_m$ generate $\mathfrak{a}$, and consider the system of $m+1$ equations in $n+1$ variables, $X_1,\dots,X_n,Y$,

$$\begin{cases} g_i(X_1,\dots,X_n) &= 0, \quad i=1,\dots,m\\ 1-Yh(X_1,\dots,X_n) &= 0.\end{cases}$$

If $(a_1,\dots,a_n,b)$ satisfies the first m equations, then $(a_1,\dots,a_n)\in Z(\mathfrak{a})$; consequently, $h(a_1,\dots,a_n)=0$, and $(a_1,\dots,a_n,b)$ doesn't satisfy the last equation. Therefore, the equations are inconsistent, and so, according to the Nullstellensatz (10.2), there exist $f_i\in k[X_1,\dots,X_n,Y]$ such that

$$1=\sum_{i=1}^{m} f_i\cdot g_i + f_{m+1}\cdot(1-Yh)$$

in $k[X_1,\dots,X_n,Y]$. On applying the homomorphism

$$\begin{cases} X_i\mapsto X_i\\ Y\mapsto h^{-1}\end{cases} : k[X_1,\dots,X_n,Y]\to k(X_1,\dots,X_n)$$

to the above equality, we obtain the identity

$$1=\textstyle\sum_i f_i(X_1,\dots,X_n,h^{-1})\cdot g_i(X_1,\dots,X_n) \tag{15}$$

in $k(X_1,\dots,X_n)$. Clearly

$$f_i(X_1,\dots,X_n,h^{-1})=\frac{\text{polynomial in } X_1,\dots,X_n}{h^{N_i}}$$

for some N_i. Let N be the largest of the N_i. On multiplying (15) by h^N we obtain an identity

$$h^N = \sum\nolimits_i (\text{polynomial in } X_1,\dots,X_n)\cdot g_i(X_1,\dots,X_n),$$

which shows that $h^N \in \mathfrak{a}$. □

PROPOSITION 10.5. *The radical of an ideal $\mathfrak{a}$ in a finitely generated k-algebra A is equal to the intersection of the maximal ideals containing it:* $\operatorname{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{m}\supset\mathfrak{a}} \mathfrak{m}$. *In particular, if A is reduced, then* $\bigcap_{\mathfrak{m}\ \text{maximal}} \mathfrak{m} = 0$.

PROOF. Because of the correspondence (2), p.3, it suffices to prove this for $A = k[X_1,\dots,X_n]$.

Let $\mathfrak{a}$ be an ideal in $k[X_1,\dots,X_n]$. Because $\operatorname{rad}(\mathfrak{a})$ is the smallest radical ideal containing $\mathfrak{a}$ and maximal ideals are radical $\operatorname{rad}(\mathfrak{a}) \subset \bigcap_{\mathfrak{m}\supset\mathfrak{a}} \mathfrak{m}$. Conversely, suppose h is contained in all maximal ideals containing $\mathfrak{a}$, and let $(a_1,\dots,a_n) \in Z(\mathfrak{a})$. The evaluation map

$$f \mapsto f(a_1,\dots,a_n): k[X_1,\dots,X_n] \to k^{\mathrm{al}}$$

has image a subring of k^{al} which is algebraic over k, and hence is a field (see §1). Therefore, the kernel of the map is a maximal ideal, which contains $\mathfrak{a}$, and therefore also contains h. This shows that $h(a_1,\dots,a_n) = 0$, and we conclude from the strong Nullstellensatz that $h \in \operatorname{rad}(\mathfrak{a})$. □

11 The max spectrum of a ring

Let A be a ring, and let V be the set of maximal ideals in A. For an ideal $\mathfrak{a}$ in A, let

$$V(\mathfrak{a}) = \{\mathfrak{m} \in V \mid \mathfrak{m} \supset \mathfrak{a}\}.$$

PROPOSITION 11.1. *There are the following relations:*

(a) $\mathfrak{a} \subset \mathfrak{b} \implies V(\mathfrak{a}) \supset V(\mathfrak{b})$;
(b) $V(0) = V$; $\quad V(A) = \emptyset$;
(c) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$;
(d) $V(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} V(\mathfrak{a}_i)$ *for any family of ideals* $(\mathfrak{a}_i)_{i\in I}$.

PROOF. The first two statements are obvious. For (c), note that

$$\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}\cap\mathfrak{b} \subset \mathfrak{a}, \mathfrak{b} \implies V(\mathfrak{a}\mathfrak{b}) \supset V(\mathfrak{a}\cap\mathfrak{b}) \supset V(\mathfrak{a}) \cup V(\mathfrak{b}).$$

For the reverse inclusions, observe that if $\mathfrak{m} \notin V(\mathfrak{a}) \cup V(\mathfrak{b})$, then there exist an $f \in \mathfrak{a} \setminus \mathfrak{m}$ and a $g \in \mathfrak{b} \setminus \mathfrak{m}$; but then $fg \in \mathfrak{a}\mathfrak{b} \setminus \mathfrak{m}$, and so $\mathfrak{m} \notin V(\mathfrak{a}\mathfrak{b})$. For (d) recall that, by definition, $\sum \mathfrak{a}_i$ consists of all finite sums of the form $\sum f_i$, $f_i \in \mathfrak{a}_i$. Thus (d) is obvious. □

Statements (b), (c), and (d) show that the sets $V(\mathfrak{a})$ satisfy the axioms to be the closed subsets for a topology on V: both the whole space and the empty set are closed; a finite union of closed sets is closed; an arbitrary intersection of closed sets is closed. This topology is called the ***Zariski topology*** on V. We let $\operatorname{spm}(A)$ denote the set of maximal ideals in A endowed with its Zariski topology.

For $h \in A$, let

$$D(h) = \{\mathfrak{m} \in V \mid h \notin \mathfrak{m}\}.$$

Then $D(h)$ is open in V, being the complement of $V((h))$. If S is a set of generators for an ideal $\mathfrak{a}$, then

$$V \smallsetminus V(\mathfrak{a}) = \bigcup\nolimits_{h\in S} D(h),$$

and so the sets $D(h)$ form a base for the topology on V. Note that, because maximal ideals are prime,

$$D(h_1\cdots h_n) = D(h_1)\cap\cdots\cap D(h_n).$$

For any element h of A, $\operatorname{spm}(A_h) \simeq D(h)$ (see 6.4), and for any ideal $\mathfrak{a}$ in A, $\operatorname{spm}(A)/\mathfrak{a} \simeq V(\mathfrak{a})$ (isomorphisms of topological spaces).

The ideals in a finite product of rings $A = A_1\times\cdots\times A_n$ are all of the form $\mathfrak{a}_1\times\cdots\times\mathfrak{a}_n$ with $\mathfrak{a}_i$ an ideal in A_i (cf. p.7). The prime (resp. maximal) ideals are those of the form

$$A_1\times\cdots\times A_{i-1}\times\mathfrak{a}_i\times A_{i+1}\times\cdots\times A_n$$

with $\mathfrak{a}_i$ prime (resp. maximal). It follows that $\operatorname{spm}(A) = \bigsqcup_i \operatorname{spm}(A_i)$ (disjoint union of open subsets).

THE MAX SPECTRUM OF A FINITELY GENERATED k-ALGEBRA

Let k be a field, and let A be a finitely generated k-algebra. For any maximal ideal $\mathfrak{m}$ of A, the field $k(\mathfrak{m}) \stackrel{\text{def}}{=} A/\mathfrak{m}$ is a finitely generated k-algebra, and so $k(\mathfrak{m})$ is finite over k (Zariski's lemma, 10.1). In particular, it equals $k(\mathfrak{m}) = k$ when k is algebraically closed.

Now fix an algebraic closure k^{al}. The image of any k-algebra homomorphism $A\to k^{\mathrm{al}}$ is a subring of k^{al} which is an integral domain algebraic over k and therefore a field (see §1). Hence the kernel of the homomorphism is a maximal ideal in A. In this way, we get a surjective map

$$\operatorname{Hom}_{k\text{-alg}}(A,k^{\mathrm{al}}) \to \operatorname{spm}(A). \tag{16}$$

Two homomorphisms $A\to k^{\mathrm{al}}$ with the same kernel $\mathfrak{m}$ factor as

$$A\to k(\mathfrak{m})\to k^{\mathrm{al}},$$

and so differ by an automorphism[15] of k^{al}. Therefore, the fibres of (16) are exactly the orbits of $\operatorname{Gal}(k^{\mathrm{al}}/k)$. When k is perfect, each extension $k(\mathfrak{m})/k$ is separable, and so each orbit has $[k(\mathfrak{m}):k]$ elements, and when k is algebraically closed, the map (16) is a bijection.

Set $A = k[X_1,\dots,X_n]/\mathfrak{a}$. Then to give a homomorphism $A\to k^{\mathrm{al}}$ is the same as giving an n-tuple $(a_1,\dots,a_n)$ of elements of k^{al} (the images of the X_i) such that $f(a_1,\dots,a_n) = 0$ for all $f\in\mathfrak{a}$, i.e., an element of the zero-set $Z(\mathfrak{a})$ of $\mathfrak{a}$. The homomorphism corresponding to $(a_1,\dots,a_n)$ maps $k(\mathfrak{m})$ isomorphically onto the subfield of k^{al} generated by the a_i's. Therefore, we have a canonical surjection

$$Z(\mathfrak{a})\to\operatorname{spm}(A) \tag{17}$$

whose fibres are the orbits of $\operatorname{Gal}(k^{\mathrm{al}}/k)$. When the field k is perfect, each orbit has $[k[a_1,\dots,a_n]:k]$-elements, and when k is algebraically closed, $Z(\mathfrak{a})\simeq\operatorname{spm}(A)$.

[15]Let f and g be two k-homomorphisms from a finite field extension k' of k into k^{al}. We consider the set of pairs (K,α) in which α is a k-homomorphism from a subfield K of k^{al} containing $f(k')$ into k^{al} such that $\alpha\circ f = g$. The set is nonempty, and Zorn's lemma can be applied to show that it has a maximal element (K',α'). For such an element K' will be algebraically closed, and hence equal to k^{al}.

ASIDE 11.2. Let $k = \mathbb{R}$ or $\mathbb{C}$. Let X be a set and let A be a k-algebra of k-valued functions on X. In analysis, X is called the ***spectrum*** of A if, for each k-algebra homomorphism $\varphi: A \to k$, there exists a unique $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in A$, and every x arises from a φ (cf. Cartier 2007, 3.3.1, footnote).

Let A be a finitely generated algebra over an arbitrary algebraically closed field k, and let $X = \operatorname{spm}(A)$. An element f of A defines a k-valued function

$$\mathfrak{m} \mapsto f \mod \mathfrak{m}$$

on X. When A is reduced, Proposition 10.5 shows that this realizes A as a ring of k-valued functions on X. Moreover, because (17) is an isomorphism in this case, for each k-algebra homomorphism $\varphi: A \to k$, there exists a unique $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in A$. In particular, when $k = \mathbb{C}$ and A is reduced, $\operatorname{spm}(A)$ is the spectrum of A in the sense of analysis.

JACOBSON RINGS

DEFINITION 11.3. A ring A is ***Jacobson*** if every prime ideal in A is an intersection of maximal ideals.

A field is Jacobson. The ring $\mathbb{Z}$ is Jacobson because every nonzero prime ideal is maximal and $(0) = \bigcap_{p=2,3,5,\dots}(p)$. A principal ideal domain (more generally, a Dedekind domain) is Jacobson if it has an infinite number of maximal ideals.[16] A local ring is Jacobson if and only if its maximal ideal is its only prime ideal. Proposition 10.5 shows that every finitely generated algebra over a field is Jacobson.

PROPOSITION 11.4. *The radical of an ideal in a Jacobson ring is equal to the intersection of the maximal ideals containing it. (Therefore, the radical ideals are precisely the intersections of maximal ideals.)*

PROOF. Proposition 2.6 says that the radical of an ideal is an intersection of prime ideals, and so this follows from the definition of a Jacobson ring. □

ASIDE 11.5. Any ring of finite type over a Jacobson ring is a Jacobson ring (EGA IV 10.4.6). Moreover, if B is of finite type over A and A is Jacobson, then the map $A \to B$ defines a continuous map $\operatorname{spm}(B) \to \operatorname{spm}(A)$.

THE TOPOLOGICAL SPACE $\operatorname{spm}(A)$

We study more closely the Zariski topology on $\operatorname{spm}(A)$. For each subset S of A, let $V(S)$ denote the set of maximal ideals containing S, and for each subset W of $\operatorname{spm}(A)$, let $I(W)$ denote the intersection of the maximal ideals in W:

$$\begin{aligned} &S \subset A, && V(S) = \{\mathfrak{m} \in \operatorname{spm}(A) \mid S \subset \mathfrak{m}\},\\ &W \subset \operatorname{spm}(A), && I(W) = \textstyle\bigcap_{\mathfrak{m}\in W} \mathfrak{m}. \end{aligned}$$

Thus $V(S)$ is a closed subset of $\operatorname{spm}(A)$ and $I(W)$ is a radical ideal in A. If $V(\mathfrak{a}) \supset W$, then $\mathfrak{a} \subset I(W)$, and so $V(\mathfrak{a}) \supset VI(W)$. Therefore $VI(W)$ is the closure of W (smallest closed subset of $\operatorname{spm}(A)$ containing W); in particular, $VI(W) = W$ if W is closed.

[16] In a principal ideal domain, a nonzero element a factors as $a = up_1^{r_1}\cdots p_s^{r_s}$ with u a unit and the p_i prime. The only prime divisors of a are $p_1, \dots, p_s$, and so a is contained in only finitely many prime ideals. Similarly, in a Dedekind domain, a nonzero ideal $\mathfrak{a}$ factors as $\mathfrak{a} = \mathfrak{p}_1^{r_1}\cdots\mathfrak{p}_s^{r_s}$ with the $\mathfrak{p}_i$ prime ideals (cf. 13.17 below), and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the only prime ideals containing $\mathfrak{a}$. On taking $\mathfrak{a} = (a)$, we see that again a is contained in only finitely many prime ideals.

PROPOSITION 11.6. *Let V be a closed subset of* $\operatorname{spm}(A)$.

(a) The points of V are closed for the Zariski topology.

(b) If A is noetherien, then every ascending chain of open subsets $U_1 \subset U_2 \subset \cdots$ of V eventually becomes constant; equivalently, every descending chain of closed subsets of V eventually becomes constant.

(c) If A is noetherian, every open covering of V has a finite subcovering.

PROOF. (a) Clearly $\{\mathfrak{m}\} = V(\mathfrak{m})$, and so it is closed.

(b) We prove the second statement. A sequence $V_1 \supset V_2 \supset \cdots$ of closed subsets of V gives rise to a sequence of ideals $I(V_1) \subset I(V_2) \subset \ldots$, which eventually becomes constant. If $I(V_m) = I(V_{m+1})$, then $VI(V_m) = VI(V_{m+1})$, i.e., $V_m = V_{m+1}$.

(c) Let $V = \bigcup_{i\in I} U_i$ with each U_i open. Choose an $i_0 \in I$; if $U_{i_0} \neq V$, then there exists an $i_1 \in I$ such that $U_{i_0} \subsetneq U_{i_0} \cup U_{i_1}$. If $U_{i_0} \cup U_{i_1} \neq V$, then there exists an $i_2 \in I$ etc.. Because of (b), this process must eventually stop. □

A topological space V having the property (b) is said to be ***noetherian***. This condition is equivalent to the following: every nonempty set of closed subsets of V has a minimal element. A topological space V having property (c) is said to be ***quasicompact*** (by Bourbaki at least; others call it compact, but Bourbaki requires a compact space to be Hausdorff). The proof of (c) shows that every noetherian space is quasicompact. Since an open subspace of a noetherian space is again noetherian, it will also be quasicompact.

DEFINITION 11.7. A nonempty topological space is said to be ***irreducible*** if it is not the union of two proper closed subsets. Equivalent conditions: any two nonempty open subsets have a nonempty intersection; every nonempty open subset is dense.

If an irreducible space W is a finite union of closed subsets, $W = W_1 \cup \ldots \cup W_r$, then $W = W_1$ or $W_2 \cup \ldots \cup W_r$; if the latter, then $W = W_2$ or $W_3 \cup \ldots \cup W_r$, etc.. Continuing in this fashion, we find that $W = W_i$ for some i.

The notion of irreducibility is not useful for Hausdorff topological spaces, because the only irreducible Hausdorff spaces are those consisting of a single point — two points would have disjoint open neighbourhoods.

PROPOSITION 11.8. *Let W be a closed subset of* $\operatorname{spm}(A)$. *If W is irreducible, then $I(W)$ is prime; the converse is true if A is a Jacobson ring. In particular, the max spectrum of a Jacobson ring A is irreducible if and only if the nilradical of A is prime.*

PROOF. $\Rightarrow$: Let W be an irreducible closed subset of $\operatorname{spm}(A)$, and suppose $fg \in I(W)$. Then fg lies in each $\mathfrak{m}$ in W, and so either $f \in \mathfrak{m}$ or $g \in \mathfrak{m}$; hence $W \subset V(f) \cup V(g)$, and so

$$W = (W \cap V(f)) \cup (W \cap V(g)).$$

As W is irreducible, one of these sets, say $W \cap V(f)$, must equal W. But then $f \in I(W)$. We have shown that $I(W)$ is prime.

$\Leftarrow$: Assume $I(W)$ is prime, and suppose $W = V(\mathfrak{a}) \cup V(\mathfrak{b})$ with $\mathfrak{a}$ and $\mathfrak{b}$ radical ideals — we have to show that W equals $V(\mathfrak{a})$ or $V(\mathfrak{b})$. Recall that $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$ (see 11.1c) and that $\mathfrak{a} \cap \mathfrak{b}$ is radical; hence $I(W) = \mathfrak{a} \cap \mathfrak{b}$ (by 11.4). If $W \neq V(\mathfrak{a})$, then there exists an $f \in \mathfrak{a} \smallsetminus I(W)$. For all $g \in \mathfrak{b}$,

$$fg \in \mathfrak{a} \cap \mathfrak{b} = I(W).$$

Because $I(W)$ is prime, this implies that $\mathfrak{b} \subset I(W)$; therefore $W \subset V(\mathfrak{b})$. □

Thus, in the max spectrum of a Jacobson ring, there are one-to-one correspondences

$$\begin{aligned} \text{radical ideals} &\leftrightarrow \text{closed subsets} \\ \text{prime ideals} &\leftrightarrow \text{irreducible closed subsets} \\ \text{maximal ideals} &\leftrightarrow \text{one-point sets.} \end{aligned}$$

EXAMPLE 11.9. Let $f \in k[X_1,\dots,X_n]$. According to Theorem 4.6, $k[X_1,\dots,X_n]$ is a unique factorization domain, and so (f) is a prime ideal if and only if f is irreducible (4.1). Thus

$$V(f) \text{ is irreducible} \iff f \text{ is irreducible.}$$

On the other hand, suppose f factors,

$$f = \prod f_i^{m_i}, \quad f_i \text{ distinct irreducible polynomials.}$$

Then

$$\begin{aligned} (f) &= \bigcap (f_i^{m_i}), \quad (f_i^{m_i}) \text{ distinct ideals,} \\ \operatorname{rad}((f)) &= \bigcap (f_i), \quad (f_i) \text{ distinct prime ideals,} \\ V(f) &= \bigcup V(f_i), \quad V(f_i) \text{ distinct irreducible algebraic sets.} \end{aligned}$$

PROPOSITION 11.10. *Let V be a noetherian topological space. Then V is a finite union of irreducible closed subsets, $V = V_1 \cup \dots \cup V_m$. If the decomposition is irredundant in the sense that there are no inclusions among the V_i, then the V_i are uniquely determined up to order.*

PROOF. Suppose that V can not be written as a *finite* union of irreducible closed subsets. Then, because V is noetherian, there will be a closed subset W of V that is minimal among those that cannot be written in this way. But W itself cannot be irreducible, and so $W = W_1 \cup W_2$, with each W_i a proper closed subset of W. Because W is minimal, both W_1 and W_2 can be expressed as finite unions of irreducible closed subsets, but then so can W. We have arrived at a contradiction.

Suppose that

$$V = V_1 \cup \dots \cup V_m = W_1 \cup \dots \cup W_n$$

are two irredundant decompositions. Then $V_i = \bigcup_j (V_i \cap W_j)$, and so, because V_i is irreducible, $V_i = V_i \cap W_j$ for some j. Consequently, there exists a function $f\colon\{1,\dots,m\} \to \{1,\dots,n\}$ such that $V_i \subset W_{f(i)}$ for each i. Similarly, there is a function $g\colon\{1,\dots,n\} \to \{1,\dots,m\}$ such that $W_j \subset V_{g(j)}$ for each j. Since $V_i \subset W_{f(i)} \subset V_{gf(i)}$, we must have $gf(i) = i$ and $V_i = W_{f(i)}$; similarly $fg = \mathrm{id}$. Thus f and g are bijections, and the decompositions differ only in the numbering of the sets. □

The V_i given uniquely by the proposition are called the ***irreducible components*** of V. They are the maximal closed irreducible subsets of V. In Example 11.9, the $V(f_i)$ are the irreducible components of $V(f)$.

COROLLARY 11.11. *A radical ideal $\mathfrak{a}$ in a noetherian Jacobson ring is a finite intersection of prime ideals, $\mathfrak{a} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$; if there are no inclusions among the $\mathfrak{p}_i$, then the $\mathfrak{p}_i$ are uniquely determined up to order.*

PROOF. Write $V(\mathfrak{a})$ as a union of its irreducible components, $V(\mathfrak{a}) = \bigcup V_i$, and take $\mathfrak{p}_i = I(V_i)$. □

REMARK 11.12. (a) An irreducible topological space is connected, but a connected topological space need not be irreducible. For example, $Z(X_1X_2)$ is the union of the coordinate axes in k^2, which is connected but not irreducible. A closed subset V of $\operatorname{spm}(A)$ is not connected if and only if there exist ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} \cap \mathfrak{b} = I(V)$ and $\mathfrak{a} + \mathfrak{b} = A$.

(b) A Hausdorff space is noetherian if and only if it is finite, in which case its irreducible components are the one-point sets.

(c) In a noetherian ring, every proper ideal $\mathfrak{a}$ has a decomposition into primary ideals: $\mathfrak{a} = \bigcap \mathfrak{q}_i$ (see §13). For radical ideals, this becomes a simpler decomposition into prime ideals, as in the corollary. For an ideal (f) in $k[X_1, \dots, X_n]$ with $f = \prod f_i^{m_i}$, it is the decomposition $(f) = \bigcap (f_i^{m_i})$ noted in Example 11.9.

MAPS OF MAX SPECTRA

Let $\varphi: A \to B$ be a homomorphism of finitely generated k-algebras (k a field). Because B is finitely generated over k, its quotient $B/\mathfrak{m}$ by any maximal ideal $\mathfrak{m}$ is a finite field extension of k (Zariski's lemma, 10.1). Therefore the image of A in $B/\mathfrak{m}$ is an integral domain finite over k, and hence is a field (see §1). Since this image is isomorphic to $A/\varphi^{-1}(\mathfrak{m})$, this shows that the ideal $\varphi^{-1}(\mathfrak{m})$ is maximal in A. Therefore φ defines a map

$$\varphi^*: \operatorname{spm}(B) \to \operatorname{spm}(A), \quad \mathfrak{m} \mapsto \varphi^{-1}(\mathfrak{m}),$$

which is continuous because $(\varphi^*)^{-1}(D(f)) = D(\varphi(f))$. In this way, spm becomes a functor from finitely generated k-algebras to topological spaces.

THEOREM 11.13. *Let $\varphi: A \to B$ be a homomorphism of finitely generated k-algebras. Let U be a nonempty open subset of* $\operatorname{spm}(B)$, *and let $\varphi^*(U)^-$ be the closure of its image in* $\operatorname{spm}(A)$. *Then $\varphi^*(U)$ contains a nonempty open subset of each irreducible component of $\varphi^*(U)^-$.*

PROOF. Let $W = \operatorname{spm}(B)$ and $V = \operatorname{spm}(A)$, so that φ^* is a continuous map $W \to V$.

We first prove the theorem in the case that φ is an injective homomorphism of integral domains. For some $b \neq 0$, $D(b) \subset U$. According to Proposition 11.14 below, there exists a nonzero element $a \in A$ such that every homomorphim $\alpha: A \to k^{\mathrm{al}}$ such that $\alpha(a) \neq 0$ extends to a homomorphism $\beta: B \to k^{\mathrm{al}}$ such that $\beta(b) \neq 0$. Let $\mathfrak{m} \in D(a)$, and choose α to be a homomorphism $A \to k^{\mathrm{al}}$ with kernel $\mathfrak{m}$. The kernel of β is a maximal ideal $\mathfrak{n} \in D(b)$ such that $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$, and so $D(a) \subset \varphi^*(D(b))$.

We now prove the general case. If $W_1, \dots, W_r$ are the irreducible components of W, then $\varphi^*(W)^-$ is a union of the sets $\varphi^*(W_i)^-$, and any irreducible component C of $\varphi^*(U)^-$ is contained in one of $\varphi^*(W_i)^-$, say $\varphi^*(W_1)^-$. Let $\mathfrak{q} = I(W_1)$ and let $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Because W_1 is irreducible, they are both prime ideals. The homomorphism $\varphi: A \to B$ induces an injective homomorphism $\bar{\varphi}: A/\mathfrak{p} \to B/\mathfrak{q}$, and $\bar{\varphi}^*$ can be identified with the restriction of φ^* to W_1. From the first case, we know that $\bar{\varphi}^*(U \cap W_1)$ contains a nonempty open subset of C, which implies that $\varphi^*(U)$ does also. □

In the next two statements, A and B are arbitrary commutative rings — they need not be k-algebras.

PROPOSITION 11.14. *Let $A \subset B$ be integral domains with B finitely generated as an algebra over A, and let b be a nonzero element of B. Then there exists an element $a \neq 0$ in A with the following property: every homomorphism $\alpha: A \to \Omega$ from A into an algebraically closed field Ω such that $\alpha(a) \neq 0$ can be extended to a homomorphism $\beta: B \to \Omega$ such that $\beta(b) \neq 0$.*

We first need a lemma.

LEMMA 11.15. *Let $B \supset A$ be integral domains, and assume $B = A[t] = A[T]/\mathfrak{a}$. Let $\mathfrak{c} \subset A$ be the ideal of leading coefficients of the polynomials in $\mathfrak{a}$. Then every homomorphism $\alpha: A \to \Omega$ from A into an algebraically closed field Ω such that $\alpha(\mathfrak{c}) \neq 0$ can be extended to a homomorphism of B into Ω.*

PROOF. If $\mathfrak{a} = 0$, then $\mathfrak{c} = 0$, and every α extends. Thus we may assume $\mathfrak{a} \neq 0$. Let α be a homomorphism $A \to \Omega$ such that $\alpha(\mathfrak{c}) \neq 0$. Then there exist polynomials $a_m T^m + \cdots + a_0$ in $\mathfrak{a}$ such that $\alpha(a_m) \neq 0$, and we choose one, denoted f, of minimum degree. Because $B \neq 0$, the polynomial f is nonconstant.

Extend α to a homomorphism $A[T] \to \Omega[T]$, again denoted α, by sending T to T, and consider the subset $\alpha(\mathfrak{a})$ of $\Omega[T]$.

FIRST CASE: $\alpha(\mathfrak{a})$ DOES NOT CONTAIN A NONZERO CONSTANT. If the Ω-subspace of $\Omega[T]$ spanned by $\alpha(\mathfrak{a})$ contained 1, then so also would $\alpha(\mathfrak{a})$,[17] contrary to hypothesis. Because

$$T \cdot \textstyle\sum c_i \alpha(g_i) = \sum c_i \alpha(g_i T), \quad c_i \in \Omega, \quad g_i \in \mathfrak{a},$$

this Ω-subspace an ideal, which we have shown to be proper, and so it has a zero c in Ω. The composite of the homomorphisms

$$A[T] \xrightarrow{\alpha} \Omega[T] \longrightarrow \Omega, \qquad T \mapsto T \mapsto c,$$

factors through $A[T]/\mathfrak{a} = B$ and extends α.

SECOND CASE: $\alpha(\mathfrak{a})$ CONTAINS A NONZERO CONSTANT. This means that $\mathfrak{a}$ contains a polynomial

$$g(T) = b_n T^n + \cdots + b_0 \quad \text{such that} \quad \alpha(b_0) \neq 0, \quad \alpha(b_1) = \alpha(b_2) = \cdots = 0.$$

On dividing $f(T)$ into $g(T)$ we obtain an equation

$$a_m^d g(T) = q(T) f(T) + r(T), \quad d \in \mathbb{N}, \quad q, r \in A[T], \quad \deg r < m.$$

When we apply α, this becomes

$$\alpha(a_m)^d \alpha(b_0) = \alpha(q)\alpha(f) + \alpha(r).$$

Because $\alpha(f)$ has degree $m > 0$, we must have $\alpha(q) = 0$, and so $\alpha(r)$ is a nonzero constant. After replacing $g(T)$ with $r(T)$, we may suppose $n < m$. If $m = 1$, such a $g(T)$ can't exist, and so we may suppose $m > 1$ and (by induction) that the lemma holds for smaller values of m.

[17] Use that, if a system of linear equation with coefficients in a field k has a solution in some larger field, then it has a solution in k.

For $h(T) = c_r T^r + c_{r-1}T^{r-1} + \cdots + c_0$, let $h'(T) = c_r + \cdots + c_0 T^r$. Then the A-module generated by the polynomials $T^s h'(T)$, $s \geq 0$, $h \in \mathfrak{a}$, is an ideal $\mathfrak{a}'$ in $A[T]$. Moreover, $\mathfrak{a}'$ contains a nonzero constant if and only if $\mathfrak{a}$ contains a nonzero polynomial cT^r, which implies $t = 0$ and $A = B$ (since B is an integral domain).

When $\mathfrak{a}'$ does not contain a nonzero constant, we set $B' = A[T]/\mathfrak{a}' = A[t']$. Then $\mathfrak{a}'$ contains the polynomial $g' = b_n + \cdots + b_0 T^n$, and $\alpha(b_0) \neq 0$. Because $\deg g' < m$, the induction hypothesis implies that α extends to a homomorphism $B' \to \Omega$. Therefore, there exists a $c \in \Omega$ such that, for all $h(T) = c_r T^r + c_{r-1}T^{r-1} + \cdots + c_0 \in \mathfrak{a}$,

$$h'(c) = \alpha(c_r) + \alpha(c_{r-1})c + \cdots + c_0 c^r = 0.$$

On taking $h = g$, we see that $c = 0$, and on taking $h = f$, we obtain the contradiction $\alpha(a_m) = 0$. □

PROOF (OF 11.14) Suppose that we know the proposition in the case that B is generated by a single element, and write $B = A[t_1, \ldots, t_n]$. Then there exists an element b_{n-1} such that any homomorphism $\alpha: A[t_1, \ldots, t_{n-1}] \to \Omega$ such that $\alpha(b_{n-1}) \neq 0$ extends to a homomorphism $\beta: B \to \Omega$ such that $\beta(b) \neq 0$. Continuing in this fashion (with b_{n-1} for b), we eventually obtain an element $a \in A$ with the required property.

Thus we may assume $B = A[t]$. Let $\mathfrak{a}$ be the kernel of the homomorphism $T \mapsto t$, $A[T] \to A[t]$.

Case (i). The ideal $\mathfrak{a} = (0)$. Write

$$b = f(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n, \quad a_i \in A,$$

and take $a = a_0$. If $\alpha: A \to \Omega$ is such that $\alpha(a_0) \neq 0$, then there exists a $c \in \Omega$ such that $f(c) \neq 0$, and we can take β to be the homomorphism $\sum d_i t^i \mapsto \sum \alpha(d_i)c^i$.

Case (ii). The ideal $\mathfrak{a} \neq (0)$. Let $f(T) = a_m T^m + \cdots + a_0$, $a_m \neq 0$, be an element of $\mathfrak{a}$ of minimum degree. Let $h(T) \in A[T]$ represent b. Since $b \neq 0$, $h \notin \mathfrak{a}$. Because f is irreducible over the field of fractions of A, it and h are coprime over that field. In other words, there exist $u, v \in A[T]$ and a nonzero $c \in A$ such that

$$uh + vf = c.$$

It follows now that ca_m satisfies our requirements, for if $\alpha(ca_m) \neq 0$, then α can be extended to $\beta: B \to \Omega$ by the lemma, and $\beta(u(t) \cdot b) = \beta(c) \neq 0$, and so $\beta(b) \neq 0$. □

REMARK 11.16. In case (ii) of the last proof, both b and b^{-1} are algebraic over A, and so there exist equations

$$a_0 b^m + \cdots + a_m = 0, \quad a_i \in A, \quad a_0 \neq 0;$$
$$a_0' b^{-n} + \cdots + a_n' = 0, \quad a_i' \in A, \quad a_0' \neq 0.$$

One can show that $a = a_0 a_0'$ has the property required by the proposition.

ASIDE 11.17. The spectrum $\operatorname{spec}(A)$ of a ring A is the set of prime ideals in A endowed with the topology for which the closed subsets are those of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{p} \supset \mathfrak{a}\}, \quad \mathfrak{a} \text{ an ideal in } A.$$

Thus $\operatorname{spm}(A)$ is the subspace of $\operatorname{spec}(A)$ consisting of the closed points. When A is Jacobson, the map $U \mapsto U \cap \operatorname{spm}(A)$ is a bijection from the set of open subsets of $\operatorname{spec}(A)$ onto the set of open subsets of $\operatorname{spm}(A)$; therefore $\operatorname{spm}(A)$ and $\operatorname{spec}(A)$ have the same topologies — only the underlying sets differ.

12 Dimension theory for finitely generated k-algebras

Throughout this section, A is both a finitely generated algebra over field k and an integral domain. We define the transcendence degree of A over k, $\operatorname{tr}\deg_k A$, to be the transcendence degree over k of the field of fractions of A (see FT[18] §8). Thus A has transcendence degree d if it contains an algebraically independent set of d elements, but no larger set (FT 8.12).

PROPOSITION 12.1. *For any linear forms $\ell_1,\dots,\ell_m$ in $X_1,\dots,X_n$, the quotient ring*

$$k[X_1,\dots,X_n]/(\ell_1,\dots,\ell_m)$$

is an integral domain of transcendence degree equal to the dimension of the subspace of k^n defined by the equations

$$\ell_i = 0, \quad i = 1,\dots,m.$$

PROOF. This follows from the more precise statement:

> Let $\mathfrak{c}$ be an ideal in $k[X_1,\dots,X_n]$ generated by linearly independent linear forms $\ell_1,\dots,\ell_r$, and let $X_{i_1},\dots,X_{i_{n-r}}$ be such that
>
> $$\{\ell_1,\dots,\ell_r,X_{i_1},\dots,X_{i_{n-r}}\}$$
>
> is a basis for the linear forms in $X_1,\dots,X_n$. Then
>
> $$k[X_1,\dots,X_n]/\mathfrak{c} \simeq k[X_{i_1},\dots,X_{i_{n-r}}].$$

This is obvious if the forms ℓ_i are $X_1,\dots,X_r$. In the general case, because $\{X_1,\dots,X_n\}$ and $\{\ell_1,\dots,\ell_r,X_{i_1},\dots,X_{i_{n-r}}\}$ are both bases for the linear forms, each element of one set can be expressed as a linear combination of the elements of the other. Therefore,

$$k[X_1,\dots,X_n] = k[\ell_1,\dots,\ell_r,X_{i_1},\dots,X_{i_{n-r}}],$$

and so

$$\begin{aligned} k[X_1,\dots,X_n]/\mathfrak{c} &= k[\ell_1,\dots,\ell_r,X_{i_1},\dots,X_{i_{n-r}}]/\mathfrak{c} \\ &\simeq k[X_{i_1},\dots,X_{i_{n-r}}]. \end{aligned}$$

□

PROPOSITION 12.2. *For any irreducible polynomial f in $k[X_1,\dots,X_n]$, the quotient ring $k[X_1,\dots,X_n]/(f)$ has transcendence degree $n-1$.*

PROOF. Let

$$k[x_1,\dots,x_n] = k[X_1,\dots,X_n]/(f), \quad x_i = X_i + (f),$$

and let $k(x_1,\dots,x_n)$ be the field of fractions of $k[x_1,\dots,x_n]$. Since f is not zero, some X_i, say, X_n, occurs in it. Then X_n occurs in every nonzero multiple of f, and so no nonzero polynomial in $X_1,\dots,X_{n-1}$ belongs to (f). This means that $x_1,\dots,x_{n-1}$ are algebraically independent. On the other hand, x_n is algebraic over $k(x_1,\dots,x_{n-1})$, and so $\{x_1,\dots,x_{n-1}\}$ is a transcendence basis for $k(x_1,\dots,x_n)$ over k. □

[18] FT = Fields and Galois Theory, available on my website.

PROPOSITION 12.3. *For any nonzero prime ideal $\mathfrak{p}$ in a k-algebra A,*

$$\operatorname{tr.deg}_k(A/\mathfrak{p}) < \operatorname{tr.deg}_k(A).$$

PROOF. We may suppose

$$A = k[X_1,\dots,X_n]/\mathfrak{a} = k[x_1,\dots,x_n].$$

For $f \in A$, let $\bar{f}$ denote the image of f in $A/\mathfrak{p}$, so that $A/\mathfrak{p} = k[\bar{x}_1,\dots,\bar{x}_n]$. Let $d = \operatorname{tr.deg}_k A/\mathfrak{p}$, and number the X_i so that $\bar{x}_1,\dots,\bar{x}_d$ are algebraically independent (see FT 8.9 for the proof that this is possible). I shall show that, for any nonzero $f \in \mathfrak{p}$, the $d+1$ elements $x_1,\dots,x_d, f$ are algebraically independent, which shows that $\operatorname{tr.deg}_k A \geq d+1$.

Suppose otherwise. Then there is a nontrivial algebraic relation, which we can write

$$a_0(x_1,\dots,x_d)f^m + a_1(x_1,\dots,x_d)f^{m-1} + \cdots + a_m(x_1,\dots,x_d) = 0,$$

with $a_i \in k[X_1,\dots,X_d]$ and $a_0 \neq 0$. Because A is an integral domain, we can cancel a power of f if necessary to make $a_m(x_1,\dots,x_d)$ nonzero. On applying the homomorphism $A \to A/\mathfrak{p}$ to the above equality, we find that

$$a_m(\bar{x}_1,\dots,\bar{x}_d) = 0,$$

which contradicts the algebraic independence of $\bar{x}_1,\dots,\bar{x}_d$. □

PROPOSITION 12.4. *Let A be a unique factorization domain. If $\mathfrak{p}$ is a prime ideal in A such that $\operatorname{tr.deg}_k A/\mathfrak{p} = \operatorname{tr.deg}_k A - 1$, then $\mathfrak{p} = (f)$ for some $f \in A$.*

PROOF. The ideal $\mathfrak{p}$ is nonzero because otherwise A and $A/\mathfrak{p}$ would have the same transcendence degree. Therefore $\mathfrak{p}$ contains a nonzero polynomial, and even an irreducible polynomial f, because it is prime. According to (4.1), the ideal (f) is prime. If $(f) \neq \mathfrak{p}$, then

$$\operatorname{tr.deg}_k A/\mathfrak{p} \overset{12.3}{>} \operatorname{tr.deg}_k A/(f) \overset{12.2}{=} \operatorname{tr.deg}_k A - 1,$$

which contradicts the hypothesis. □

THEOREM 12.5. *Let $f \in A$ be neither zero nor a unit, and let $\mathfrak{p}$ be a prime ideal that is minimal among those containing (f); then*

$$\operatorname{tr.deg}_k A/\mathfrak{p} = \operatorname{tr.deg}_k A - 1.$$

We first need a lemma.

LEMMA 12.6. *Let A be an integrally closed integral domain, and let L be a finite extension of the field of fractions K of A. If $\alpha \in L$ is integral over A, then $\operatorname{Nm}_{L/K}\alpha \in A$, and α divides $\operatorname{Nm}_{L/K}\alpha$ in the ring $A[\alpha]$.*

PROOF. Let $X^r + a_{r-1}X^{r-1} + \cdots + a_0$ be the minimum polynomial of α over K. Then r divides the degree n of L/K, and $\operatorname{Nm}_{L/K}(\alpha) = \pm a_0^{\frac{n}{r}}$ (FT 5.40). Moreover, a_0 lies in A by (5.9). From the equation

$$0 = \alpha(\alpha^{r-1} + a_{r-1}\alpha^{r-2} + \cdots + a_1) + a_0$$

we see that α divides a_0 in $A[\alpha]$, and therefore it also divides $\operatorname{Nm}_{L/K}\alpha$. □

PROOF (OF THEOREM 12.5). Write $\operatorname{rad}(f)$ as an irredundant intersection of prime ideals $\operatorname{rad}(f) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ (see 11.11). Then $V(\mathfrak{a}) = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_r)$ is the decomposition of $V(\mathfrak{a})$ into its irreducible components. There exists an $\mathfrak{m}_0 \in V(\mathfrak{p}_1) \smallsetminus \bigcup_{i\geq 2} V(\mathfrak{p}_i)$ and an open neighbourhood $D(h)$ of $\mathfrak{m}_0$ disjoint from $\bigcup_{i\geq 2} V(\mathfrak{p}_i)$. The ring A_h (resp. $A_h/S^{-1}\mathfrak{p}$) is an integral domain with the same transcendance degree as A (resp. $A/\mathfrak{p}$) — in fact, with the same field of fractions. In A_h, $\operatorname{rad}(\frac{f}{1}) = \operatorname{rad}(f)^e = \mathfrak{p}_1^e$. Therefore, after replacing A with A_h, we may suppose that $\operatorname{rad}(f)$ is prime, say, equal to $\mathfrak{p}$.

According to the Noether normalization theorem (5.11), there exist algebraically independent elements $x_1, \ldots, x_d$ in A such that A is a finite $k[x_1, \ldots, x_d]$-algebra. Note that $d = \operatorname{tr\,deg}_k A$. According to the lemma, $f_0 \stackrel{\text{def}}{=} \operatorname{Nm}(f)$ lies in $k[x_1, \ldots, x_d]$, and we shall show that $\mathfrak{p} \cap k[x_1, \ldots, x_d] = \operatorname{rad}(f_0)$. Therefore, the homomorphism

$$k[x_1, \ldots, x_d]/\operatorname{rad}(f_0) \to A/\mathfrak{p}$$

is injective. As it is also finite, this implies that

$$\operatorname{tr\,deg}_k A/\mathfrak{p} = \operatorname{tr\,deg}_k k[x_1, \ldots, x_d]/\operatorname{rad}(f_0) \stackrel{12.2}{=} d - 1,$$

as required.

By assumption A is finite (hence integral) over its subring $k[x_1, \ldots, x_d]$. The lemma shows that f divides f_0 in A, and so $f_0 \in (f) \subset \mathfrak{p}$. Hence $(f_0) \subset \mathfrak{p} \cap k[x_1, \ldots, x_d]$, which implies

$$\operatorname{rad}(f_0) \subset \mathfrak{p} \cap k[x_1, \ldots, x_d]$$

because $\mathfrak{p}$ is radical. For the reverse inclusion, let $g \in \mathfrak{p} \cap k[x_1, \ldots, x_d]$. Then $g \in \operatorname{rad}(f)$, and so $g^m = fh$ for some $h \in A$, $m \in \mathbb{N}$. Taking norms, we find that

$$g^{me} = \operatorname{Nm}(fh) = f_0 \cdot \operatorname{Nm}(h) \in (f_0),$$

where e is the degree of the extension of the fields of fractions, which proves the claim. □

COROLLARY 12.7. *Let $\mathfrak{p}$ be a minimal nonzero prime ideal in A; then* $\operatorname{tr\,deg}_k(A/\mathfrak{p}) = \operatorname{tr\,deg}_k(A) - 1$.

PROOF. Let f be a nonzero element of $\mathfrak{p}$. Then f is not a unit, and $\mathfrak{p}$ is minimal among the prime ideals containing f. □

THEOREM 12.8. *The length d of any maximal (i.e., nonrefinable) chain of distinct prime ideals*

$$\mathfrak{p}_d \supset \mathfrak{p}_{d-1} \supset \cdots \supset \mathfrak{p}_0 \tag{18}$$

in A is $\operatorname{tr\,deg}_k(A)$. *In particular, every maximal ideal of A has height* $\operatorname{tr\,deg}_k(A)$, *and so the Krull dimension of A is equal to* $\operatorname{tr\,deg}_k(A)$.

PROOF. From (12.7), we find that

$$\operatorname{tr\,deg}_k(A) = \operatorname{tr\,deg}_k(A/\mathfrak{p}_1) + 1 = \cdots = \operatorname{tr\,deg}_k(A/\mathfrak{p}_d) + d.$$

But $\mathfrak{p}_d$ is maximal, and so $A/\mathfrak{p}_d$ is a finite field extension of k. In particular, $\operatorname{tr\,deg}_k(A/\mathfrak{p}_d) = 0$. □

EXAMPLE 12.9. Let $f(X,Y)$ and $g(X,Y)$ be nonconstant polynomials with no common factor. Then $k[X,Y]/(f)$ has Krull dimension 1, and so $k[X,Y]/(f,g)$ has dimension zero.

EXAMPLE 12.10. We classify the prime ideals $\mathfrak{p}$ in $A = k[X,Y]$. If $A/\mathfrak{p}$ has dimension 2, then $\mathfrak{p} = (0)$. If $A/\mathfrak{p}$ has dimension 1, then $\mathfrak{p} = (f)$ for some irreducible polynomial f of A (by 12.4). Finally, if $A/\mathfrak{p}$ has dimension zero, then $\mathfrak{p}$ is maximal. Thus, when k is algebraically closed, the prime ideals in $k[X,Y]$ are exactly the ideals (0), (f) (with f irreducible), and $(X-a, Y-b)$ (with $a, b \in k$).

REMARK 12.11. Let A be a finitely generated k-algebra (not necessarily an integral domain). Every maximal chain of prime ideals in A ending in fixed prime ideal $\mathfrak{p}$ has length $\operatorname{tr.deg}_k(A/\mathfrak{p})$, and so the Krull dimension of A is $\max(\operatorname{tr.deg}_k(A/\mathfrak{p}))$ where $\mathfrak{p}$ runs over the minimal prime ideals of A. In the next section, we show that a noetherian ring has only finitely many minimal prime ideals, and so the Krull dimension of A is finite.

If $x_1, \dots, x_m$ is an algebraically independent set of elements of A such that A is a finite $k[x_1, \dots, x_m]$-algebra, then $\dim A = m$.

13 Primary decompositions

In this section, A is an arbitrary commutative ring.

DEFINITION 13.1. An ideal $\mathfrak{q}$ in A is ***primary*** if it is proper and

$$ab \in \mathfrak{q},\ b \notin \mathfrak{q} \implies a^n \in \mathfrak{q} \text{ for some } n \geq 1.$$

Thus, a proper ideal $\mathfrak{q}$ in A is primary if and only if all zero-divisors in $A/\mathfrak{q}$ are nilpotent. A radical ideal is primary if and only if it is prime. An ideal (m) in $\mathbb{Z}$ is primary if and only if m is a power of a prime.

PROPOSITION 13.2. *The radical of a primary ideal $\mathfrak{q}$ is a prime ideal containing $\mathfrak{q}$, and it is contained in every other prime ideal containing $\mathfrak{q}$ (i.e., it is the smallest prime ideal containing $\mathfrak{p}$).*

PROOF. Suppose $ab \in \operatorname{rad}(\mathfrak{q})$ but $b \notin \operatorname{rad}(\mathfrak{q})$. Then some power, say $a^n b^n$, of ab lies in $\mathfrak{q}$, but $b^n \notin \mathfrak{q}$, and so $a \in \operatorname{rad}(\mathfrak{q})$. The shows that $\operatorname{rad}(\mathfrak{q})$ is primary, and hence prime (because it is radical).

Let $\mathfrak{p}$ be a second prime ideal containing $\mathfrak{q}$, and let $a \in \operatorname{rad}(\mathfrak{q})$. For some n, $a^n \in \mathfrak{q} \subset \mathfrak{p}$, which implies that $a \in \mathfrak{p}$. □

When $\mathfrak{q}$ is a primary ideal and $\mathfrak{p}$ is its radical, we say that $\mathfrak{q}$ is ***$\mathfrak{p}$-primary.***

PROPOSITION 13.3. *Every ideal $\mathfrak{q}$ whose radical is a maximal ideal $\mathfrak{m}$ is primary (in fact, $\mathfrak{m}$-primary); in particular, every power of a maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-primary.*

PROOF. Every prime ideal containing $\mathfrak{q}$ contains its radical $\mathfrak{m}$, and therefore equals $\mathfrak{m}$. This shows that $A/\mathfrak{a}$ is local with maximal ideal $\mathfrak{m}/\mathfrak{a}$. Therefore, every element of $A/\mathfrak{a}$ is either a unit, and hence is not a zero-divisor, or it lies in $\mathfrak{m}/\mathfrak{a}$, and hence is nilpotent. □

PROPOSITION 13.4. *Let $\varphi: A \to B$ be a homomorphism of rings. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in B, then $\mathfrak{q}^c \overset{\text{def}}{=} \varphi^{-1}(\mathfrak{q})$ is a $\mathfrak{p}^c$-primary ideal in A.*

PROOF. The map $A/\mathfrak{q}^c \to B/\mathfrak{q}$ is injective, and so every zero-divisor in $A/\mathfrak{q}^c$ is nilpotent. This shows that $\mathfrak{q}^c$ is primary, and therefore $\mathrm{rad}(\mathfrak{q}^c)$-primary. But (see 2.11), $\mathrm{rad}(\mathfrak{q}^c) = \mathrm{rad}(\mathfrak{q})^c = \mathfrak{p}^c$, as claimed. □

LEMMA 13.5. *Let $\mathfrak{q}$ and $\mathfrak{p}$ be a pair of ideals in A such that $\mathfrak{q} \subset \mathfrak{p} \subset \mathrm{rad}(\mathfrak{q})$ and*

$$ab \in \mathfrak{q} \implies a \in \mathfrak{p} \text{ or } b \in \mathfrak{q}. \tag{19}$$

Then $\mathfrak{p}$ is a prime ideal and $\mathfrak{q}$ is $\mathfrak{p}$-primary.

PROOF. Clearly $\mathfrak{q}$ is primary, hence $\mathrm{rad}(\mathfrak{q})$-primary, and $\mathrm{rad}(\mathfrak{q})$ is prime. By assumption $\mathfrak{p} \subset \mathrm{rad}(\mathfrak{q})$, and it remains to show that they are equal. Let $a \in \mathrm{rad}(\mathfrak{q})$, and let n be the smallest positive integer such that $a^n \in \mathfrak{q}$. If $n = 1$, then $a \in \mathfrak{q} \subset \mathfrak{p}$; on the other hand, if $n > 1$, then $a^n = aa^{n-1} \in \mathfrak{q}$ and $a^{n-1} \notin \mathfrak{q}$, and so $a \in \mathfrak{p}$ by (19). □

PROPOSITION 13.6. *A finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.*

PROOF. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ be $\mathfrak{p}$-primary, and let $\mathfrak{q} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$. We show that the pair of ideals $\mathfrak{q} \subset \mathfrak{p}$ satisfies the conditions of (13.5).

Let $a \in \mathfrak{p}$; since some power of a belongs to each $\mathfrak{q}_i$, a sufficiently high power of it will belong to all of them, and so $\mathfrak{p} \subset \mathrm{rad}(\mathfrak{q})$.

Let $ab \in \mathfrak{q}$ but $a \notin \mathfrak{p}$. Then $ab \in \mathfrak{q}_i$ but $a \notin \mathfrak{p}$, and so $b \in \mathfrak{q}_i$. Since this is true for all i, we have that $b \in \mathfrak{q}$. □

The ***minimal prime ideals*** of an ideal $\mathfrak{a}$ are the minimal elements of the set of prime ideals containing $\mathfrak{a}$.

DEFINITION 13.7. A ***primary decomposition*** of an ideal $\mathfrak{a}$ is a finite set of primary ideals whose intersection is $\mathfrak{a}$. A primary decomposition S of $\mathfrak{a}$ is ***minimal*** if

(a) the prime ideals $\mathrm{rad}(\mathfrak{q})$, $\mathfrak{q} \in S$, are distinct, and
(b) no element of S can be omitted, i.e., for no $\mathfrak{q}_0 \in S$ is $\mathfrak{q}_0 \supset \bigcap\{\mathfrak{q} \mid \mathfrak{q} \in S, \mathfrak{q} \neq \mathfrak{q}_0\}$.

If $\mathfrak{a}$ admits a primary decomposition, then it admits a minimal primary decomposition, because Proposition 13.6 can be used to combine primary ideals with the same radical, and any $\mathfrak{q}_i$ that fails (b) can simply be omitted. The prime ideals occurring as the radical of an ideal in a minimal primary decomposition of $\mathfrak{a}$ are said to ***belong to*** $\mathfrak{a}$.

PROPOSITION 13.8. *Suppose $\mathfrak{a} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary for $i = 1, \dots, n$. Then the minimal prime ideals of $\mathfrak{a}$ are the minimal elements of the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.*

PROOF. Let $\mathfrak{p}$ be a prime ideal containing $\mathfrak{a}$, and let $\mathfrak{q}_i'$ be the image of $\mathfrak{q}_i$ in the integral domain $A/\mathfrak{p}$. Then $\mathfrak{p}$ contains $\mathfrak{q}_1 \cdots \mathfrak{q}_n$, and so $\mathfrak{q}_1' \cdots \mathfrak{q}_n' = 0$. This implies that, for some i, $\mathfrak{q}_i' = 0$, and so $\mathfrak{p}$ contains $\mathfrak{q}_i$. Now (13.2) shows that $\mathfrak{p}$ contains $\mathfrak{p}_i$. □

In particular, if $\mathfrak{a}$ admits a primary decomposition, then it has only finitely many minimal prime ideals, and so its radical is a *finite* intersection of prime ideals.

For an ideal $\mathfrak{a}$ in A and an element $x \in A$, we let

$$(\mathfrak{a}:x) = \{a \in A \mid ax \in \mathfrak{a}\}.$$

It is again an ideal in A, which equals A if $x \in \mathfrak{a}$.

LEMMA 13.9. *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and let $x \in A \setminus \mathfrak{q}$. Then $(\mathfrak{q}:x)$ is $\mathfrak{p}$-primary (and hence $\operatorname{rad}(\mathfrak{q}:x) = \mathfrak{p}$).*

PROOF. For any $a \in (\mathfrak{q}:x)$, we know that $ax \in \mathfrak{q}$ and $x \notin \mathfrak{q}$, and so $a \in \mathfrak{p}$. Hence $(\mathfrak{q}:x) \subset \mathfrak{p}$. On taking radicals, we find that $\operatorname{rad}(\mathfrak{q}:x) = \mathfrak{p}$. Let $ab \in (\mathfrak{q}:x)$. Then $xab \in \mathfrak{q}$, and so either $a \in \mathfrak{p}$ or $xb \in \mathfrak{q}$ (because $\mathfrak{q}$ is $\mathfrak{p}$-primary); in the second case, $b \in (\mathfrak{q}:x)$ as required. □

THEOREM 13.10. *Let $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be a minimal primary decomposition of $\mathfrak{a}$, and let $\mathfrak{p}_i = \operatorname{rad}(\mathfrak{q}_i)$. Then*

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \{\operatorname{rad}(\mathfrak{a}:x) \mid x \in A, \quad \operatorname{rad}(\mathfrak{a}:x) \text{ prime}\}.$$

In particular, the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal primary decomposition.

PROOF. For any $a \in A$,

$$(\mathfrak{a}:a) = (\textstyle\bigcap \mathfrak{q}_i : a) = \bigcap (\mathfrak{q}_i : a),$$

and so

$$\operatorname{rad}(\mathfrak{a}:a) = \operatorname{rad} \textstyle\bigcap (\mathfrak{q}_i:a) \overset{(13.9)}{=} \bigcap_{a \notin \mathfrak{q}_i} \mathfrak{p}_i. \tag{20}$$

If $\operatorname{rad}(\mathfrak{a}:a)$ is prime, then it equals one of the $\mathfrak{p}_i$ (otherwise, for each i there exists an $a_i \in \mathfrak{p}_i \setminus \mathfrak{p}$, and $a_1 \cdots a_n \in \bigcap_{a \notin \mathfrak{q}_i} \mathfrak{p}_i$ but not $\mathfrak{p}$, which is a contradiction). Hence RHS$\supset$LHS. For each i, there exists an $a \in \bigcap_{j \neq i} \mathfrak{q}_j \setminus \mathfrak{q}_i$ because the decomposition is minimal, and (20) shows that $\operatorname{rad}(\mathfrak{a}:a) = \mathfrak{p}_i$. □

THEOREM 13.11. *In a noetherian ring, every ideal admits a primary decomposition.*

The theorem is a consequence of the following more precise statement, but first we need a definition: an ideal $\mathfrak{a}$ is said to be ***irreducible*** if

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \ (\mathfrak{b}, \mathfrak{c} \text{ ideals}) \implies \mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c}.$$

PROPOSITION 13.12. *Let A be a noetherian ring.*

(a) *Every ideal in A can be expressed as a finite intersection of irreducible ideals.*

(b) *Every irreducible ideal in A is primary.*

PROOF. (a) Suppose (a) fails, and let $\mathfrak{a}$ be maximal among the ideals for which it fails. Then, in particular, $\mathfrak{a}$ itself is not irreducible, and so $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ with $\mathfrak{b}$ and $\mathfrak{c}$ properly containing $\mathfrak{a}$. Because $\mathfrak{a}$ is maximal, both $\mathfrak{b}$ and $\mathfrak{c}$ can be expressed as finite intersections of irreducible ideals, but then so can $\mathfrak{a}$.

(b) Let $\mathfrak{a}$ be irreducible in A, and consider the quotient ring $A' \overset{\text{def}}{=} A/\mathfrak{a}$. Let a be a zero-divisor in A', say $ab = 0$ with $b \neq 0$. We have to show that a is nilpotent. As A' is noetherian, the chain of ideals

$$((0):a) \subset ((0):a^2) \subset \cdots$$

becomes constant, say, $((0):a^m) = ((0):a^{m+1})) = \cdots$. Let $c \in (a^m) \cap (b)$. Then $c \in (b)$ implies $ca = 0$, and $c \in (a^m)$ implies that $c = da^m$ for some $d \in A$. Now

$$(da^m)a = 0 \Rightarrow d \in (0:a^{m+1}) = (0:a^m) \Rightarrow c = 0.$$

Hence $(a^m) \cap (b) = (0)$. Because $\mathfrak{a}$ is irreducible, so also is the zero ideal in A', and it follows that $a^m = 0$. □

A $\mathfrak{p}$-primary ideal $\mathfrak{a}$ in a noetherian ring contains a power of $\mathfrak{p}$ by Proposition 3.13. The next result proves a converse when $\mathfrak{p}$ is maximal.

PROPOSITION 13.13. *Let $\mathfrak{m}$ be a maximal ideal of a noetherian ring. Any proper ideal $\mathfrak{a}$ of A that contains a power of a maximal ideal $\mathfrak{m}$ is $\mathfrak{m}$-primary.*

PROOF. Suppose that $\mathfrak{m}^r \subset \mathfrak{a}$, and let $\mathfrak{p}$ be a prime ideal belonging to $\mathfrak{a}$. Then $\mathfrak{m}^r \subset \mathfrak{a} \subset \mathfrak{p}$, so that $\mathfrak{m} \subset \mathfrak{p}$, which implies that $\mathfrak{m} = \mathfrak{p}$. Thus $\mathfrak{m}$ is the only prime ideal belonging to $\mathfrak{a}$, which means that $\mathfrak{a}$ is $\mathfrak{m}$-primary. □

EXAMPLE 13.14. We give an example of a power of a prime ideal $\mathfrak{p}$ that is not $\mathfrak{p}$-primary. Let

$$A = k[X, Y, Z]/(Y^2 - XZ) = k[x, y, z].$$

The ideal (X, Y) in $k[X, Y, Z]$ is prime and contains $(Y^2 - XZ)$, and so the ideal $\mathfrak{p} = (x, y)$ in A is prime. Now $xz = y^2 \in \mathfrak{p}^2$, but one checks easily that $x \notin \mathfrak{p}^2$ and $z \notin \mathfrak{p}$, and so $\mathfrak{p}^2$ is not $\mathfrak{p}$-primary.

REMARK 13.15. Let $\mathfrak{a}$ be an ideal in a noetherian ring, and let $\mathfrak{b} = \bigcap_{n\geq 1} \mathfrak{a}^n$. We give another proof that $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$ (see p.12). Let

$$\mathfrak{a}\mathfrak{b} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s, \quad \operatorname{rad}(\mathfrak{q}_i) = \mathfrak{p}_i,$$

be a minimal primary decomposition of $\mathfrak{a}\mathfrak{b}$. We shall show that $\mathfrak{b} \subset \mathfrak{a}\mathfrak{b}$ by showing that $\mathfrak{b} \subset \mathfrak{q}_i$ for each i.

If there exists a $b \in \mathfrak{b} \smallsetminus \mathfrak{q}_i$, then

$$\mathfrak{a}b \subset \mathfrak{a}\mathfrak{b} \subset \mathfrak{q}_i,$$

from which it follows that $\mathfrak{a} \subset \mathfrak{p}_i$. We know that $\mathfrak{p}_i^r \subset \mathfrak{q}_i$ for some r (see 3.13), and so

$$\mathfrak{b} = \bigcap \mathfrak{a}^n \subset \mathfrak{a}^r \subset \mathfrak{p}_i^r \subset \mathfrak{q}_i,$$

which is a contradiction. This completes the proof.

DEFINITION 13.16. A ***Dedekind domain*** is a noetherian integrally closed integral domain of dimension 1.

THEOREM 13.17. *Every proper nonzero ideal $\mathfrak{a}$ in a Dedekind domain can be written in the form*

$$\mathfrak{a} = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_s^{r_s}$$

with the $\mathfrak{p}_i$ distinct prime ideals and the $r_i > 0$; the ideals $\mathfrak{p}_i$ are exactly the prime ideals containing $\mathfrak{a}$, and the exponents r_i are uniquely determined.

PROOF. For the proof, which is quite elementary, see Chapter 3 of my notes Algebraic Number Theory. □

14 Artinian rings

A ring A is ***artinian*** if every descending chain of ideals $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots$ in A eventually becomes constant; equivalently, if every nonempty set of ideals has a minimal element. Similarly, a module M over a ring A is ***artinian*** if every descending chain of submodules $N_1 \supset N_2 \supset \cdots$ in M eventually becomes constant.

PROPOSITION 14.1. *An artinian ring has Krull dimension zero; in other words, every prime ideal is maximal.*

PROOF. Let $\mathfrak{p}$ be a prime ideal of an artinian ring A, and let $A' = A/\mathfrak{p}$. Then A' is an artinian integral domain. For any nonzero element a of A', the chain $(a) \supset (a^2) \supset \cdots$ eventually becomes constant, and so $a^n = a^{n+1}b$ for some $b \in A'$ and $n \geq 1$. We can cancel a^n to obtain $1 = ab$. Thus a is a unit, A' is a field, and $\mathfrak{p}$ is maximal. □

COROLLARY 14.2. *In an artinian ring, the nilradical and the Jacobson radical coincide.*

PROOF. The first is the intersection of the prime ideals (2.6), and the second is the intersection of the maximal ideals (2.7). □

PROPOSITION 14.3. *An artinian ring has only finitely many maximal ideals.*

PROOF. Let $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ be a minimal element in the set of all finite intersections of maximal ideals in the artinian ring A, and let $\mathfrak{m}$ be a maximal ideal in A. Then $\mathfrak{m}$ equals one of the $\mathfrak{m}_i$, because otherwise there exists an $a_i \in \mathfrak{m}_i \smallsetminus \mathfrak{m}$ for each i, and $a_1 \cdots a_n$ lies in $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ but not $\mathfrak{m}$ (because $\mathfrak{m}$ is prime); thus $\mathfrak{m} \cap \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ is smaller than $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$, which contradicts the definition of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$. □

PROPOSITION 14.4. *In an artinian ring, some power of the nilradical is zero.*

PROOF. Let $\mathfrak{N}$ be the nilradical of the artinian ring A. The chain $\mathfrak{N} \supset \mathfrak{N}^2 \supset \cdots$ eventually becomes constant, and so $\mathfrak{N}^n = \mathfrak{N}^{n+1} = \cdots$ for some $n \geq 1$. Suppose $\mathfrak{N}^n \neq 0$. Then there exist ideals $\mathfrak{a}$ such that $\mathfrak{a} \cdot \mathfrak{N}^n \neq 0$, for example $\mathfrak{N}$, and we may suppose that $\mathfrak{a}$ has been chosen to be minimal among such ideals. There exists an $a \in \mathfrak{a}$ such that $a \cdot \mathfrak{N}^n \neq 0$, and so $\mathfrak{a} = (a)$ (by minimality). Now $(a\mathfrak{N}^n)\mathfrak{N}^n = a\mathfrak{N}^{2n} = a\mathfrak{N}^n \neq 0$ and $a\mathfrak{N}^n \subset (a)$, and so $a\mathfrak{N}^n = (a)$ (by minimality again). Hence $a = ax$ for some $x \in \mathfrak{N}^n$. Now $a = ax = ax^2 = \cdots = a0 = 0$ because $x \in \mathfrak{N}$. This contradicts the definition of a, and so $\mathfrak{N}^n = 0$. □

LEMMA 14.5. *Let A be a ring in which some finite product of maximal ideals is zero. Then A is artinian if and only if it is noetherian.*

PROOF. Suppose $\mathfrak{m}_1 \cdots \mathfrak{m}_n = 0$ with the $\mathfrak{m}_i$ maximal ideals (not necessarily distinct), and consider

$$A \supset \mathfrak{m}_1 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_{r-1} \supset \mathfrak{m}_1 \cdots \mathfrak{m}_r \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n = 0.$$

The action of A on the quotient $M_r \stackrel{\text{def}}{=} \mathfrak{m}_1 \cdots \mathfrak{m}_{r-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_r$ factors through the field $A/\mathfrak{m}_r$, and the subspaces of the vector space M_r are in one-to-one correspondence with the ideals of A contained between $\mathfrak{m}_1 \cdots \mathfrak{m}_{r-1}$ and $\mathfrak{m}_1 \cdots \mathfrak{m}_r$. If A is either artinian or noetherian, then M_r satisfies a chain condition on subspaces and so it is finite-dimensional as a vector space and both artinian and noetherian as an A-module. Now repeated applications of Proposition 3.3 (resp. its analogue for artinian modules) show that if A is artinian (resp. noetherian), then it is noetherian (resp. artinian) as an A-module, and hence as a ring. □

THEOREM 14.6. *A ring is artinian if and only if it is noetherian of dimension zero.*

PROOF. $\Rightarrow$: Let A be an artinian ring. After (14.1), it remains to show that A is noetherian, but according to (14.2), (14.3), and (14.4), some finite product of maximal ideals is zero, and so this follows from the lemma.

$\Leftarrow$: Let A be a noetherian ring of dimension zero. The zero ideal admits a primary decomposition (13.11), and so A has only finitely many minimal prime ideals, which are all maximal because $\dim A = 0$. Hence $\mathfrak{N}$ is a finite intersection of maximal ideals (2.6), and since some power of $\mathfrak{N}$ is zero (3.13), we again have that some finite product of maximal ideals is zero, and so can apply the lemma. □

THEOREM 14.7. *Every artinian ring is (uniquely) a product of local artinian rings.*

PROOF. Let A be artinian, and let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the distinct maximal ideals in A. We saw in the proof of (14.6) that some product $\mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r} = 0$. For $i \neq j$, the ideal $\mathfrak{m}_i^{n_i} + \mathfrak{m}_j^{n_j}$ is not contained in any maximal ideal, and so equals A. Now the Chinese remainder theorem 2.13 shows that

$$A \simeq A/\mathfrak{m}_1^{n_1} \times \cdots \times A/\mathfrak{m}_r^{n_r},$$

and each ring $A/\mathfrak{m}_i^{n_i}$ is obviously local. □

PROPOSITION 14.8. *Let A be a local artinian ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{m}$ is principal, so also is every ideal in A; in fact, if $\mathfrak{m} = (t)$, then every ideal is of the form (t^r) for some $r \geq 0$.*

PROOF. Because $\mathfrak{m}$ is the Jacobson radical of A, some power of $\mathfrak{m}$ is zero (by 14.4); in particular, $(0) = (t^r)$ for some r. Let $\mathfrak{a}$ be a nonzero ideal in A. There exists an integer $r \geq 0$ such that $\mathfrak{a} \subset \mathfrak{m}^r$ but $\mathfrak{a} \not\subset \mathfrak{m}^{r+1}$. Therefore there exists an element a of $\mathfrak{a}$ such that $a = ct^r$ for some $c \in A$ but $a \notin (t^{r+1})$. The second condition implies that $c \notin \mathfrak{m}$, and so it is a unit; therefore $\mathfrak{a} = (a)$. □

15 Dimension theory for noetherian rings

Let A be a noetherian ring and let $\mathfrak{p}$ be a prime ideal in A. Let $A_{\mathfrak{p}} = S^{-1}A$ where $S = A \smallsetminus \mathfrak{p}$. We begin by studying extension and contraction of ideals with respect to the homomorphism $A \to A_{\mathfrak{p}}$ (cf. 2.10). Recall (6.6) that $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}^e \stackrel{\text{def}}{=} \mathfrak{p}A_{\mathfrak{p}}$. The ideal

$$\left(\mathfrak{p}^n\right)^{ec} = \{a \in A \mid sa \in \mathfrak{p}^n \text{ for some } s \in S\}$$

is called the nth ***symbolic power*** of $\mathfrak{p}$, and is denoted $\mathfrak{p}^{(n)}$. If $\mathfrak{m}$ is maximal, then $\mathfrak{m}^{(n)} = \mathfrak{m}^n$ (see 6.7).

LEMMA 15.1. *The ideal $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary.*

PROOF. According to Proposition 13.3, the ideal $(\mathfrak{p}^e)^n$ is $\mathfrak{p}^e$-primary. Hence (see 13.4), $((\mathfrak{p}^e)^n)^c$ is $(\mathfrak{p}^e)^c$-primary. But $\mathfrak{p}^{ec} = \mathfrak{p}$ (see 6.4), and

$$(((\mathfrak{p}^e)^n)^c \stackrel{2.11}{=} ((\mathfrak{p}^n)^e)^c \stackrel{\text{def}}{=} \mathfrak{p}^{(n)}. \tag{21}$$

□

LEMMA 15.2. *Consider ideals $\mathfrak{a} \subset \mathfrak{p}' \subset \mathfrak{p}$ with $\mathfrak{p}'$ prime. If $\mathfrak{p}'$ is a minimal prime ideal of $\mathfrak{a}$, then $\mathfrak{p}'^e$ is a minimal prime ideal of $\mathfrak{a}^e$ (extension relative to $A \to A_\mathfrak{p}$).*

PROOF. If not, there exists a prime ideal $\mathfrak{p}'' \neq \mathfrak{p}'^e$ such that $\mathfrak{p}'^e \supset \mathfrak{p}'' \supset \mathfrak{a}^e$. Now, by (6.4), $\mathfrak{p}' = \mathfrak{p}'^{ec}$ and $\mathfrak{p}''^c \neq \mathfrak{p}'^{ec}$, and so

$$\mathfrak{p}' = \mathfrak{p}'^{ec} \supsetneq \mathfrak{p}''^c \supset \mathfrak{a}^{ec} \supset \mathfrak{a}$$

contradicts the minimality of $\mathfrak{p}'$. □

THEOREM 15.3 (KRULL'S PRINCIPAL IDEAL THEOREM). *Let A be a noetherian ring. For any nonunit $b \in A$, the height of a minimal prime ideal $\mathfrak{p}$ of (b) is at most one.*

PROOF. Consider $A \to A_\mathfrak{p}$. According to Lemma 15.2, $\mathfrak{p}^e$ is a minimal prime ideal of $(b)^e = (\frac{b}{1})$, and (6.4) shows that the theorem for $A_\mathfrak{p} \supset \mathfrak{p}^e \supset (\frac{b}{1})$ implies it for $A \supset \mathfrak{p} \supset (b)$. Therefore, we may replace A with $A_\mathfrak{p}$, and so assume that A is a noetherian local ring with maximal ideal $\mathfrak{p}$.

Suppose that $\mathfrak{p}$ properly contains a prime ideal $\mathfrak{p}_1$: we have to show that $\mathfrak{p}_1 \supset \mathfrak{p}_2 \implies \mathfrak{p}_1 = \mathfrak{p}_2$.

Let $\mathfrak{p}_1^{(r)}$ be the rth symbolic power of $\mathfrak{p}_1$. The only prime ideal of the ring $A/(b)$ is $\mathfrak{p}/(b)$, and so $A/(b)$ is artinian (apply 14.6). Therefore the descending chain of ideals

$$\left(\mathfrak{p}_1^{(1)} + (b)\right)/(b) \supset \left(\mathfrak{p}_1^{(2)} + (b)\right)/(b) \supset \left(\mathfrak{p}_1^{(3)} + (b)\right)/(b) \supset \cdots$$

eventually becomes constant: there exists an s such that

$$\mathfrak{p}_1^{(s)} + (b) = \mathfrak{p}_1^{(s+1)} + (b) = \mathfrak{p}_1^{(s+2)} + (b) = \cdots. \tag{22}$$

We claim that, for any $m \geq s$,

$$\mathfrak{p}_1^{(m)} \subset (b)\mathfrak{p}_1^{(m)} + \mathfrak{p}_1^{(m+1)}. \tag{23}$$

Let $x \in \mathfrak{p}_1^{(m)}$. Then

$$x \in (b) + \mathfrak{p}_1^{(m)} \overset{(22)}{=} (b) + \mathfrak{p}_1^{(m+1)},$$

and so $x = ab + x'$ with $a \in A$ and $x' \in \mathfrak{p}_1^{(m+1)}$. As $\mathfrak{p}_1^{(m)}$ is $\mathfrak{p}_1$-primary (see 15.1) and $ab = x - x' \in \mathfrak{p}_1^{(m)}$ but $b \notin \mathfrak{p}_1$, we have that $a \in \mathfrak{p}_1^{(m)}$. Now $x = ab + x' \in (b)\mathfrak{p}_1^{(m)} + \mathfrak{p}_1^{(m+1)}$ as claimed.

We next show that, for any $m \geq s$,

$$\mathfrak{p}_1^{(m)} = \mathfrak{p}_1^{(m+1)}.$$

As $b \in \mathfrak{p}$, (23) shows that $\mathfrak{p}_1^{(m)}/\mathfrak{p}_1^{(m+1)} = \mathfrak{p} \cdot \left(\mathfrak{p}_1^{(m)}/\mathfrak{p}_1^{(m+1)}\right)$, and so $\mathfrak{p}_1^{(m)}/\mathfrak{p}_1^{(m+1)} = 0$ by Nakayama's lemma (3.7).

Now

$$\mathfrak{p}_1^s \subset \mathfrak{p}_1^{(s)} = \mathfrak{p}_1^{(s+1)} = \mathfrak{p}_1^{(s+2)} = \cdots$$

and so $\mathfrak{p}_1^s \subset \bigcap_{m \geq s} \mathfrak{p}_1^{(m)}$. Note that

$$\bigcap\nolimits_{m \geq s} \mathfrak{p}_1^{(m)} \overset{(21)}{=} \bigcap\nolimits_{m \geq s} ((\mathfrak{p}_1^e)^m)^c = (\bigcap\nolimits_{m \geq s} (\mathfrak{p}_1^e)^m)^c \overset{3.12}{=} (0)^c,$$

and so for any $x \in \mathfrak{p}_1^s$, there exists an $a \in A \smallsetminus \mathfrak{p}_1$ such that $ax = 0$. Let $x \in \mathfrak{p}_1$; then $ax^s = 0$ for some $a \in A \smallsetminus \mathfrak{p}_1 \supset A \smallsetminus \mathfrak{p}_2$, and so $x \in \mathfrak{p}_2$ (because $\mathfrak{p}_2$ is prime). We have shown that $\mathfrak{p}_1 = \mathfrak{p}_2$, as required. □

In order to extend Theorem 15.6 to non principal ideals, we shall need a lemma.

LEMMA 15.4. *Let $\mathfrak{p}$ be a prime ideal in a noetherian ring A, and let S be a finite set of prime ideals in A, none of which contains $\mathfrak{p}$. If there exists a chain of distinct prime ideals*

$$\mathfrak{p} \supset \mathfrak{p}_{d-1} \supset \cdots \supset \mathfrak{p}_0,$$

then there exists such a chain with $\mathfrak{p}_1$ not contained in any ideal in S.

PROOF. We first prove this in the special case that the chain has length 2. Suppose that $\mathfrak{p} \supset \mathfrak{p}_1 \supset \mathfrak{p}_0$ are distinct prime ideals and that $\mathfrak{p}$ is not contained in any prime ideal in S. According to Proposition 2.9, there exists an element

$$a \in \mathfrak{p} \setminus (\mathfrak{p}_0 \cup \textstyle\bigcup \{\mathfrak{p}' \in S\}).$$

As $\mathfrak{p}$ contains $(a)+\mathfrak{p}_0$, it also contains a minimal prime ideal $\mathfrak{p}_1'$ of $(a)+\mathfrak{p}_0$. Now $\mathfrak{p}_1'/\mathfrak{p}_0$ is a minimal prime ideal of the *principal ideal* $((a)+\mathfrak{p}_0)/\mathfrak{p}_0$ in $A/\mathfrak{p}_0$, and so has height 1, whereas the chain $\mathfrak{p}/\mathfrak{p}_0 \supset \mathfrak{p}_1/\mathfrak{p}_0 \supset \mathfrak{p}_0/\mathfrak{p}_0$ shows that $\mathfrak{p}/\mathfrak{p}_0$ has height at least 2. Therefore $\mathfrak{p} \supset \mathfrak{p}_1' \supset \mathfrak{p}_0$ are distinct primes, and $\mathfrak{p}_1' \notin S$ because it contains a. This completes the proof of the special case.

Now consider the general case. On applying the special case to $\mathfrak{p} \supset \mathfrak{p}_{d-1} \supset \mathfrak{p}_{d-2}$, we see that there exists a chain of distinct prime ideals $\mathfrak{p} \supset \mathfrak{p}'_{d-1} \supset \mathfrak{p}_{d-2}$ such that $\mathfrak{p}'_{d-1}$ is not contained in any ideal in S. Then on applying the special case to $\mathfrak{p}'_{d-1} \supset \mathfrak{p}_{d-2} \supset \mathfrak{p}_{d-1}$, we we see that there exists a chain of distinct prime ideals $\mathfrak{p} \supset \mathfrak{p}'_{d-1} \supset \mathfrak{p}'_{d-2} \supset \mathfrak{p}_{d-2}$ such that $\mathfrak{p}'_{d-2}$ is not contained in any ideal in S. Repeat the argument until the proof is complete. □

THEOREM 15.5. *Let A be a noetherian ring. For any proper ideal $\mathfrak{a} = (a_1,\dots,a_m)$, the height of a minimal prime ideal of $\mathfrak{a}$ is at most m.*

PROOF. For $m = 1$, this was just proved. Thus, we may suppose $m \geq 2$ and that the theorem has been proved for ideals generated by $m-1$ elements. Let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$, and let $\mathfrak{p}_1',\dots,\mathfrak{p}_t'$ be the minimal prime ideals of $(a_2,\dots,a_m)$. Each $\mathfrak{p}_i'$ has height at most $m-1$. If $\mathfrak{p}$ is contained in one of the $\mathfrak{p}_i'$, it will have height $\leq m-1$, and so we may suppose that it isn't.

Let $\mathfrak{p}$ have height d. We have to show that $d \leq m$. According to the lemma, there exists a chain of distinct prime ideals

$$\mathfrak{p} = \mathfrak{p}_d \supset \mathfrak{p}_{d-1} \supset \cdots \supset \mathfrak{p}_0, \quad d \geq 1,$$

with $\mathfrak{p}_1$ not contained in any $\mathfrak{p}_i'$, and so Proposition 2.9 shows that there exists a

$$b \in \mathfrak{p}_1 \setminus \textstyle\bigcup_{i=1}^{r} \mathfrak{p}_i'.$$

We next show that $\mathfrak{p}$ is a minimal prime ideal of $(b,a_2,\dots,a_m)$. Certainly $\mathfrak{p}$ contains a minimal prime ideal $\mathfrak{p}'$ of this ideal. As $\mathfrak{p}' \supset (a_2,\dots,a_m)$, $\mathfrak{p}$ contains one of the $\mathfrak{p}_i'$s, but, by construction, it cannot equal it. If $\mathfrak{p} \neq \mathfrak{p}'$, then

$$\mathfrak{p} \supset \mathfrak{p}' \supset \mathfrak{p}_i$$

are distinct ideals, which shows that $\bar{\mathfrak{p}} \stackrel{\text{def}}{=} \mathfrak{p}/(a_2,\dots,a_m)$ has height at least 2 in $\bar{A} \stackrel{\text{def}}{=} A/(a_2,\dots,a_m)$. But $\bar{\mathfrak{p}}$ is a minimal ideal in $\bar{A}$ of the principal ideal $(a_1,\dots,a_n)/(a_2,\dots,a_n)$, which contradicts Theorem 15.3. Hence $\mathfrak{p}$ is minimal, as claimed.

But now $\mathfrak{p}/(b)$ is a minimal prime ideal of $(b,a_2,\dots,a_m)$ in $R/(b)$, and so the height of $\mathfrak{p}/(b)$ is at most $m-1$ (by induction). The prime ideals

$$\mathfrak{p}/(b) = \mathfrak{p}_d/(b) \supset \mathfrak{p}_{d-1}/(b) \supset \cdots \supset \mathfrak{p}_1/(b)$$

are distinct, and so $d-1 \le m-1$. This completes the proof that $d = m$. □

The ***height*** of an ideal $\mathfrak{a}$ in a noetherian ring is the minimum height of a prime ideal containing it,

$$\operatorname{ht}(\mathfrak{a}) = \min_{\mathfrak{p}\supset\mathfrak{a},\ \mathfrak{p}\text{ prime}} \operatorname{ht}(\mathfrak{p}).$$

The theorem shows that $\operatorname{ht}(\mathfrak{a})$ is finite.

The following provides a (strong) converse to Theorem 15.5.

THEOREM 15.6. *Let A be a noetherian ring, and let $\mathfrak{a}$ be a proper ideal of A of height r. Then there exist r elements $a_1,\dots,a_r$ of $\mathfrak{a}$ such that, for each $i \le r$, $(a_1,\dots,a_i)$ has height i.*

PROOF. If $r = 0$, then we take the empty set of a_is. Thus, suppose $r \ge 1$. There are only finitely many prime ideals of height 0, because such an ideal is a minimal prime ideal of (0), and none of these ideals can contain $\mathfrak{a}$ because it has height ≥ 1. Proposition 2.9 shows that there exists an

$$a_1 \in \mathfrak{a} \smallsetminus \bigcup\{\text{prime ideals of height } 0\}.$$

By construction, (a_1) has height at least 1, and so Theorem 15.3 shows it has height exactly 1.

This completes the proof when $r = 1$, and so suppose that $r \ge 2$. There are only finitely many prime ideals of height 1 containing (a_1) because such an ideal is a minimal prime ideal of (a_1), and none of these ideals can contain $\mathfrak{a}$ because it has height ≥ 2. Choose

$$a_2 \in \mathfrak{a} \smallsetminus \bigcup\{\text{prime ideals of height } 1 \text{ containing } (a_1)\}.$$

By construction, (a_1,a_2) has height at least 2, and so Theorem 15.5 shows that it has height exactly 2.

This completes the proof when $r = 2$, and when $r > 2$ we can continue in this fashion until it is complete.

COROLLARY 15.7. *Every prime ideal of height r in a noetherian ring arises as a minimal prime ideal for an ideal generated by r elements.*

PROOF. According to the theorem, an ideal $\mathfrak{a}$ of height r contains an ideal $(a_1,\dots,a_r)$ of height r. If $\mathfrak{a}$ is prime, then it is a minimal ideal of $(a_1,\dots,a_r)$. □

COROLLARY 15.8. *Let A be a commutative noetherian ring, and let $\mathfrak{a}$ be an ideal in A that can be generated by n elements. For any prime ideal $\mathfrak{p}$ in A containing $\mathfrak{a}$,*

$$\operatorname{ht}(\mathfrak{p}/\mathfrak{a}) \le \operatorname{ht}(\mathfrak{p}) \le \operatorname{ht}(\mathfrak{p}/\mathfrak{a}) + n.$$

PROOF. The first inequality follows immediately from the correspondence between ideals in A and in $A/\mathfrak{a}$.

Denote the quotient map $A \to A' \stackrel{\text{def}}{=} A/\mathfrak{a}$ by $a \mapsto a'$. Let $\text{ht}(\mathfrak{p}/\mathfrak{a}) = d$. Then there exist elements $a_1, \dots, a_d$ in A such that $\mathfrak{p}/\mathfrak{a}$ is a minimal prime ideal of $(a_1', \dots, a_d')$. Let $b_1, \dots, b_n$ generate $\mathfrak{a}$. Then $\mathfrak{p}$ is a minimal prime ideal of $(a_1, \dots, a_d, b_1, \dots, b_n)$, and hence has height $\leq d + n$. □

We now use dimension theory to prove a stronger version of "generic flatness" (9.8).

THEOREM 15.9 (GENERIC FREENESS). *Let A be a noetherian integral domain, and let B be a finitely generated A-algebra. For any finitely generated B-module M, there exists a nonzero element a of A such that M_a is a free A_a-module.*

PROOF. Let F be the field of fractions of A. We prove the theorem by induction on the Krull dimension of $F \otimes_A B$, starting with the case of Krull dimension -1. Recall that this means that $F \otimes_A B = 0$, and so $a1_B = 0$ for some nonzero $a \in A$. Then $M_a = 0$, and so the theorem is trivially true (M_a is the free A_a-module generated by the empty set).

In the general case, and argument as in (9.9) shows that, after replacing A, B, and M with A_a, B_a, and M_a for a suitable $a \in A$, we may suppose that the map $B \to F \otimes_A B$ is injective — we identify B with its image. The Noether normalization shows that there exist algebraically independent elements $x_1, \dots, x_m$ of $F \otimes_A B$ such that $F \otimes_A B$ is a finite $F[x_1, \dots, x_m]$-algebra. As in the proof of (9.8), there exists a nonzero $a \in A$ such that B_a is a finite $A_a[x_1, \dots, x_m]$-algebra. Hence M_a is a finitely generated $A_a[x_1, \dots, x_m]$-module.

As any extension of free modules is free[19], Proposition 3.5 shows that it suffices to prove the theorem for $M_a = A_a[x_1, \dots, x_m]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ in $A_a[x_1, \dots, x_m]$. If $\mathfrak{p} = 0$, then M_a is free over A_a (with basis the monomials in the x_i). Otherwise, $F \otimes_A (A_a[x_1, \dots, x_m]/\mathfrak{p})$ has Krull dimension less than that of $F \otimes_A B$, and so we can apply the induction hypothesis. □

16 Regular local rings

Throughout this section, A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. The Krull dimension d of A is equal to the height of $\mathfrak{m}$, and

$$\text{ht}(\mathfrak{m}) \stackrel{(15.5)}{\leq} \text{minimum number of generators of } \mathfrak{m} \stackrel{(3.9)}{=} \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

When equality holds, the ring A is said to be ***regular***. In other words, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq d$, and equality holds exactly when the ring is regular.

For example, when A has dimension zero, it is regular if and only if its maximal ideal can be generated by the empty set, and so is zero. This means that A is a field; in particular, it is an integral domain. The main result of this section is that all regular rings are integral domains.

LEMMA 16.1. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, and let $c \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. Denote the quotient map $A \to A' \stackrel{\text{def}}{=} A/(c)$ by $a \mapsto a'$. Then*

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{m}'/\mathfrak{m}'^2 + 1$$

where $\mathfrak{m}' \stackrel{\text{def}}{=} \mathfrak{m}/(c)$ is the maximal ideal of A'.

[19] If M' is a submodule of M such that $M'' \stackrel{\text{def}}{=} M/M'$ is free, then $M \approx M' \oplus M''$.

PROOF. Let $e_1,\dots,e_n$ be elements of $\mathfrak{m}$ such that $\{e'_1,\dots,e'_n\}$ is a k-linear basis for $\mathfrak{m}'/\mathfrak{m}'^2$. We shall show that $\{e_1,\dots,e_n,c\}$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$.

As $e'_1,\dots,e'_n$ span $\mathfrak{m}'/\mathfrak{m}'^2$, they generate the ideal $\mathfrak{m}'$ (see 3.9), and so $\mathfrak{m}=(e_1,\dots,e_n)+(c)$, which implies that $\{e_1,\dots,e_n,c\}$ spans $\mathfrak{m}/\mathfrak{m}^2$.

Suppose that $a_1,\dots,a_{n+1}$ are elements of A such that

$$a_1e_1+\cdots+a_ne_n+a_{n+1}c\equiv 0 \bmod \mathfrak{m}^2. \tag{24}$$

Then

$$a'_1e'_1+\cdots+a'_ne'_n\equiv 0 \bmod \mathfrak{m}'^2,$$

and so $a'_1,\dots,a'_n\in\mathfrak{m}'$. It follows that $a_1,\dots,a_n\in\mathfrak{m}$. Now (24) shows that $a_{n+1}c\in\mathfrak{m}^2$. If $a_{n+1}\notin\mathfrak{m}$, then it is a unit in A, and $c\in\mathfrak{m}^2$, which contradicts its definition. Therefore, $a_{n+1}\in\mathfrak{m}$, and the relation (24) is the trivial one. □

PROPOSITION 16.2. *If A is regular, then so also is $A/(a)$ for any $a\in\mathfrak{m}\smallsetminus\mathfrak{m}^2$; moreover, $\dim A=\dim A/(a)+1$.*

PROOF. With the usual notations, (15.8) shows that

$$\operatorname{ht}(\mathfrak{m}')\le\operatorname{ht}(\mathfrak{m})\le\operatorname{ht}(\mathfrak{m}')+1.$$

Therefore

$$\dim_k(\mathfrak{m}'/\mathfrak{m}'^2)\ge\operatorname{ht}(\mathfrak{m}')\ge\operatorname{ht}(\mathfrak{m})-1=\dim_k(\mathfrak{m}/\mathfrak{m}^2)-1=\dim_k(\mathfrak{m}'/\mathfrak{m}'^2).$$

Equalities must hold throughout, which proves that A' is regular with dimension $\dim A-1$.□

THEOREM 16.3. *Every regular noetherian local ring is an integral domain.*

PROOF. Let A be a regular local ring of dimension d. We have already noted that the statement is true when $d=0$.

We next prove that A is an integral domain if it contains distinct ideals $\mathfrak{a}\supset\mathfrak{p}$ with $\mathfrak{a}=(a)$ principal and $\mathfrak{p}$ prime. Let $b\in\mathfrak{p}$, and suppose $b\in\mathfrak{a}^n=(a^n)$ for some $n\ge 1$. Then $b=a^nc$ for some $c\in A$. As a is not in the prime ideal $\mathfrak{p}$, we must have that $c\in\mathfrak{p}\subset\mathfrak{a}$, and so $b\in\mathfrak{a}^{n+1}$. Continuing in this fashion, we see that $b\in\bigcap_n\mathfrak{a}^n\overset{3.12}{=}\{0\}$. Therefore $\mathfrak{p}=\{0\}$, and so A is an integral domain.

We now assume $d\ge 1$, and proceed by induction on d. Let $a\in\mathfrak{m}\smallsetminus\mathfrak{m}^2$. As $A/(a)$ is regular of dimension $d-1$, it is an integral domain, and so (a) is a prime ideal. If it has height 1, then the last paragraph shows that A is an integral domain. Thus, we may suppose that, for all $a\in\mathfrak{m}\smallsetminus\mathfrak{m}^2$, the prime ideal (a) has height 0, and so is a minimal prime ideal of A. Let S be the set of all minimal prime ideals of A — recall (§13) that S is finite. We have shown that $\mathfrak{m}\smallsetminus\mathfrak{m}^2\subset\bigcup\{\mathfrak{p}\mid\mathfrak{p}\in S\}$, and so $\mathfrak{m}\subset\mathfrak{m}^2\cup\bigcup\{\mathfrak{p}\mid\mathfrak{p}\in S\}$. It follows from Proposition 2.9 that either $\mathfrak{m}\subset\mathfrak{m}^2$ (and hence $\mathfrak{m}=0$) or $\mathfrak{m}$ is a minimal prime ideal of A, but both of these statements contradict the assumption that $d\ge 1$. □

COROLLARY 16.4. *A regular noetherian local ring of dimension 1 is a principal ideal domain (with a single nonzero prime ideal).*

PROOF. Let A be a regular local ring of dimension 1 with maximal ideal $\mathfrak{m}$,and let $\mathfrak{a}$ be a nonzero proper ideal in A. The conditions imply that $\mathfrak{m}$ is principal, say $\mathfrak{m} = (t)$. The radical of $\mathfrak{a}$ is $\mathfrak{m}$ because $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{a}$, and so $\mathfrak{a} \supset \mathfrak{m}^r$ for some r (by 3.13). The ring $A/\mathfrak{m}^r$ is local and artinian, and so $\mathfrak{a} = (t^s) + \mathfrak{m}^r$ for some $s \geq 1$ (by 14.8). This implies that $\mathfrak{a} = (t^s)$ by Nakayama's lemma (3.7). □

THEOREM 16.5. *Let A be a regular noetherian local ring.*
(a) *For any prime ideal $\mathfrak{p}$ in A, the ring $A_{\mathfrak{p}}$ is regular.*
(b) *The ring A is a unique factorization domain (hence is integrally closed).*

PROOF. The best proofs use homological algebra, and are (at present) beyond this primer. For an account of the theorems in the same spirit as this primer, see `http://www.math.uchicago.edu/~may/MISC/RegularLocal.pdf`. See also Matsumura 1986 19.3, 20.3.□

17 Connections with geometry

Throughout this section, k is a field.

AFFINE k-ALGEBRAS

Let A be a finitely generated k-algebra. Recall (10.5) that the nilradical of A is equal to the intersection of the maximal ideals of A.

PROPOSITION 17.1. *Let A be a finitely generated k-algebra over a perfect field k. If A is reduced, then so also is $K \otimes_k A$ for every field $K \supset k$.*

PROOF. Let (e_i) be a basis for K as a k-vector space, and suppose $\alpha = \sum e_i \otimes a_i$ is a nonzero nilpotent element in $K \otimes_k A$. Because A is reduced, there exists a maximal ideal $\mathfrak{m}$ in A such that some a_i do not belong to $\mathfrak{m}$. The image $\bar{\alpha}$ of α in $K \otimes_k (A/\mathfrak{m})$ is a nonzero nilpotent, but $A/\mathfrak{m}$ is a finite separable field extension of k, and so this is impossible.[20] □

When k is not perfect, Proposition 17.1 fails, because then k has characteristic $p \neq 0$ and it contains an element a that is not a pth power. The polynomial $X^p - a$ is irreducible in $k[X]$, but $X^p - a = (X - \alpha)^p$ in $k^{\mathrm{al}}[X]$. Therefore, $A = k[X]/(X^p - a)$ is a field, but $k^{\mathrm{al}} \otimes_k A = k^{\mathrm{al}}[X]/(X - \alpha)^p$ is not reduced.

DEFINITION 17.2. An ***affine*** k-algebra is a finitely generated k-algebra A such that $k^{\mathrm{al}} \otimes_k A$ is reduced.

Let A be a finitely generated k-algebra. If A is affine, then $K \otimes_k A$ is reduced for every finite extension K of k, because a k-homomorphism $K \to k^{\mathrm{al}}$ defines an injective homomorphism $K \otimes_k A \to k^{\mathrm{al}} \otimes_k A$. Conversely, if A is reduced and k is perfect, then (17.1) shows that A is affine.

PROPOSITION 17.3. *If A is an affine k-algebra and B is a reduced k-algebra, then $A \otimes_k B$ is reduced.*

[20] Every finite separable field extension of k is of the form $k[X]/(f(X))$ with $f(X)$ separable and therefore without repeated factors in any extension field of k; hence $K \otimes_k k[X]/(f(X)) \simeq K[X]/(f(X))$ is a product of fields.

PROOF. Let (e_i) be a basis for A as a k-vector space, and suppose $\alpha = \sum e_i \otimes b_i$ is a nonzero nilpotent element of $A \otimes_k B$. Let B' be the k-subalgebra of B generated by the (finitely many) nonzero b_i. Because B' is reduced, there exists a maximal ideal $\mathfrak{m}$ in B' such that some b_i do not belong to $\mathfrak{m}$. Then the image $\bar{\alpha}$ of α in $A \otimes_k (B'/\mathfrak{m})$ is a nonzero nilpotent, but $B'/\mathfrak{m}$ is a finite field extension of k (Zariski's lemma, 10.1), and so this is impossible. □

COROLLARY 17.4. *If A and B are affine k-algebras, then so also is $A \otimes_k B$.*

PROOF. By definition, $k^{\mathrm{al}} \otimes_k A$ is reduced, and $k^{\mathrm{al}} \otimes_k (A \otimes_k B) \simeq (k^{\mathrm{al}} \otimes_k A) \otimes_k B$, which is reduced by (17.2). □

LOCALLY RINGED SPACES

Let V be a topological space, and let k be a k-algebra. A ***presheaf*** $\mathcal{O}$ of k-algebras on V assigns to each open subset U of V a k-algebra $\mathcal{O}(U)$ and to each inclusion $U' \subset U$ a "restriction" map

$$f \mapsto f|U' : \mathcal{O}(U) \to \mathcal{O}(U');$$

when $U = U'$ the restriction map is required to be the identity map, and if

$$U'' \subset U' \subset U,$$

then the composite of the restriction maps

$$\mathcal{O}(U) \to \mathcal{O}(U') \to \mathcal{O}(U'')$$

is required to be the restriction map $\mathcal{O}(U) \to \mathcal{O}(U'')$. In other words, a presheaf is a contravariant functor to the category of k-algebras from the category whose objects are the open subsets of V and whose morphisms are the inclusions. A ***homomorphism of presheaves*** $\alpha : \mathcal{O} \to \mathcal{O}'$ is a family of homomorphisms of k-algebras

$$\alpha(U) : \mathcal{O}(U) \to \mathcal{O}'(U)$$

commuting with the restriction maps, i.e., a natural transformation.

A presheaf $\mathcal{O}$ is a ***sheaf*** if for every open covering $\{U_i\}$ of an open subset U of V and family of elements $f_i \in \mathcal{O}(U_i)$ agreeing on overlaps (that is, such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ for all i, j), there is a unique element $f \in \mathcal{O}(U)$ such that $f_i = f|U_i$ for all i.[21] A ***homomorphism of sheaves*** on V is a homomorphism of presheaves.

For $v \in V$, the ***stalk*** of a sheaf $\mathcal{O}$ (or presheaf) at v is

$$\mathcal{O}_v = \varinjlim \mathcal{O}(U) \quad \text{(limit over open neighbourhoods of } v\text{)}.$$

In other words, it is the set of equivalence classes of pairs (U, f) with U an open neighbourhood of v and $f \in \mathcal{O}(U)$; two pairs (U, f) and (U', f') are equivalent if $f|U'' = f|U''$ for some open neighbourhood U'' of v contained in $U \cap U'$.

A ***ringed space*** is a pair $(V, \mathcal{O})$ consisting of topological space V together with a sheaf of rings. If the stalk $\mathcal{O}_v$ of $\mathcal{O}$ at v is a local ring for all $v \in V$, then $(V, \mathcal{O})$ is called a ***locally ringed space***.

[21] This condition implies that $\mathcal{O}(\emptyset) = 0$.

A ***morphism*** $(V,\mathcal{O}) \to (V',\mathcal{O}')$ ***of ringed spaces*** is a pair (φ,ψ) with φ a continuous map $V \to V'$ and ψ a family of maps

$$\psi(U'):\mathcal{O}'(U') \to \mathcal{O}(\varphi^{-1}(U')), \quad U' \text{ open in } V',$$

commuting with the restriction maps. Such a pair defines homomorphism of rings $\psi_v:\mathcal{O}'_{\varphi(v)} \to \mathcal{O}_v$ for all $v \in V$. A ***morphism of locally ringed spaces*** is a morphism of ringed space such that ψ_v is a local homomorphism for all v.

Let $\mathcal{B}$ be a base for the topology on V that is closed under finite intersections. A sheaf on $\mathcal{B}$ can be defined in the obvious way, and such a sheaf $\mathcal{O}$ extends to a sheaf $\mathcal{O}'$ on V: for any open subset U of V, define $\mathcal{O}'(U)$ to be the set of families

$$(f_{U'})_{U'\subset U, U'\in\mathcal{B}}, \quad f_{U'} \in \mathcal{O}(U'),$$

agreeing on overlaps. Then $\mathcal{O}'$ is a sheaf of k-algebras on V, and there is a canonical isomorphism $\mathcal{O} \to \mathcal{O}'|\mathcal{B}$.

AFFINE ALGEBRAIC SPACES AND VARIETIES

Let A be a finitely generated k-algebra, and let $V = \operatorname{spm}(A)$. Recall (§11) that the set of principal open subsets of V

$$\mathcal{B} = \{D(f) \mid f \in A\}$$

is a base for the topology on V which is closed under finite intersections. If $D(g) \subset D(f)$, then $V(g) \supset V(f)$, and so some power of g lies in (f) (by 10.4), say, $g^r = cf$ with $c \in A$. Therefore f becomes a unit in A_g, and so there is a well-defined "restriction" homomorphism $A_f \to A_g$ of k-algebras. When $D(g) = D(f)$ this homomorphism is an isomorphism. For each principal open subset D of $\operatorname{spm}(A)$, we choose an f_D such that $D = D(f_D)$.

PROPOSITION 17.5. *There exists a sheaf $\mathcal{O}$ of k-algebras on* $\operatorname{spm}(A)$ *such that*

(a) *for all basic open subsets D, the k-algebra $\mathcal{O}(D) = A_{f_D}$, and*
(b) *for all inclusions $D' \subset D$ of basic open subsets, the restriction map $\mathcal{O}(D) \to \mathcal{O}(D')$ is the canonical map $A_{f_D} \to A_{f_{D'}}$.*

For any other sheaf $\mathcal{O}'$ satisfying (a) and (b), there exists a unique isomorphism $\mathcal{O} \to \mathcal{O}'$ inducing the identity map $\mathcal{O}(D) \to \mathcal{O}'(D)$ for every basic open subset.

PROOF. It suffices to check that $D \rightsquigarrow A_{f_D}$ is a sheaf on the base of basic open subsets. This is straightforward but tedious, and so is left as an exercise. □

We write $\operatorname{Spm}(A)$ for $\operatorname{spm}(A)$ endowed with this sheaf of k-algebras. It is independent of the choice of the elements f_D (up to a unique isomorphism).

PROPOSITION 17.6. *For every $\mathfrak{m} \in \operatorname{spm}(A)$, the stalk $\mathcal{O}_{\mathfrak{m}}$ is canonically isomorphic to $\mathcal{O}_{\mathfrak{m}}$.*

PROOF. Apply (7.3). □

Thus $\operatorname{Spm}(A)$ is a locally ringed space. An ***affine algebraic space*** is topological space V together with a sheaf of k-algebras $\mathcal{O}$ such that $(V,\mathcal{O})$ is isomorphic to $\operatorname{Spm}(A)$ for some finitely generated k-algebra A. A ***regular map***of affine algebraic spaces is morphism of locally ringed spaces.

EXAMPLE 17.7. Affine n-space $\mathbb{A}^n = \mathrm{Spm}(k[X_1,\dots,X_n])$. To give a regular map $V \to \mathbb{A}^1$ is the same as giving a homomorphism of k-algebras $k[X] \to \mathcal{O}(V)$, i.e., an element of $\mathcal{O}(V)$. For this reason, $\mathcal{O}(V)$ is often called the ***ring*** (or ***k-algebra***) ***of regular functions*** on V.

PROPOSITION 17.8. *For any affine algebraic space $(V,\mathcal{O}_V)$ and locally ringed space $(W,\mathcal{O}_W)$, the canonical map*

$$\mathrm{Hom}(V,W) \to \mathrm{Hom}_{k\text{-}alg}(\mathcal{O}_W(W),\mathcal{O}_V(V))$$

is an isomorphism.

PROOF. Exercise for the reader. □

An affine algebraic space V defines a functor

$$R \rightsquigarrow V(R) \stackrel{\text{def}}{=} \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}(V),R). \tag{25}$$

from k-algebras to sets. For example, $\mathbb{A}^n(R) \simeq R^n$ for all k-algebras R.

An ***affine algebraic variety*** is an affine algebraic space V such that $\mathcal{O}_V(V)$ is an affine algebra.

TANGENT SPACES; NONSINGULAR POINTS; REGULAR POINTS

Let $k[\varepsilon]$ be the ring of dual numbers (so $\varepsilon^2 = 0$). For an affine algebraic space V over k, the map $\varepsilon \mapsto 0: k[\varepsilon] \to k$ defines a map

$$V(k[\varepsilon]) \to V(k).$$

For any $a \in V(k)$, we define the ***tangent space*** to V at a, $\mathrm{Tgt}_a(V)$, to be the inverse image of a under this map.

PROPOSITION 17.9. *There is a canonical isomorphism*

$$\mathrm{Tgt}_a(V) \simeq \mathrm{Hom}_{k\text{-}lin}(\mathfrak{m}_a/\mathfrak{m}_a^2,k).$$

This follows from the next two lemmas.

Let $V = V(\mathfrak{a}) \subset k^n$, and assume that the origin o lies on V. Let $\mathfrak{a}_\ell$ be the ideal generated by the linear terms f_ℓ of the $f \in \mathfrak{a}$. By definition, $T_o(V) = V(\mathfrak{a}_\ell)$. Let $A_\ell = k[X_1,\dots,X_n]/\mathfrak{a}_\ell$, and let $\mathfrak{m}$ be the maximal ideal in $k[V]$ consisting of the functions zero at o; thus $\mathfrak{m} = (x_1,\dots,x_n)$.

LEMMA 17.10. *There is a canonical isomorphism*

$$\mathrm{Hom}_{k\text{-}lin}(\mathfrak{m}/\mathfrak{m}^2,k) \xrightarrow{\simeq} \mathrm{Hom}_{k\text{-}alg}(A_\ell,k).$$

PROOF. Let $\mathfrak{n} = (X_1,\dots,X_n)$ be the maximal ideal at the origin in $k[X_1,\dots,X_n]$. Then $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{n}/(\mathfrak{n}^2+\mathfrak{a})$, and as $f - f_\ell \in \mathfrak{n}^2$ for every $f \in \mathfrak{a}$, it follows that $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{n}/(\mathfrak{n}^2 + \mathfrak{a}_\ell)$. Let $f_{1,\ell},\dots,f_{r,\ell}$ be a basis for the vector space $\mathfrak{a}_\ell$. From linear algebra we know that there are $n-r$ linear forms $X_{i_1},\dots,X_{i_{n-r}}$ forming with the $f_{i,\ell}$ a basis for the linear forms on k^n. Then $X_{i_1}+\mathfrak{m}^2,\dots,X_{i_{n-r}}+\mathfrak{m}^2$ form a basis for $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space, and the lemma shows that $A_\ell \simeq k[X_{i_1}\dots,X_{i_{n-r}}]$. A homomorphism $\alpha: A_\ell \to k$ of k-algebras is determined by its values $\alpha(X_{i_1}),\dots,\alpha(X_{i_{n-r}})$, and they can be arbitrarily given. Since the k-linear maps $\mathfrak{m}/\mathfrak{m}^2 \to k$ have a similar description, the first isomorphism is now obvious. □

LEMMA 17.11. *There is a canonical isomorphism*

$$\mathrm{Hom}_{k\text{-}alg}(A_\ell, k) \xrightarrow{\simeq} T_o(V).$$

PROOF. To give a k-algebra homomorphism $A_\ell \to k$ is the same as to give an element $(a_1,\dots,a_n) \in k^n$ such that $f(a_1,\dots,a_n) = 0$ for all $f \in A_\ell$, which is the same as to give an element of $T_P(V)$. □

REMARK 17.12. Let $V = \mathrm{Spm}\, k[X_1,\dots,X_n]/(f_1,\dots,f_m)$, and let $(a_1,\dots,a_n) \in V(k)$. Then $\mathrm{Tgt}_a(V)$ is canonically isomorphic to the subspace of k^n defined by the equations

$$\left.\frac{\partial f_i}{\partial X_1}\right|_a X_1 + \cdots + \left.\frac{\partial f_i}{\partial X_n}\right|_a X_n, \quad i = 1,\dots,m.$$

When a is the origin, this is a restatement of (17.11), and the general case can be deduced from this case by a translation.

The ***dimension*** of an affine algebraic space V is the Krull dimension of $\mathcal{O}(V)$. If V is irreducible, then $\mathcal{O}(V)/\mathfrak{N}$ is an integral domain, and the dimension of V is equal to the transcendence degree over k of the field of fractions of $\mathcal{O}(V)/\mathfrak{N}$; moreover, all maximal ideals have height $\dim V$ (12.11).

PROPOSITION 17.13. *Let V be an affine algebraic space over k, and let $a \in V(k)$. Then $\dim \mathrm{Tgt}_a(V) \geq \dim V$, and equality holds if and only if $\mathcal{O}(V)_{\mathfrak{m}_a}$ is regular.*

PROOF. Let $\mathfrak{n}$ be the maximal ideal of the local ring $A = \mathcal{O}(V)_{\mathfrak{m}_a}$. Then $A/\mathfrak{n} = k$, and $\dim_k \mathfrak{n}/\mathfrak{n}^2 \geq \mathrm{ht}(\mathfrak{n})$, with equality if and only if A is regular. As $\mathfrak{m}_a/\mathfrak{m}_a^2 \simeq \mathfrak{n}/\mathfrak{n}^2$ (6.7), Proposition 17.9 implies that $\dim \mathrm{Tgt}_a(V) = \dim_k \mathfrak{n}/\mathfrak{n}^2$, from which the statement follows. □

An $a \in V(k)$ is ***nonsingular*** if $\dim \mathrm{Tgt}_a(V) = \dim V$; otherwise it is ***singular***. An affine algebraic space V is ***regular*** if all of its local rings $\mathcal{O}(V)_{\mathfrak{m}}$ are regular, and it is ***smooth*** if $V_{k^{\mathrm{al}}}$ is regular. Thus an algebraic space over an algebraically closed field is smooth if and only if all $a \in V(k)$ are nonsingular. A smooth algebraic space is regular, but the converse is false. For example, let k' be a finite inseparable extension of k, and let V be a smooth algebraic space over k'; when we regard V is an algebraic space over k, it is regular, but not smooth.

PROPOSITION 17.14. *A smooth affine algebraic space V is a regular affine algebraic variety; in particular, $\mathcal{O}(V)$ is an integral domain. Conversely, if k is perfect, then every regular affine algebraic space over k is smooth.*

PROOF. Let $A = \mathcal{O}(V)$. If V is smooth, then all the local the local rings of $k^{\mathrm{al}} \otimes_k A$ are regular; in particular, they are integral domains (16.3). This implies that $k^{\mathrm{al}} \otimes_k A$ is reduced, because it implies that the annihilator of any nilpotent element is not contained in any maximal ideal, and so is the whole ring. Therefore A is an affine algebra, and so V is an affine algebraic variety. Let $\mathfrak{m}$ be a maximal ideal in A, and let $\mathfrak{n} = \mathfrak{m}(k^{\mathrm{al}} \otimes_k A)$. Then $\mathfrak{n}$ is a maximal ideal of $k^{\mathrm{al}} \otimes_k A$, and

$$\mathfrak{n}/\mathfrak{n}^2 \simeq k^{\mathrm{al}} \otimes (\mathfrak{m}/\mathfrak{m}^2),$$

and so $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim_{k^{\mathrm{al}}}(\mathfrak{n}/\mathfrak{n}^2)$. This implies that $A_{\mathfrak{m}}$ is regular. In particular, $A_{\mathfrak{m}}$ is an integral domain for all maximal ideals of A, which implies that A is integral domain,

because it implies that the annihilator of any zero-divisor is not contained in any maximal ideal. Conversely, if V is regular, A is an integral domain, and hence an affine k-algebra if k is perfect. □

PROPOSITION 17.15. *Let V be an irreducible affine algebraic space over an algebraically closed field k, and identify V with $V(k)$. The set of nonsingular points of V is open, and it is nonempty if V is an algebraic variety.*

PROOF. We may suppose $V = \operatorname{Spm} k[X_1,\dots,X_n]/(f_1,\dots,f_m)$. Let $d = \dim V$. According to Remark 17.12, the set of singular points of V is the zero-set of the ideal generated by the $(n-d)\times(n-d)$ minors of the matrix

$$\operatorname{Jac}(f_1,\dots,f_m)(a) = \begin{pmatrix} \frac{\partial f_1}{\partial X_1}(a) & \cdots & \frac{\partial f_1}{\partial X_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial X_1}(a) & \cdots & \frac{\partial f_m}{\partial X_n}(a) \end{pmatrix},$$

which is closed. Therefore the set of nonsingular points is open.

Now suppose that V is an algebraic variety. The next two lemmas allow us to suppose that $V = k[X_1,\dots,X_n]/(f)$ where f is a nonconstant irreducible polynomial. Then $\dim V = n-1$, and so we have to show that the equations

$$f = 0, \quad \frac{\partial f}{\partial X_1} = 0, \quad \cdots \quad , \frac{\partial f}{\partial X_n} = 0$$

have no common zero. If $\frac{\partial f}{\partial X_1}$ is identically zero on $V(f)$, then f divides it. But $\frac{\partial f}{\partial X_1}$ has degree less than that of f and f is irreducible, and so this implies that $\frac{\partial f}{\partial X_1} = 0$. Therefore f is a polynomial in $X_2,\dots,X_n$ (characteristic zero) or $X_1^p, X_2,\dots,X_n$ (characteristic p). Continuing in this fashion, we find that either f is constant (characteristic zero) or a pth power (characteristic p), which contradict the hypothesis. □

Let V be an irreducible affine algebraic variety. Then $\mathcal{O}(V)$ is an integral domain, and we let $k(V)$ denote its field of fractions. Two irreducible affine algebraic varieties V and W are said to be ***birationally equivalent*** if $k(V) \approx k(W)$.

LEMMA 17.16. *Two irreducible varieties V and W are birationally equivalent if and only if there are open subsets U and U' of V and W respectively such that $U \approx U'$.*

PROOF. Assume that V and W are birationally equivalent. We may suppose that $A = \mathcal{O}(V)$ and $B = \mathcal{O}(W)$ have a common field of fractions K. Write $B = k[x_1,\dots,x_n]$. Then $x_i = a_i/b_i$, $a_i, b_i \in A$, and $B \subset A_{b_1\dots b_r}$. Since $\operatorname{Spm}(A_{b_1\dots b_r})$ is a basic open subvariety of V, we may replace A with $A_{b_1\dots b_r}$, and suppose that $B \subset A$. The same argument shows that there exists a $d \in B \subset A$ such $A \subset B_d$. Now

$$B \subset A \subset B_d \Longrightarrow B_d \subset A_d \subset (B_d)_d = B_d,$$

and so $A_d = B_d$. This shows that the open subvarieties $D(b) \subset V$ and $D(b) \subset W$ are isomorphic. This proves the "only if" part, and the "if" part is obvious. □

LEMMA 17.17. *Every irreducible algebraic variety of dimension d is birationally equivalent to a hypersurface in $\mathbb{A}^{d+1}$.*

PROOF. Let V be an irreducible variety of dimension d. According to FT 8.21, there exist algebraically independent elements $x_1, \dots, x_d \in k(V)$ such that $k(V)$ is finite and separable over $k(x_1, \dots, x_d)$. By the primitive element theorem (FT 5.1), $k(V) = k(x_1, \dots, x_d, x_{d+1})$ for some x_{d+1}. Let $f \in k[X_1, \dots, X_{d+1}]$ be an irreducible polynomial satisfied by the x_i, and let H be the hypersurface $f = 0$. Then $k(V) \approx k(H)$. □

ALGEBRAIC SCHEMES, SPACES, AND VARIETIES

An algebraic space over k is a locally ringed space that admits a finite open covering by affine algebraic spaces. An algebraic variety over k is a locally ringed space $(X, \mathcal{O}_X)$ that admits a finite open covering by affine algebraic spaces and satisfies the following separation condition: for every pair $\varphi_1, \varphi_2 : Z \to X$ of locally ringed space with Z and affine algebraic variety, the subset of Z on which φ_1 and φ_2 agree is closed.

Let $(X, \mathcal{O}_X)$ be an algebraic scheme over k, i.e., a scheme of finite type over k, and let X' be the subset of X obtained by omitting all the nonclosed points. Then $(X', \mathcal{O}_X|X')$ is an algebraic space over k. Conversely, let $(X, \mathcal{O}_X)$ be an algebraic space over k; for each open subset U of X, let U' be the set of irreducible closed subsets of U, and regard U' as a subset of X' in the obvious way; then $(X', \mathcal{O}_{X'})$ where $\mathcal{O}_{X'}(U') = \mathcal{O}_X(U)$ is an algebraic scheme over k.

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