

Algebraic groups as automorphism groups of algebras

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Abstract

We show that every algebraic group scheme over a field with at least 8 elements can be realized as the group of automorphisms of a nonassociative algebra. This is only a modest improvement of the theorem of Gordeev and Popov (2003), but it allows us to give a new characterization of algebraic Lie algebras and to simplify the standard descriptions of Mumford–Tate domains and Shimura varieties as moduli spaces. Once the original argument of Gordeev and Popov has been rewritten in the language of schemes, we find that it also applies to algebraic groups over Dedekind domains.

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Introduction

Let k be a field. We use the following conventions: an algebra A over k is a k -vector space V equipped with a k -linear map $t : V \otimes_k V \rightarrow V$ (no conditions);¹ a commutative k -algebra is a commutative associative k -algebra with an identity element; an algebraic group over k is an affine group scheme of finite type over k . When V is a vector space over k and R is a commutative k -algebra, V_R denotes the R -module $R \otimes_k V$.

Let $A = (V, t)$ be a finite-dimensional algebra over k . The functor $R \rightsquigarrow \text{Aut}_R(A_R)$ of commutative k -algebras is represented by an algebraic subgroup of GL_{V_R} , which we denote by $\underline{\text{Aut}}(A)$. In general, $\underline{\text{Aut}}(A)$ need not be smooth.

In the remainder of the introduction, k is a field with at least 8 elements.

THEOREM 1. *Every algebraic group over k is isomorphic to $\underline{\text{Aut}}(A)$ for some finite-dimensional algebra A over k .*

This statement is almost the same as that of Theorem 1 in Gordeev and Popov 2003. However, there “algebraic group” is meant in the sense of Borel 1991, not schemes. In the language of schemes, they prove that, for each smooth algebraic group G over k , there exists a finite-dimensional algebra A over k such that $G(K) = \text{Aut}(K \otimes A)$ for all

¹This is Bourbaki’s definition. Note that we do not require an algebra to have a two-sided identity.

fields K containing k (ibid., Corollary 1). By contrast, we prove that, for each algebraic group G (not necessarily smooth) over k , there exists a finite-dimensional algebra A over k such that $G(R) = \text{Aut}(R \otimes A)$ for all commutative k -algebras R .

If $G = \underline{\text{Aut}}(A)$, then, in particular, $G(R) = \text{Aut}(R \otimes A)$ for R the ring of dual numbers over k . From this it follows that $\text{Lie}(G)$ is the Lie algebra $\text{Der}(A)$ of derivations of A . We now have the following simple criterion for a Lie algebra to be algebraic.

COROLLARY 1. *Let \mathfrak{g} be a finite-dimensional Lie algebra over k . Then $\mathfrak{g} = \text{Lie}(G)$ for some algebraic group G over k if and only if $\mathfrak{g} = \text{Der}(A)$ for some finite-dimensional algebra A over k .*

PROOF. If $\mathfrak{g} = \text{Der}(A)$, we define G to be $\underline{\text{Aut}}(A)$, and then $\text{Lie}(G) = \text{Der}(A) = \mathfrak{g}$. Conversely, if $\mathfrak{g} = \text{Lie}(G)$, we use Theorem 1 to set $G = \underline{\text{Aut}}(A)$, and then $\text{Der}(A) = \text{Lie}(G) = \mathfrak{g}$. ■

REMARK 1. When k has characteristic $p \neq 0$, both $\text{Lie}(G)$ and $\text{Der}(A)$ have natural p -Lie algebra structures, and Corollary 1 holds with “Lie algebra” replaced by “ p -Lie algebra”.

Theorem 1 extends to neutral tannakian categories. An *algebra* in a tensor category is an object X equipped with an algebra structure, i.e., a morphism $t : X \otimes X \rightarrow X$.

COROLLARY 2. *Let \mathcal{C} be a neutral algebraic² tannakian category over k . There exists an algebra (X, t) in \mathcal{C} such that, for every fibre functor ω with values in a field $k' \supset k$,*

$$\underline{\text{Aut}}^{\otimes}(\omega) = \underline{\text{Aut}}(\omega(X), \omega(t)).$$

PROOF. As \mathcal{C} is neutral, there exists a k -valued fibre functor ω_0 , and ω_0 defines an equivalence of tensor categories $\mathcal{C} \rightarrow \text{Rep}(G)$, where $G = \underline{\text{Aut}}^{\otimes}(\omega_0)$. According to Theorem 1, $G = \underline{\text{Aut}}(A)$ for some algebra $A = (V, t)$ in $\text{Rep}(G)$. There exists an algebra (X, t) in \mathcal{C} such that $\omega_0(X, t)$ is isomorphic to (V, t) . For any k' -valued fibre functor ω ,

$$\underline{\text{Aut}}^{\otimes}(\omega) \subset \underline{\text{Aut}}(\omega(X), \omega(t)),$$

but ω becomes isomorphic to ω_0 over some field containing k' , and so the inclusion is an equality. ■

QUESTION 1. Does Corollary 2 hold for nonneutral tannakian categories?

Let G be an algebraic group over k . A standard result says that there exists a finite-dimensional k -vector space V and a family of tensors for V such that G is isomorphic to the subgroup of GL_V fixing the tensors. Theorem 1 gives a more precise statement.

COROLLARY 3. *Let G be an algebraic group over k . There exists a finite-dimensional k -vector space V and a $t \in V \otimes V^{\vee} \otimes V^{\vee}$ such that G is isomorphic to the subgroup of GL_V fixing t . Here V^{\vee} is the linear dual of V .*

²A tannakian category over a field is said to be algebraic if it corresponds to an algebraic gerbe. This amounts to saying that the affine group scheme attached to a fibre functor over some extension field of the base field is algebraic, i.e., of finite type. See Saavedra 1972, III, 3.3.1.

PROOF. Let V be a finite-dimensional k -vector space, and let $t' : V \otimes V \rightarrow V$ be the linear map corresponding to $t \in V \otimes V^\vee \otimes V^\vee$. Let R be a commutative k -algebra and α an R -linear automorphism of V_R . Then $\alpha(t) = t$ if and only if α is an algebra automorphism of (V, t') . Thus, the corollary is a restatement of Theorem 1. ■

Once the proof of Theorem 1 of Gordeev and Popov has been rewritten in the language of schemes, one sees that it in fact applies over more general bases. In particular, we prove the following statement.

THEOREM 2. *Let G be a flat algebraic group over a Dedekind domain R . If R has enough units, then there exists an algebra A over R , flat and finitely generated as an R -module, such that G is isomorphic to $\underline{\text{Aut}}(A)$ (i.e., G represents the functor of commutative R -algebras $R' \rightsquigarrow \text{Aut}(A_{R'})$).*

See Theorem 4 for a precise statement. In the course of proving Theorem 2, we obtain the following result (Corollary 1 to Proposition 8), which has applications to Shimura varieties.³

THEOREM 3. *Let G be a flat algebraic group over a Dedekind domain R . There exists a finite flat R -submodule V of $\mathcal{O}(G)$, stable under G , such that the homomorphism $G \rightarrow \text{GL}_V$ is a closed immersion. If R is principal and the generic fibre of G over R is linearly reductive, then G is the subgroup of GL_V fixing a finite collection of tensors in spaces $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$.*

As polarizable rational Hodge structures form a tannakian category, it is possible to equip such a Hodge structure with an algebra structure. Theorem 1 allows us to realize Mumford–Tate domains as a moduli spaces for polarized rational Hodge structures with an algebra structure. This is simpler than the usual description in terms of polarized rational Hodge structures equipped with some family of Hodge tensors.

Similarly, Theorem 1 and its corollaries allow us to realize Shimura varieties of abelian type with rational weight as moduli schemes for abelian motives with an algebra structure. This is simpler than the description in Theorems 3.13 and 3.31 of Milne 1994. As this depends on Deligne’s theory of absolute Hodge classes on abelian varieties, at present it applies only in characteristic zero. However, once Deligne’s theory has been extended to mixed characteristic (cf. Milne 2009), Theorem 2 will allow us to obtain a new moduli description of Shimura varieties in mixed characteristic. This should allow a significant simplification of the theory. It was this that sparked the author’s interest in the topic. We do not explain these applications here as we plan to return to them in a future work. For a brief explanation, see the last two sections of arXiv:2012.05708v1.

In Section 1 of the article we explain, following Gordeev and Popov, the construction of some algebras, and in Section 2 we prove our main theorems.

Notation and conventions.

Throughout, R is a commutative ring with 1. By a *finite flat R -module*, we mean an R -module that is finitely presented and flat. Unadorned tensor products are over R . We say that an R -module S is a *direct summand* of an R -module W if it is a submodule of W and admits a complement, i.e., $W = S \oplus W'$ for some W' .

³The second statement should be considered folklore. Gabber has proved a similar result over noetherian regular base schemes of dimension ≤ 2 — see the revised 2011 version of SGA 3, Exposé VI_B, Prop. 13.2.

An algebra A over R is an R -module $V = \text{mod}(A)$ together with an R -linear map $t : V \otimes_k V \rightarrow V$. We say that A is finitely presented, flat, ... if the R -module $\text{mod}(A)$ is finitely presented, flat, For an element a of an algebra, r_a denotes right multiplication by a . We let $\langle S \rangle$ denote the linear span of a subset S of a module.

For a finite flat R -module V , we let $T(V)$ denote the tensor algebra of V ,

$$T(V) = \bigoplus_{i \geq 0} V^{\otimes i},$$

and we let $T(V)_+$ denote the following ideal in $T(V)$,

$$T(V)_+ = \bigoplus_{i \geq 1} V^{\otimes i}.$$

Both are algebras over R equipped with a natural action of the algebraic group GL_V :

$$g \cdot t_i = g^{\otimes i}(t_i), \quad g \in \text{GL}(V_R), \quad t_i \in V_R^{\otimes i}, \quad R \text{ a commutative } k\text{-algebra.}$$

By an algebraic group over R , we mean an affine group scheme of finite presentation over R . An embedding of algebraic groups is a morphism that is both a homomorphism and a closed immersion. For an R -module V with an action of an algebraic group G , we let V_0 denote V equipped with the trivial action of G .

1 Some special algebras

This section is adapted from Gordeev and Popov 2003 and Perepechko 2009.

The algebra $A(V, S)$

PROPOSITION 1. *Let V be a nonzero finite flat R -module. Let S be a finite flat R -submodule of $V^{\otimes r}$, some $r > 1$, such that $V^{\otimes r}/S$ is flat. Then there exists a finite flat graded algebra⁴ $A = V \oplus A^2 \oplus \dots$ over R such that*

$$(\text{GL}_V)_S = \underline{\text{Aut}}(A, V) \quad (\text{automorphisms of } A \text{ stabilizing } V).$$

Here $(\text{GL}_V)_S$ represents the functor $R' \rightsquigarrow \{g \in \text{GL}(V_{R'}) \mid g^{\otimes r}(S_{R'}) = S_{R'}\}$.

PROOF. Let

$$I(S) = S \oplus \left(\bigoplus_{i > r} V^{\otimes i} \right).$$

It is an ideal in the algebra $T(V)_+$, and we define

$$A(V, S) = T(V)_+/I(S).$$

This is a finite flat algebra over R with

$$\text{mod}(A(V, S)) = \left(\bigoplus_{i=1}^{r-1} V^{\otimes i} \right) \oplus (V^{\otimes r}/S)$$

as a graded R -module. Let R' be a commutative R -algebra. When we replace V and S with $V_{R'}$ and $S_{R'}$ in the above definition, we obtain an algebra $A(V_{R'}, S_{R'})$ over R' , and

$$A(V_{R'}, S_{R'}) \simeq R' \otimes_R A(V, S).$$

⁴Here A^2 is the part of degree 2 of the graded algebra A .

The ideal $I(S)$ is stable under the natural action of $(\mathrm{GL}_V)_S$ on $\mathrm{T}(V)_+$, and so $(\mathrm{GL}_V)_S$ acts on $A(V, S)$ by algebra automorphisms. The quotient map $\pi : \mathrm{T}(V)_+ \rightarrow A(V, S)$ is $(\mathrm{GL}_V)_S$ -equivariant. The condition $r > 1$ ensures that $V = V^{\otimes 1}$ is a submodule of $A(V, S)$. Hence $(\mathrm{GL}_V)_S$ acts faithfully on $A(V, S)$, and it stabilizes V . It remains to show that the algebraic group $(\mathrm{GL}_V)_S$ represents the functor

$$R' \rightsquigarrow \{\sigma \in \mathrm{Aut}(A(V, S)_{R'}) \mid \sigma(V_{R'}) = V_{R'}\}.$$

Let R' be a commutative R -algebra. We have seen that

$$(\mathrm{GL}_V)_S(R') \subset \{\sigma \in \mathrm{Aut}(A(V_{R'}, S_{R'})) \mid \sigma(V_{R'}) = V_{R'}\}$$

and it remains to prove equality. Let σ be an element $\mathrm{Aut}(A(V_{R'}, S_{R'}))$ such that $\sigma(V_{R'}) = V_{R'}$. Put $g = \sigma|_{V_{R'}}$, and let g^* denote the canonical extension of g to an automorphism of $\mathrm{T}(V_{R'})_+$. Then $g^*|_{V_{R'}} = g = \sigma|_{V_{R'}}$, and the diagram

$$\begin{array}{ccc} \mathrm{T}(V_{R'})_+ & \xrightarrow{g^*} & \mathrm{T}(V_{R'})_+ \\ \downarrow \pi_R & & \downarrow \pi_R \\ A(V_{R'}, S_{R'}) & \xrightarrow{\sigma} & A(V_{R'}, S_{R'}) \end{array}$$

commutes because it does on $V_{R'}$, which generates the algebra $\mathrm{T}(V_{R'})_+$. The commutativity of the diagram implies that $g^*(\mathrm{Ker}(\pi_{R'})) = \mathrm{Ker}(\pi_{R'})$. As $\mathrm{Ker}(\pi_{R'}) = I(S_{R'})$, it follows that g is an element of $\mathrm{GL}(V_{R'})$ such that $g^{\otimes r}(S_{R'}) = S_{R'}$. The diagram shows that its image in $\mathrm{Aut}(A(V_{R'}, S_{R'}))$ is σ . ■

Two lemmas

LEMMA 1. *Let V be a finite flat R -module and ϕ an endomorphism of V . Suppose that V decomposes into a direct sum of eigenspaces*

$$V = V_1 \oplus \cdots \oplus V_n$$

for ϕ with eigenvalues $\alpha_1, \dots, \alpha_n \in R$ that are distinct modulo every maximal ideal of R . For any commutative R -algebra R' , $V_{R'} = V_{1R'} \oplus \cdots \oplus V_{nR'}$ with

$$V_{iR'} = \{x \in V_{R'} \mid \phi_{R'}(x) = \alpha_i x\}. \quad (1)$$

PROOF. Certainly, $V_{R'} \stackrel{\mathrm{def}}{=} R' \otimes_R V$ is the direct sum of the R' -modules $V_{iR'} \stackrel{\mathrm{def}}{=} R' \otimes_R V_i$ and $V_{iR'}$ is contained in the right-hand side of (1). For the opposite inclusion, let $x \in V_{R'}$ be such that $\phi_{R'}(x) = \alpha_i x$, and write $x = x_1 + \cdots + x_n$ with $x_j \in V_{jR'}$. Then

$$\phi_{R'}(x) = \alpha_1 x_1 + \cdots + \alpha_n x_n$$

and so

$$0 = \phi_{R'}(x) - \alpha_i x = \sum_j (a_j - \alpha_i) x_j.$$

It follows that $(a_j - \alpha_i) x_j = 0$ for all $j \neq i$. As $(\alpha_j - \alpha_i) \in R^\times \subset R'^\times$, this implies that $x_j = 0$ for all $j \neq i$. ■

LEMMA 2. Let A be an algebra over R with a left identity element $e \in A$. Suppose that $\text{mod}(A)$ decomposes into a direct sum of eigenspaces

$$\text{mod}(A) = Re \oplus A_1 \oplus \cdots \oplus A_r$$

for r_e with eigenvalues $1, \alpha_1, \dots, \alpha_r \in R$ such that $0, 1, \alpha_1, \dots, \alpha_r$ are distinct modulo every maximal ideal of R . For any commutative R -algebra R' ,

- (a) e is the unique left identity element in $A_{R'}$;
- (b) if $\sigma \in \text{Aut}(A_{R'})$, then $\sigma(e) = e$ and $\sigma(A_{iR'}) = A_{iR'}$ for all i .

PROOF. According to Lemma 1,

$$\text{mod}(A_{R'}) = R'e \oplus A_{1R'} \oplus \cdots \oplus A_{rR'}$$

with $A_{iR'} = \{x \in A_{R'} \mid xe = \alpha_i x\}$.

(a) Let e' be a left identity element of $A_{R'}$, and write $e' = \alpha e + a_1 + \cdots + a_r$ with $\alpha \in R'$ and $a_i \in A_{iR'}$. Then $e = e'e = (\alpha e + a_1 + \cdots + a_r)e = \alpha e + \alpha_1 a_1 + \cdots + \alpha_r a_r$. As $\alpha_i \in R^\times \subset R'^\times$ and $\alpha_i a_i \in A_{iR'}$, this implies that $a_i = 0$ for all i . Therefore $e' = \alpha e$ and $e = \alpha e$.

(b) We have $\sigma(e) = e$ because both are left identity elements in $A_{R'}$. Moreover, $\sigma(A_{iR'})$ is the submodule of $A_{R'}$ on which $r_{\sigma(e)}$ acts as multiplication by α_i . As $r_{\sigma(e)} = r_e$, we deduce that $\sigma(A_{iR'}) = A_{iR'}$. ■

The algebra $D(L, U, S, \gamma)$

PROPOSITION 2. Let V be a finite flat R -module of the form $V = L \oplus U$ with L free of rank 2. Let S be a finite flat R -submodule of $V^{\otimes r}$, some $r > 1$, such that $V^{\otimes r}/S$ is flat. Extend the action of GL_U on U to V by letting it act trivially on L . If there exist $\gamma_1, \dots, \gamma_6 \in R$ such that the elements $0, 1, \gamma_1, \dots, \gamma_6$ are distinct modulo every maximal ideal of R , then there exists a finite flat algebra D over R such that

$$(\text{GL}_U)_S \simeq \underline{\text{Aut}}(D).$$

PROOF. We define the underlying R -module of $D = D(L, U, S, \gamma)$ to be

$$\begin{aligned} \text{mod}(D) &= Re \oplus Rb \oplus Rc \oplus Rd \oplus \text{mod}(A(V, S)) \\ &= Re \oplus Rb \oplus Rc \oplus Rd \oplus L \oplus U \oplus \left(\bigoplus_{i=2}^{r-1} V^{\otimes i} \right) \oplus (V^{\otimes r}/S). \end{aligned}$$

Let $\{\ell_1, \ell_2\}$ be a basis for L . The multiplication map on D is determined by the following rules:

- (a) e is a left identity element for D ;
- (b) each submodule in the top row of the following table is an eigenspace for r_e with eigenvalue the element in the row below it,

$$\begin{array}{c|c|c|c|c|c|c} \langle e \rangle & \langle b \rangle & \langle c \rangle & \langle d \rangle & L & U & \left(\bigoplus_{i=2}^{r-1} V^{\otimes i} \right) \oplus (V^{\otimes r}/S) \\ \hline 1 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \end{array} ;$$

(c) the multiplication table for b, c, d is

	b	c	d
b	0	$c + \frac{\gamma_2 - \gamma_1}{\gamma_2 - \gamma_3} b$	0
c	$-c$	b	e
d	ℓ_1	d	ℓ_2

(d) $\langle b, c, d \rangle \cdot A(V, S) = 0 = A(V, S) \cdot \langle b, c, d \rangle$;

(e) $A(V, S)$ is a subalgebra of D .

The action of $(\mathrm{GL}_U)_S$ on $T(V)_+$ leaves the ideal $I(S)$ stable, and so it passes to the quotient $A(V, S)$ (see the proof of Proposition 1). We extend this action on $\mathrm{mod}(A(V, S))$ to an action on $\mathrm{mod}(D)$ by letting $(\mathrm{GL}_U)_S$ act trivially on $\langle e, b, c, d \rangle$. In this way, we get a homomorphism

$$(\mathrm{GL}_U)_S \rightarrow \underline{\mathrm{Aut}}(D). \quad (2)$$

It remains to show that this is an isomorphism.

Let R' be a commutative R -algebra. We have to show that the map

$$(\mathrm{GL}_U)_S(R') \rightarrow \mathrm{Aut}(D_{R'})$$

is an isomorphism. It is clearly injective. On the other hand, let σ be an automorphism of the algebra $D_{R'}$ over R' . According to Lemma 2, $\sigma(e) = e$ and σ stabilizes each of the R' -submodules $R'b, R'c, R'd, L_{R'}, U_{R'}$, and $\left(\bigoplus_{i=2}^{r-1} V^{\otimes i}\right) \oplus (V^{\otimes r}/S)_{R'}$ of $\mathrm{mod}(D)_{R'}$. Let $\sigma(b) = \gamma_b b$, $\sigma(c) = \gamma_c c$, and $\sigma(d) = \gamma_d d$, where the γ lie in R' . Now

$$\begin{aligned} c \cdot d = e &\Rightarrow \gamma_c \gamma_d = 1 \\ d \cdot c = d &\Rightarrow \gamma_c \gamma_d = \gamma_d \\ c \cdot b = -c &\Rightarrow \gamma_c \gamma_b = \gamma_c. \end{aligned}$$

From the first equation, we see that γ_c and γ_d are units in R' , and so the remaining equations show that $\gamma_c = 1 = \gamma_b$. Therefore $\gamma_d = 1$ also, and so σ acts as the identity map on $\langle e, b, c, d \rangle_{R'}$. Next

$$\begin{aligned} d \cdot b = \ell_1 &\Rightarrow \sigma(\ell_1) = \ell_1 \\ d \cdot d = \ell_2 &\Rightarrow \sigma(\ell_2) = \ell_2, \end{aligned}$$

and so σ acts as the identity on $L_{R'}$. Finally, σ acts on $\mathrm{mod}(A(V, S))_{R'}$ as an automorphism of $A(V, S)_{R'}$. As it maps $V_{R'}$ into $V_{R'}$, Proposition 2 shows that σ arises from an element of $(\mathrm{GL}_U)_S(R')$. ■

Note that D is not associative: we need not have $xe \cdot y = x \cdot ey$.

2 Algebraic groups as stabilizers

In this section, we explain how to realize algebraic groups as the stabilizers of submodules or of families of tensors, and we prove Theorems 2 and 3.

Preliminaries

In this subsection, we extend some standard results from base fields to more general rings.

2.1. An R -module V is finite flat if it satisfies the following equivalent conditions (see the author's notes on commutative algebra, 12.6):

- ◊ V is finitely generated and projective;
- ◊ V is finitely presented and flat;
- ◊ V is locally free of finite rank.

Assume that R is noetherian, and let W be an R -submodule of an R -module V . If V is finitely generated and V/W is flat, then V/W is projective, and so W is a direct summand of V , i.e., $V = W \oplus W'$ for some R -submodule W' of V . Conversely, if V is finite flat and W is a direct summand of V , then V/W is isomorphic to a direct summand of V , and hence is (finite) flat (ibid., 11.3).

2.2. Let $f_1, \dots, f_m \in R$ be such that $f_1 + \dots + f_m = 1$. For any R -module V ,

$$0 \longrightarrow V \longrightarrow \prod_i V_{f_i} \rightrightarrows \prod_{i,j} V_{f_i f_j}$$

is exact (ibid., 11.22). When V is finite flat, the f_i may be chosen so that V_{f_i} is free as an R_{f_i} -module. This often allows us in proofs to suppose that V is free.

2.3. Assume that R is an integral domain, and let V and W be finite flat R -modules. If v and w are nonzero elements of V and W , then $v \otimes w$ is a nonzero element of $V \otimes W$. This becomes obvious once we tensor with the field of fractions of R . Note that the hypothesis on R is necessary: if R contains nonzero elements a, b such that $ab = 0$, then a and b are nonzero elements of the R -module R , but $a \otimes b = 0$ in $R \otimes R \simeq R$.

2.4. Assume that R is noetherian. Let V be an R -module, and let $TV = \bigoplus_n V^{\otimes n}$ be its tensor algebra. The exterior algebra $\bigwedge V$ of V is TV/I , where I is the two-sided ideal generated by the elements $x \otimes x$, $x \in V$. The antisymmetrization map is

$$a_n : V^{\otimes n} \rightarrow V^{\otimes n}, \quad a_n(t) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma(t).$$

If V is finite flat, then the kernel of a_n is I_n , and so a_n defines an isomorphism

$$\bigwedge^n V \rightarrow A_n''(V) \subset V^{\otimes n}, \quad A_n''(V) \stackrel{\text{def}}{=} \text{Im}(a_n);$$

moreover, $A_n''(V)$ is locally a direct summand of $T^n V$, and so it is finite flat. See Bourbaki, Algebra, III, §7, no. 4, Remark, and Exercise 8.

2.5. Let V be a finite flat R -module. Then GL_V is a flat algebraic group over R , locally isomorphic to GL_n , $n = \text{rank } V$.

2.6. Let G be an algebraic group over R and V an R -module. By an action of G on V , we mean an action of $G(R')$ on $V(R')$ functorial in the R -algebra R' . When V is finite flat, to give an action of G on V is the same as giving a homomorphism of algebraic groups $G \rightarrow \text{GL}_V$.

2.7. Let V be a finite flat R -module. An action $r : G \rightarrow \mathrm{GL}_V$ of G on V maps the universal element in $G(\mathcal{O}(G))$ to an $\mathcal{O}(G)$ -linear endomorphism of $V \otimes \mathcal{O}(G)$, which is determined by its restriction to V ,

$$\rho : V \rightarrow V \otimes \mathcal{O}(G).$$

The map ρ is a co-action of the Hopf algebra $\mathcal{O}(G)$ on V , i.e.,

$$\begin{cases} (\mathrm{id}_V \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}_{\mathcal{O}(G)}) \circ \rho \\ (\mathrm{id}_V \otimes \epsilon) \circ \rho = \mathrm{id}_V. \end{cases} \quad (3)$$

In this way, we get a one-to-one correspondence $r \leftrightarrow \rho$ between the actions of G on V and the co-actions of $\mathcal{O}(G)$ on V (Milne 2017, 4.1).

LEMMA 3. *Let G be an algebraic group over R . Let W be a finite flat R -module with an action of G , and let $\rho : W \rightarrow W_0 \otimes \mathcal{O}(G)$ be the corresponding co-action map. Then ρ is G -equivariant, and realizes W as a direct summand of $W_0 \otimes \mathcal{O}(G)$. (Here W_0 denotes W with the trivial action of G .)*

PROOF. The first equality in (3) says that ρ is a homomorphism of $\mathcal{O}(G)$ -comodules (and hence a homomorphism of G -modules). The second equality in (3) says that the composite of ρ with $\mathrm{id}_{V_0} \otimes \epsilon$ is the identity map. \blacksquare

2.8. Let G be an algebraic group over R and V an R -module on which G acts. When $i : S \rightarrow V$ is an R -submodule of V , we define G_S (stabilizer of S in G) to be the functor

$$R' \rightsquigarrow \{\alpha \in \mathrm{Aut}_{R'}(V_{R'}) \mid \alpha(i_{R'}(S_{R'})) = i_{R'}(S_{R'})\}.$$

When S is a subset of V , we define G_S to be the functor

$$R' \rightsquigarrow \{\alpha \in \mathrm{Aut}_{R'}(V_{R'}) \mid \alpha(s) = s \text{ for all } s \in S\}.$$

If S is contained in an R -submodule V' of V , stable under G , and V/V' is flat, then the group functor G_S is the same for $S \subset V'$ as for $S \subset V$ (because the map $V'_{R'} \rightarrow V_{R'}$ is injective for all R -algebras R').

2.9. When R is noetherian, every comodule over a flat R -coalgebra is a filtered union of finitely generated subcomodules (Serre 1993, 1.4). In particular, every G -module, where G is a flat algebraic group over R , is a filtered union of finite generated G -submodules.

LEMMA 4. *Let R be an integral domain and G an algebraic group over R .*

- (a) *Let V_1 and V_2 be finite flat R -modules on which G acts, and let $S_1 \subset V_1$ and $S_2 \subset V_2$ be nonzero submodules such that V_1/S_1 and V_2/S_2 are flat. Then the stabilizer of $S_1 \otimes S_2 \subset V_1 \otimes V_2$ in $\mathrm{GL}_{V_1} \times \mathrm{GL}_{V_2}$ is equal to the stabilizer of $S_1 \oplus S_2 \subset V_1 \oplus V_2$ in $\mathrm{GL}_{V_1} \times \mathrm{GL}_{V_2}$.*
- (b) *Let $V_1 = V = V_2$ in (a). Then the stabilizer of $S_1 \otimes S_2 \subset V \otimes V$ in GL_V is equal to the intersection of the stabilizers of $S_1 \subset V$ and $S_2 \subset V$ in GL_V .*
- (c) *Let V be a finite flat R -module on which G acts, and let L be a line (i.e., one-dimensional subspace) in V such that V/L is flat. For every $r > 0$, the stabilizer of $L \subset V$ in GL_V is equal to the stabilizer of $L^{\otimes r} \subset V^{\otimes r}$ in GL_V .*

PROOF. (a) As V_1/S_1 and V_2/S_2 are flat and finitely generated and R is integral domain, they are finitely presented. The hypotheses imply that $V_1 = S_1 \oplus W_1$ and $V_2 = S_2 \oplus W_2$ for some finite flat R -submodules W_1 and W_2 of V_1 and V_2 . Then

$$V_1 \otimes V_2 = (S_1 \otimes S_2) \oplus (S_1 \otimes W_2) \oplus (W_1 \otimes S_2) \oplus (W_1 \otimes W_2).$$

Let R' be an R -algebra and α_1 and α_2 automorphisms of $V_{1R'}$ and $V_{2R'}$. We have to show that

$$(\alpha_1 \otimes \alpha_2)(S_1 \otimes S_2) \subset S_1 \otimes S_2 \Rightarrow \alpha_1(S_1) \subset S_1 \text{ and } \alpha_2(S_2) \subset S_2,$$

the reverse implication being obvious. Let s_1 and s_2 be nonzero elements of S_1 and S_2 , and let $\alpha_1(s_1) = s'_1 + w_1$ and $\alpha_2(s_2) = s'_2 + w_2$. Then

$$S_1 \otimes S_2 \ni (\alpha_1 \otimes \alpha_2)(s_1 \otimes s_2) = s'_1 \otimes s'_2 + s'_1 \otimes w_2 + w_1 \otimes s'_2 + w_1 \otimes w_2.$$

If $w_1 \neq 0$, then $s'_2 = 0 = w_2$ (see 2.3), contradicting $s_2 \neq 0$. Hence $w_1 = 0$, and similarly, $w_2 = 0$.

Statement (b) follows from (a), and (c) follows from (b).

LEMMA 5. *Assume that R is noetherian. Let V be a finite flat R -module and S an R -submodule such that V/S is flat. If S is locally free of rank d , then the stabilizer of $S \subset V$ in GL_V is equal to the stabilizer of $\bigwedge^d S \subset \bigwedge^d V$ in GL_V .⁵*

PROOF. If $S = V$, this is obvious, and so we assume that $S \neq V$. Because V/S is flat, $V = S \oplus W$ for some R -submodule W of V (here we use that R is noetherian). As V is finite flat, so also is W .

Let $L = \bigwedge^d S$. Let R' be an R -algebra and α an automorphism of $V_{R'}$. We shall show that

$$\alpha L_{R'} = L_{R'} \Rightarrow \alpha S_{R'} = S_{R'},$$

the reverse implication being obvious. We may suppose that S and W are free (2.2).

Let $(e_j)_{1 \leq j \leq d}$ be a basis for S , and extend it to a basis $(e_i)_{1 \leq i \leq n}$ of V . Let $s = e_1 \wedge \cdots \wedge e_d$. Then

$$S_R = \{v \in V_R \mid s \wedge v = 0 \text{ (in } \bigwedge^{d+1} V_R)\}.$$

To see this, let $v = \sum_{i=1}^n a_i e_i$, $a_i \in R$, be an element of V_R . Then

$$s \wedge v = \sum_{d+1 \leq i \leq n} a_i e_1 \wedge \cdots \wedge e_d \wedge e_i.$$

As the elements $e_1 \wedge \cdots \wedge e_d \wedge e_i$, $d+1 \leq i \leq n$, are part of a basis for $\bigwedge^{d+1} V$, we see that

$$s \wedge v = 0 \iff a_i = 0 \text{ for all } d+1 \leq i \leq n \iff v \in S.$$

Let $\alpha \in \mathrm{GL}(V_R)$. If $(\bigwedge^d \alpha)(L_R) = L_R$, then $(\bigwedge^d \alpha)s = cs$ for some $c \in R^\times$. If $v \in S_R$, then $s \wedge v = 0$, and so

$$0 = (\bigwedge^{d+1} \alpha)(s \wedge v) = (\bigwedge^d \alpha)s \wedge \alpha v = c(s \wedge (\alpha v)),$$

which implies that $\alpha v \in S_R$. ■

⁵This is a standard fact, implicit in the proof of the projectivity of Grassmanians.

Recall that there is a natural left action of G on $\mathcal{O}(G)$ (the regular representation), namely,

$$(gf)(x) = f(xg), \quad f \in \mathcal{O}(G), \quad g \in G, \quad x \in G.$$

LEMMA 6. *Let G be an algebraic group over R and H a closed algebraic subgroup of G . Let $I \subset \mathcal{O}(G)$ be the ideal of H . Then H is the stabilizer of I in $\mathcal{O}(G)$, i.e., for all R -algebras R' ,*

$$H(R') = \{g \in G(R') \mid gI_{R'} \subset I_{R'}\}.$$

PROOF. Let $h \in H(R')$ some R' , and let $f \in I_{R'}$. Then, for all R' -algebras R'' and $x \in H(R'')$,

$$(hf)(x) \stackrel{\text{def}}{=} f(xh) = 0$$

because $xh \in H(R'')$. Hence $hf \in I_{R'}$.

Let $g \in G(R')$ be such that $gI_{R'} \subset I_{R'}$, and let $f \in I$. Then

$$f(g) = f(e \cdot g) = (gf)(e) = 0,$$

because $gf \in I_{R'}$. Hence $g \in H(R')$. ■

2.10. Let V be a finite flat R -module. We let GL_V act on the (finite flat) R -module $\text{End}(V)$ by setting

$$g\alpha = g\circ\alpha, \quad g \in \text{GL}(V_{R'}), \quad \alpha \in \text{End}(V_{R'}), \quad \text{some } R'.$$

Then the canonical isomorphism $\text{End}(V) \simeq V_0^\vee \otimes V$ of R -modules is GL_V -equivariant. Here V_0 denotes V with the trivial action of G .

2.11. Let V be a finite flat R -module and G a closed algebraic subgroup of GL_V . Then GL_V is a schematically dense open subscheme of End_V (multiplicative monoid scheme). Correspondingly

$$\text{Sym}(\text{End}(V)) = \mathcal{O}(\text{End}_V) \subset \mathcal{O}(\text{GL}_V).$$

For example, if V is free, then the choice of a basis for V identifies the inclusion with

$$R[X_{ij}] \subset R[X_{ij}][1/\det(X_{ij})].$$

The inclusion $\text{Sym}(\text{End}(V)) \hookrightarrow \mathcal{O}(\text{GL}_V)$ is GL_V -equivariant for the actions considered in 2.10. Let I be the ideal of G in $\mathcal{O}(\text{GL}_V)$, and let $I' = I \cap \text{Sym}(\text{End}(V))$. Then I' generates the ideal I , and so G is the stabilizer of $I' \subset \text{Sym}(\text{End}(V))$ in GL_V (Lemma 6).

2.12. Recall that an algebraic group over a field is said to be linearly reductive if every finite-dimensional representation is semisimple. In characteristic zero, G is linearly reductive if and only if G° is reductive. In characteristic $p \neq 0$, G is linearly reductive if and only if G° is of multiplicative type and p does not divide the index $(G : G^\circ)$ (Nagata's theorem). See Milne 2017, 12.56.

Are algebraic groups linear?

Let G be a flat algebraic group over a ring R . Does there exist an embedding of G into GL_n for some n ? Apparently the answer is not known even for R the ring of dual numbers over a field. However, there is the following result.⁶

PROPOSITION 3. *Let G be a flat algebraic group over a Dedekind domain R . There exists a finite flat R -submodule V of $\mathcal{O}(G)$, stable under G , such that the homomorphism $G \rightarrow \mathrm{GL}_V$ is a closed immersion.*

PROOF. There exists a finitely generated R -submodule V of $\mathcal{O}(\mathrm{GL}_V)$, stable under G , containing a set of generators for $\mathcal{O}(G)$ (see 2.9). Now G flat over $R \Rightarrow \mathcal{O}(G)$ is torsion-free (as an R -module) $\Rightarrow V$ is torsion-free $\Rightarrow V$ is flat (because R is a Dedekind domain). It remains to show that the homomorphism $\alpha : \mathcal{O}(\mathrm{GL}_V) \rightarrow \mathcal{O}(G)$ defined by the action of G on V is surjective.

We may suppose that V is free (see 2.2). Let $\Delta : \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ be the comultiplication map and $\epsilon : \mathcal{O}(G) \rightarrow R$ the co-identity map. Let $(e_i)_{1 \leq i \leq n}$ be a basis for V , and write $\Delta(e_j) = \sum_i e_i \otimes a_{ij}$, $a_{ij} \in \mathcal{O}(G)$. The image of α contains the a_{ij} (the choice of the basis (e_i) , determines an isomorphism $\mathcal{O}(\mathrm{GL}_V) \simeq R[T_{ij}]$, and α maps T_{ij} to a_{ij} ; see Milne 2017, 4.1). As $\epsilon : \mathcal{O}(G) \rightarrow R$ is the co-identity,

$$e_j = (\epsilon \otimes \mathrm{id}_A)\Delta(e_j) = \sum_i \epsilon(e_i)a_{ij},$$

and so the image of α contains V , which we chose to generate $\mathcal{O}(G)$. ■

Thus, if R is a Dedekind domain and G is flat, then there is an embedding of G into GL_V for some finite flat R -module V . Such a V is a direct summand $F = V \oplus W$ of a free R -module F of finite rank. Extend the action of G on V to F by letting it act trivially on W and choose a basis for F . Now G is a closed algebraic subgroup of GL_n , $n = \mathrm{rank} F$.

Expressing all representations in terms of one faithful representation

Let G be an algebraic group over a field k , and let (V, r) be a faithful representation of G . Then V generates the tannakian category of finite-dimensional representations of G . This means that every finite-dimensional representation of G can be constructed from V by forming tensor products, direct sums, duals, and subquotients (Milne 2017, 4.14).

In this section, we present variants of this statement. For a finite flat R -module of rank r , we let $\det = \bigwedge^r V$ and $\det^{-1} = \det^\vee$. For an R -module V , we let $T^{m,n}(V) = V^{\otimes m} \otimes (V^\vee)^{\otimes n}$.

PROPOSITION 4. *Let G be a closed algebraic subgroup of GL_V , where V is a free R -module of finite rank. Let W be a G -module that is free of finite rank as an R -module. For some s , $W \cdot \det^s$ is isomorphic to a submodule of a quotient of a direct sum of tensor powers of V .*

PROOF. The choice of a basis for W_0 realizes W as a G -submodule of $\mathcal{O}(G)^m$, $m = \mathrm{rank} W$ (see Lemma 3). The embedding $G \hookrightarrow \mathrm{GL}_V$ corresponds to a surjective homomorphism $\mathcal{O}(\mathrm{GL}_V) \rightarrow \mathcal{O}(G)$. Recall that $\mathcal{O}(\mathrm{GL}_V) = \mathrm{Sym}(\mathrm{End}(V))[1/\det]$ and that $\mathrm{End}(V) \simeq$

⁶This should be considered folklore. See an earlier footnote.

$V_0^\vee \otimes V$ as a G -module (2.10). The choice of a basis for V_0 determines a G -isomorphism $\text{End}(V) \simeq nV$, $n = \text{rank } V$. We have G -homomorphisms

$$T(nV)^m \rightarrow \text{Sym}(nV)^m \subset \mathcal{O}(\text{GL}_V)^m \twoheadrightarrow \mathcal{O}(G)^m.$$

For some $s \geq 0$, $W \cdot \det^s$ is contained in the image of $T(nV)^m$ in $\mathcal{O}(G)^m$. Hence $W \cdot \det^s$ is contained in a quotient of $T^{\leq h}(nV)^m$ for some h , and $T^{\leq h}(nV)^m$ is a sum of tensor powers of V . ■

COROLLARY 1. *Let G , V , and W be as in the proposition. Then W is isomorphic to a submodule of a quotient of a direct sum of modules $T^{m,n}(V)$.*

PROOF. Let $n = \text{rank } V$. As \det is a direct summand of $V^{\otimes n}$ (see 2.4), its dual \det^{-1} is a direct summand of $(V^\vee)^{\otimes n}$. In the proof of Proposition 4, we constructed a diagram

$$W \otimes \det^s \hookrightarrow Q \leftarrow T^{\leq h}(nV)^m.$$

On tensoring this with $(V^\vee)^{\otimes ns}$, we get a diagram

$$W \subset W \otimes \det^s \otimes (V^\vee)^{\otimes ns} \hookrightarrow Q \otimes (V^\vee)^{\otimes ns} \leftarrow T^{\leq h}(nV)^m \otimes (V^\vee)^{\otimes ns},$$

as required. ■

REMARK 2. If R is a field and G is linearly reductive, then “of a quotient” can be omitted from the statements of Proposition 4 and Corollary 1.

When $G \subset \text{SL}_V$, the proof of Proposition 4 simplifies.

PROPOSITION 5. *Let G be a closed algebraic subgroup of SL_V , where V is a free R -module of finite rank. Let W be a G -module that is free of finite rank as an R -module. Then W is isomorphic to a submodule of a quotient of a direct sum of tensor powers of V .*

PROOF. As before, $W \subset \mathcal{O}(G)^m$, $m = \text{rank } W$. In this case, we get G -homomorphisms

$$T(nV)^m \rightarrow \text{Sym}(nV)^m \twoheadrightarrow \mathcal{O}(\text{SL}_V)^m \twoheadrightarrow \mathcal{O}(G)^m.$$

For some h , W is contained in a quotient of $T^{\leq h}(nV)^m$. ■

When V is a finite-dimensional vector space over a field k of characteristic zero, Proposition 5 shows that every finite-dimensional SL_V -module W is isomorphic to a submodule of $T(nV)^m$, where $n = \dim V$ and $m = \dim W$. In fact, a stronger result holds.

PROPOSITION 6 (GORDEEV–POPOV). *Let V be a finite-dimensional vector space over a field k . Every finite-dimensional SL_V -module is isomorphic to a submodule of $T(V)_+$.*

PROOF. See Gordeev and Popov 2003, Proposition 11. ■

Algebraic groups as stabilizers

PROPOSITION 7. *Let R be a noetherian ring. Let G be a closed algebraic subgroup of GL_V , where V is a finite flat R -module. For some $h \geq 0$, there exists an R -submodule $S \subset T^{\leq h}(V_0^\vee \otimes V)$ such that G is the stabilizer of S in GL_V .*

PROOF. Let I be the kernel of the homomorphism of R -algebras

$$\mathrm{Sym}(V_0^\vee \otimes V) \hookrightarrow \mathcal{O}(\mathrm{GL}_V) \twoheadrightarrow \mathcal{O}(G).$$

Then G is the stabilizer of I in GL_V (see 2.11). For some $h \geq 0$, $\mathrm{Sym}^{\leq h}(V_0^\vee \otimes V)$ contains a set of generators for the ideal I (here we use that R is noetherian), and G is the stabilizer of

$$I \cap \mathrm{Sym}^{\leq h}(V_0^\vee \otimes V) \subset \mathrm{Sym}^{\leq h}(V_0^\vee \otimes V)$$

in GL_V . Now G is the stabilizer in GL_V of the preimage S of $I \cap \mathrm{Sym}^{\leq h}(V_0^\vee \otimes V)$ under the quotient map

$$T^{\leq h}(V_0^\vee \otimes V) \twoheadrightarrow \mathrm{Sym}^{\leq h}(V_0^\vee \otimes V). \quad \blacksquare$$

REMARK 3. If R is a Dedekind domain and G is flat, then the R -submodule S constructed in the proof of the proposition has the property that $T^{\leq h}(V_0^\vee \otimes V)/S$ is flat. To see this, note that the hypotheses imply that $\mathrm{Sym}(V_0^\vee \otimes V)/I$ is torsion-free, and so I is saturated as an R -submodule of $\mathrm{Sym}(V_0^\vee \otimes V)$. It follows that $I \cap \mathrm{Sym}^{\leq h}(V_0^\vee \otimes V)$ is saturated, and so

$$T^{\leq h}(V_0^\vee \otimes V)/S \simeq \mathrm{Sym}^{\leq h}(V_0^\vee \otimes V)/I \cap \mathrm{Sym}^{\leq h}(V_0^\vee \otimes V)$$

is flat.

The next statement improves results of Deligne (1982, 3.1) and Kisin (2010, 1.3.1). It has applications to Shimura varieties in mixed characteristic.

PROPOSITION 8. *Let R be a Dedekind domain. Let G be a closed algebraic subgroup of GL_V , where V is a finite flat R -module. If the generic fibre of G is linearly reductive, then, locally on $\mathrm{Spec} R$, G is the algebraic subgroup of GL_V fixing a finite collection of tensors in spaces $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$, $m, n \geq 0$.*

PROOF. By ‘‘locally on $\mathrm{Spec} R$ ’’ we mean that there exist $f_i \in R$ such that $f_1 + \cdots + f_m = 1$ and the statement holds after a base change $R \rightarrow R_{f_i}$. Thus, we may suppose that V is free, say, of rank n , and replace $V_0^\vee \otimes V$ with nV in Proposition 7. Let $S \subset T^{\leq h}(nV) \stackrel{\mathrm{def}}{=} W$ be as in that proposition. Then W is free of finite rank, and so S is finite flat (here we use that R is Dedekind). Let $r = \mathrm{rank} S$. Then G is the stabilizer of $L \stackrel{\mathrm{def}}{=} \bigwedge^r S \subset \bigwedge^r W$ in GL_V (Lemma 5). Note that L is locally free of rank 1 and that $\bigwedge^r W$ is a direct summand of $\bigotimes^r W$ (see Lemma 8), which is a direct sum of tensor powers of V .

As the generic fibre of G is linearly reductive, the quotient map $(\bigwedge^r W)^\vee \rightarrow L^\vee$ has a G -equivariant section over the generic point η of $\mathrm{Spec} R$. It follows that there exists a G -stable line $L^* \subset (\bigwedge^r W)^\vee$ that maps isomorphically to L^\vee over η . Now G acts trivially on $L \otimes_R L^*$ because this is so over η , and the stabilizer of

$$L \otimes_R L^* \subset \bigwedge^r W \otimes (\bigwedge^r W^\vee) \subset \bigotimes^r W \otimes (\bigotimes^r W^\vee)$$

in GL_V is equal to G .

After a base change $R \rightarrow R_{f_i}$, the module $L \otimes_R L^*$ will be free. Let $\{s\}$ be a basis for $L \otimes_R L^*$, and write $s = \sum_{i \in I} s_i$ with each s_i an element of a module $T^{m,n}(V)$. Then $G = (\mathrm{GL}_V)_S$ with $S = \{s_i \mid i \in I\}$. \blacksquare

COROLLARY 1. *Let G be a flat algebraic group over a Dedekind domain R . There exists a finite flat R -module V and an embedding $G \hookrightarrow \mathrm{GL}_V$. If R is a principal ideal domain and the generic fibre of G over R is linearly reductive, then G is the algebraic subgroup of GL_V fixing a finite collection of tensors in spaces $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$, $m, n \geq 0$.*

PROOF. This follows from Propositions 3 and 8. \blacksquare

REMARK 4. The condition that G_η is linearly reductive can be replaced by the following condition: the map $\mathrm{Hom}_\eta(G_\eta, \mathbb{G}_m) \rightarrow \mathrm{Hom}_\eta(\mathrm{GL}_{U_\eta}, \mathbb{G}_m)$ has finite cokernel. The proof requires Lemma 4(c).

Algebraic groups as automorphism groups of algebras

The next two lemmas are adapted from Gordeev and Popov 2003.⁷

LEMMA 7. *Let U (resp. L) be a free R -module of finite rank m (resp. rank 1). There exists an injective homomorphism of graded GL_U -modules*

$$\iota: \mathrm{T}(mU) \hookrightarrow \mathrm{T}(L \oplus U)$$

realizing $\mathrm{T}(mU)$ as a direct summand of $\mathrm{T}(L \oplus U)$. Here GL_U acts trivially on L .

PROOF. Let U_i be the i th summand of mU considered as a subspace of mU , and choose a basis $\{f_{ij} \mid j = 1, \dots, m\}$ of U_i . Let $\{l\}$ be a basis for L , and set

$$\iota(f_{i_1 j_1} \otimes \cdots \otimes f_{i_r j_r}) = l^{\otimes i_1} \otimes f_{i_1 j_1} \otimes \cdots \otimes l^{\otimes i_r} \otimes f_{i_r j_r}.$$

The map $\mathrm{T}(mU) \rightarrow \mathrm{T}(L \oplus U)$, defined on a basis of $\mathrm{T}(mU)_+$ by this formula and sending 1 to 1, has the claimed properties. \blacksquare

When R is a field, there even exists an injective homomorphism $\mathrm{T}^{\leq h}(mU) \hookrightarrow \mathrm{T}_+(U)$ (Proposition 6).

LEMMA 8. *Let U be a finite flat R -module and L a free R -module of rank 1. For all $r \geq h \geq 2$, there is an injective homomorphism of GL_U -modules*

$$\mathrm{T}^{\leq h}(U) \hookrightarrow (L \oplus U)^{\otimes r}$$

realizing $\mathrm{T}^{\leq h}(U)$ as a direct summand of $(L \oplus U)^{\otimes r}$.

PROOF. For any $r \geq 1$,

$$\begin{aligned} (L \oplus U)^{\otimes r} &\simeq \bigoplus_{i+j=r} L^{\otimes i} \otimes U^{\otimes j} \oplus \text{other terms} \\ &\simeq \mathrm{T}^{\leq r} U \oplus \text{other terms} \end{aligned}$$

(the second isomorphism depends on a choice of a basis for L). \blacksquare

⁷Readers should be careful not to confuse the tensor algebra with the symmetric algebra, for which the proof of Lemma 7 fails.

PROPOSITION 9. *Let R be a Dedekind domain. Let G be a closed algebraic subgroup of GL_U flat over R , where U is a free R -module of finite rank. Let L be a free R -module of rank 2 with G acting trivially. There exists a finite flat R -module S of $(L \oplus U)^{\otimes r}$, some $r \geq 2$, such that $(L \oplus U)/S$ is flat and $G = (\mathrm{GL}_U)_S$.*

PROOF. Let $m = \mathrm{rank} U$. According to Proposition 7 and Remark 3, $G = (\mathrm{GL}_U)_S$ with S a finite flat R -submodule of $T^{\leq h}(mU)$ such that $T^{\leq h}(mU)/S$ is flat. According to Lemmas 7 and 8, for all $r \geq h$, there exists an injective homomorphism $T^{\leq h}(mU) \hookrightarrow (L \oplus U)^{\otimes r}$ making $T^{\leq h}(mU)$ a direct summand of $(L \oplus U)^{\otimes r}$.

On combining the last two statements, we find that $G = (\mathrm{GL}_U)_S$, where S is a finite flat R -submodule of $(L \oplus U)^{\otimes r}$ such that $(L \oplus U)^{\otimes r}/S$ is flat. ■

THEOREM 4. *Let G be an algebraic group flat over a Dedekind domain R . If there exist $\gamma_1, \dots, \gamma_6 \in R$ such that the elements $0, 1, \gamma_1, \dots, \gamma_6$ are distinct modulo every maximal ideal of R , then there exists a finite flat algebra D over R such that $G = \underline{\mathrm{Aut}}(D)$.*

PROOF. According to Proposition 3, there exists a finite flat R -module U and an embedding $G \rightarrow \mathrm{GL}_U$. Now we can apply Proposition 9 and Proposition 2. ■

Theorem 4 leaves open the question: given an algebraic group G over R , what can be said about the algebras A over R such that $G = \underline{\mathrm{Aut}}(A)$. When R is a field, Gordeev and Popov (2003) prove a number of results about this, for example, that A can be chosen to be simple.

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