Addendum to: Values of zeta functions of varieties over finite fields, Amer. J. Math. 108, (1986), 297-360.

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The original article expressed the special values of the zeta function of a variety over a finite field in terms of the $\hat{\mathbb{Z}}$-cohomology of the variety. As the article was being completed,[1] Lichtenbaum conjectured the existence of complexes of sheaves $\mathbb{Z}(r)$ extending the sequence $\mathbb{Z}$, $\mathbb{G}_m[-1], \ldots$. The complexes given by Bloch's higher Chow groups are known to satisfy most of the axioms for $\mathbb{Z}(r)$. Using Lichtenbaum's Weil-étale topology, we can now give a beautiful restatement of the main theorem of the original article in terms of $\mathbb{Z}$-cohomology groups.

Notations

We use the notations of Milne 1986. For example,

$$M^{(n)} = M/nM, \quad TM = \varprojlim_n \mathrm{Ker}(n\colon M \to M), \quad z(f) = \frac{[\mathrm{Ker}(f)]}{[\mathrm{Coker}(f)]},$$

and $\nu_s(r)$ denotes the sheaf of logarithmic de Rham-Witt differentials on $X_{\text{ét}}$ (ibid., p. 307). The symbol l denotes a prime number, possibly p.

Review of abelian groups

In this subsection, we review some elementary results on abelian groups. An abelian group N is said to be ***bounded*** if $nN = 0$ for some $n \geq 1$, and a subgroup M of N is ***pure*** if $M \cap mN = mM$ for all $n \geq 1$.

LEMMA 1. *(a) Every bounded abelian group is a direct sum of cyclic groups.*

(b) Every bounded pure subgroup M of an abelian group N is a direct summand of N.

PROOF. (a) Fuchs 1970, 17.2.

(b) Kaplansky 1954, Theorem 7, p. 18, or Fuchs 1970, 27.5. □

LEMMA 2. *Let M be a subgroup of N, and let l^n be a prime power. If $M \cap l^n N = 0$ and M is maximal among the subgroups with this property, then M is a direct summand of N.*

PROOF. The subgroup M is bounded because $l^n M \subset M \cap l^n N = 0$. To prove that it is pure, one shows by induction on $r \geq 0$ that $M \cap l^r N \subset l^r M$. See Fuchs 1970, 27.7. □

[1] It was submitted in September 1983. This addendum was originally posted on the author's website in 2009.

NOTES. (Fuchs 1970, 9.8.) Let B and C be subgroups of an abelian group A. Assume that $C \cap B = 0$ and that C is maximal among the subgroups of A with this property. Let $a \in A$. If $pa \in C$ (p prime), then $a \in B + C$.

Proof: We may suppose that $a \notin C$. Then $\langle C, a \rangle$ contains a nonzero element b of B, say, $b = c + ma$ with $c \in C$ and $m \in \mathbb{Z}$. Here $(m, p) = 1$ because otherwise $b = c + m_0(pa) \in B \cap C = 0$. Thus $rm + sp = 1$ for some $r, s \in \mathbb{Z}$, and

$$a = r(ma) + s(pa) = rb - rc + s(pa) \in B + C.$$

(Fuchs 1970, 27.7). We prove that $M \cap l^r N \subset l^r M$ for all $r \geq 0$. This being trivially true for $r = 0$, we may apply induction on r. Let $m = l^{r+1}a \neq 0$, $m \in M$, $a \in N$. Then $r \leq n-1$, because otherwise $l^{r+1}a \in M \cap l^n N = 0$. By (9.8), $l^r a \in l^n N + M$, say, $l^r a = l^n c + d$ with $c \in N$, $d \in M$. Then $d = l^r a - l^n c \in M \cap l^r N$, which equals $l^r M$ by the induction hypothesis. From $m = l^{r+1}a = l^{n+1}c + ld$, we find that $(m - ld) \in M \cap l^{n+1}N = 0$, and so $m = ld \in l^{r+1}M$.

Every abelian group M contains a largest divisible subgroup M_{div}, which is obviously contained the first Ulm subgroup of M, $U(M) \stackrel{\mathrm{def}}{=} \bigcap_{n \geq 1} nM$. Note that $U(M/U(M)) = 0$.

NOTES. A sum of divisible subgroups is obviously divisible. For the last statement, let $x \in M$ map to the first Ulm subgroup of $M/U(M)$. Then, for each $n \geq 1$, there exists a $y \in M$ such that $ny - x \in U(M)$, and so $ny - x = ny'$ for some $y' \in M$. Now $x = n(y - y')$, and so x is divisible by n in M, i.e., $x \in U(M)$.

PROPOSITION 3. *If M/nM is finite for all $n \geq 1$, then $U(M) = M_{\mathrm{div}}$.*

PROOF. (Cf. Milne 1988, 3.3.) If $U(M)$ is not divisible, then there exists a prime l such that $U(M) \neq lU(M)$. Fix such an l, and let $x \in U(M) \smallsetminus lU(M)$. For each $n \geq 1$, there exists an element x_n of M such that $l^n x_n = x$. In fact x_n has order exactly l^n in $M/U(M)$, and so $M/U(M)$ contains elements of arbitrary high l-power order.

Let S be a finite l-subgroup of $M/U(M)$. As $U(M/U(M)) = 0$ and S is finite, there exists an n such that $S \cap l^n(M/U(M)) = 0$. By Zorn's lemma, there exists a subgroup N of $M/U(M)$ that is maximal among those satisfying (a) $N \supset S$ and (b) $N \cap l^n(M/U(M)) = 0$. Moreover, N is maximal with respect to (b) alone. Therefore N is a direct summand of $M/U(M)$ (Lemma 2). As N is bounded (in fact, $l^n N = 0$), it is a direct sum of cyclic groups (Lemma 1). We conclude that S is contained in a finite l-subgroup S' of $M/U(M)$ that is a direct summand of $M/U(M)$. Note that

$$S'^{(l)} \hookrightarrow (M/U(M))^{(l)} \simeq M^{(l)},$$

and so $\dim_{\mathbb{F}_l} M^{(l)} \geq \dim_{\mathbb{F}_l} S'^{(l)}$. But is clear (from the first paragraph) that $\dim_{\mathbb{F}_l} S'^{(l)}$ is unbounded, and so this contradicts the hypothesis on M. $\square$

NOTES. Cf. Fuchs, Vol II, 65.1.

COROLLARY 4. *If $TM = 0$ and all quotients M/nM are finite, then $U(M)$ is uniquely divisible (= divisible and torsion-free = a $\mathbb{Q}$-vector space).*

PROOF. The first condition implies that M_{div} is torsion-free, and the second that $U(M) = M_{\mathrm{div}}$. $\square$

For an abelian group M, we let M_l denote the completion of M with respect to the l-adic topology. Every continuous homomorphism from M into a complete separated group factors uniquely through M_l. In particular, the quotient maps $M \to M/l^n M$ extend to homomorphisms $M_l \to M/l^n M$, and these induce an isomorphism $M_l \to \varprojlim_n M/l^n M$. The kernel of $M \to M_l$ is $\bigcap_n l^n M$. See Fuchs 1970, §13.

LEMMA 5. *Let N be a torsion-free abelian group. If N/lN is finite, then the l-adic completion of N is a free finitely generated $\mathbb{Z}_l$-module.*

PROOF. Let $y_1,\ldots,y_r$ be elements of N that form a basis for N/lN. Then

$$N = \sum \mathbb{Z}y_i + lN = \sum \mathbb{Z}y_i + l(\sum \mathbb{Z}y_i + lN) = \cdots = \sum \mathbb{Z}y_i + l^n N,$$

and so $y_1,\ldots,y_r$ generate N/l^nN. As N/l^nN has order l^{nr}, it is in fact a free $\mathbb{Z}/l^n\mathbb{Z}$-module with basis $\{y_1,\ldots,y_r\}$. Let $a \in N_l$, and let a_n be the image of a in $N/l^{n+1}N$. Then

$$a_n = c_{n,1}y_1 + \cdots + c_{n,r}y_r$$

for some $c_{n,i} \in \mathbb{Z}/l^{n+1}\mathbb{Z}$. As a_n maps to a_{n-1} in N/l^nN and the $c_{n,i}$ are unique, $c_{n,i}$ maps to $c_{n-1,i}$ in $\mathbb{Z}/l^n\mathbb{Z}$. Hence $(c_{n,i})_{n\in\mathbb{N}} \in \mathbb{Z}_l$, and it follows that $\{y_1,\ldots,y_r\}$ is a basis for N_l as a $\mathbb{Z}_l$-module. □

PROPOSITION 6. *Let $\phi: M \times N \to \mathbb{Z}$ be a bi-additive pairing of abelian groups whose extension $\phi_l: M_l \times N_l \to \mathbb{Z}_l$ to the l-adic completions has trivial left kernel. If N/lN is finite and $\bigcap_n l^n M = 0$, then M is free and finitely generated.*

PROOF. We may suppose that N is torsion-free. As $\bigcap_n l^n M = 0$, the map $M \to M_l$ is injective. Choose elements $y_1,\ldots,y_r$ of N that form a basis for N/lN. According to the proof of Lemma 5, they form a basis for N_l as a $\mathbb{Z}_l$-module. Consider the map

$$x \mapsto (\phi(x,y_1),\ldots,\phi(x,y_r)): M \to \mathbb{Z}^r.$$

If x is in the kernel of this map, then $\phi_l(x,y) = 0$ for all $y \in N_l$, and so $x = 0$. Therefore the map M injects into $\mathbb{Z}^r$, which completes the proof. □

Review of Bloch's complex

Let X be a smooth variety over a field k. We take $\mathbb{Z}(r)$ to be the complex of sheaves on X defined by Bloch's higher Chow groups. For the definition of Bloch's complex, and a review of its basic properties, we refer the reader to the survey article Geisser 2005.

The properties of $\mathbb{Z}(r)$ that we shall need are the following.

(A)$_{n_0}$ For all integers n_0 prime to the characteristic of k, the cycle class map

$$\left(\mathbb{Z}(r) \xrightarrow{n_0} \mathbb{Z}(r)\right) \to \mu_{n_0}^{\otimes r}[0]$$

is a quasi-isomorphism (Geisser and Levine 2001, 1.5).

(A)$_p$ For all integers $s \geq 1$, the cycle class map

$$\left(\mathbb{Z}(r) \xrightarrow{p^s} \mathbb{Z}(r)\right) \to \nu_s(r)[-r-1]$$

is a quasi-isomorphism (Geisser and Levine 2000, Theorem 8.5).

(B) There exists a cycle class map $\mathrm{CH}^r(X) \to H^{2r}(X_{\text{ét}}, \mathbb{Z}(r))$ compatible (via (A)) with the cycle class map into $H^{2r}(X_{\text{ét}}, \widehat{\mathbb{Z}}(r))$. Here $\mathrm{CH}^r(X)$ is the Chow group.

(C) There exist pairings

$$\mathbb{Z}(r) \otimes^L \mathbb{Z}(s) \to \mathbb{Z}(r+s)$$

compatible (via (A)$_n$) with the natural pairings

$$\mu_n^{\otimes r} \times \mu_n^{\otimes s} \to \mu_n^{\otimes r+s}, \quad \gcd(n,p) = 1.$$

When k is algebraically closed, there exists a trace map $H^{2d}(X_{\text{ét}}, \mathbb{Z}(d)) \to \mathbb{Z}$ compatible (via (A)$_n$) with the usual trace map in étale cohomology.

Values of zeta functions

Throughout this section, X is a smooth projective variety over a finite field k with q elements, r is an integer, and ρ_r is the rank of the group of numerical equivalence classes of algebraic cycles of codimension r on X.

We list the following conjectures for reference.

$T^r(X)$ (Tate conjecture): The order of the pole of the zeta function $Z(X,t)$ at $t=q^{-r}$ is equal to ρ_r.

$T^r(X,l)$ (l-Tate conjecture): The map $\mathrm{CH}^r(X)\otimes\mathbb{Q}_l \to H^{2r}(\bar{X}_{\text{ét}},\mathbb{Q}_l(r))^{\Gamma}$ is surjective.

$S^r(X,l)$ (semisimplicity at 1): The map $H^{2r}(\bar{X}_{\text{ét}},\mathbb{Q}_l(r))^{\Gamma} \to H^{2r}(\bar{X}_{\text{ét}},\mathbb{Q}_l(r))_{\Gamma}$ induced by the identity map is bijective.

The statement $T^r(X)$ is implied by the conjunction of $T^r(X,l)$, $T^{d-r}(X,l)$, and $S^r(X,l)$ for a single l, and implies $T^r(X,l)$, $T^{d-r}(X,l)$, $S^r(X,l)$, $S^{d-r}(X,l)$ for all l (see Tate 1994, 2.9; Milne 2007, 1.11).

Let V be a variety over a finite field k. To give a sheaf on $V_{\text{ét}}$ is the same as giving a sheaf on $\bar{V}_{\text{ét}}$ together with a continuous action of $\Gamma \overset{\text{def}}{=} \mathrm{Gal}(\bar{k}/k)$. Let Γ_0 be the subgroup of Γ generated by the Frobenius element (so $\Gamma_0 \simeq \mathbb{Z}$). The Weil-étale topology is defined so that to give a sheaf on $V_{\text{Wét}}$ is the same as giving a sheaf on $\bar{V}_{\text{ét}}$ together with an action of Γ_0 (Lichtenbaum 2005). For example, for $V=\operatorname{Spec} k$, the sheaves on $V_{\text{ét}}$ are the discrete Γ-modules, and the sheaves on $V_{\text{Wét}}$ are the Γ_0-modules. In the Weil-étale topology, the Hochschild-Serre spectral sequence becomes

$$H^i(\Gamma_0, H^j(\bar{V}_{\text{ét}},F)) \implies H^{i+j}(V_{\text{Wét}},F). \tag{1}$$

Since

$$H^i(\Gamma_0,M) = M^{\Gamma_0},\ M_{\Gamma_0}, 0, 0, \ldots \text{ for } i=0,1,2,3,\ldots, \tag{2}$$

this gives exact sequences

$$0 \to H^{i-1}(\bar{V}_{\text{ét}},F)_{\Gamma_0} \to H^i(V_{\text{Wét}},F) \to H^i(\bar{V}_{\text{ét}},F)^{\Gamma_0} \to 0, \quad \text{all } i \geq 0.$$

If F is a sheaf on $V_{\text{ét}}$ such that the groups $H^j(\bar{V}_{\text{ét}},F)$ are torsion, then the Hochschild-Serre spectral sequence for the étale topology gives exact sequences

$$0 \to H^{i-1}(\bar{V}_{\text{ét}},F)_{\Gamma} \to H^i(V_{\text{ét}},F) \to H^i(\bar{V}_{\text{ét}},F)^{\Gamma} \to 0, \quad \text{all } i \geq 0.$$

The two spectral sequences are compatible, and so, for such a sheaf F, the canonical maps $H^i(V_{\text{ét}},F) \to H^i(V_{\text{Wét}},F)$ are isomorphisms.

Let X be a smooth projective variety over a finite field, and let

$$e^{2r}\colon H^{2r}(X_{\text{Wét}},\mathbb{Z}(r)) \to H^{2r+1}(X_{\text{Wét}},\mathbb{Z}(r))$$

denote cup-product with the canonical element of $H^1(\Gamma_0,\mathbb{Z}) = H^1(k_{\text{Wet}},\mathbb{Z})$, and let

$$\chi(X_{\text{Wét}},\mathbb{Z}(r)) = \prod\nolimits_{i\neq 2r,2r+1}[H^i(X_{\text{Wét}},\mathbb{Z}(r))]^{(-1)^i} z(e^{2r})$$

when all terms are defined and finite. Let

$$\chi(X,\mathcal{O}_X,r) = \sum\nolimits_{i\leq r,\,j}(-1)^{i+j}(r-i)\dim H^j(X,\Omega^i_{X/k}).$$

We define $\chi'(X_{\text{Wét}},\mathbb{Z}(r))$ as for $\chi(X_{\text{Wét}},\mathbb{Z}(r))$, but with each group $H^i(X_{\text{Wét}},\mathbb{Z}(r))$ replaced by its quotient

$$H^i(X_{\text{Wét}},\mathbb{Z}(r))' \overset{\text{def}}{=} \frac{H^i(X_{\text{Wét}},\mathbb{Z}(r))}{U(H^i(X_{\text{Wét}},\mathbb{Z}(r)))}.$$

THEOREM 7. *Let X be a smooth projective variety over a finite field such that the Tate conjecture $T^r(X)$ is true for some integer $r \geq 0$. Then $\chi'(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))$ is defined, and*

$$\lim_{t \to q^{-r}} Z(X,t) \cdot (1-q^r t)^{\rho_r} = \pm \chi'(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r)) \cdot q^{\chi(X, \mathcal{O}_X, r)}. \tag{3}$$

In particular, the groups $H^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))'$ are finite for $i \neq 2r, 2r+1$. For $i = 2r, 2r+1$, they are finitely generated. For all i, $U(H^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r)))$ is uniquely divisible.

PROOF. We begin with a brief review of Milne 1986. For an integer $n = n_0 p^s$ with $\gcd(p, n_0) = 1$,

$$H^i(X_{\mathrm{\acute{e}t}}, (\mathbb{Z}/n\mathbb{Z})(r)) \overset{\text{def}}{=} H^i(X_{\mathrm{\acute{e}t}}, \mu_{n_0}^{\otimes r}) \times H^{i-r}(X_{\mathrm{\acute{e}t}}, \nu_s(r)), \text{ and}$$
$$H^i(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r)) \overset{\text{def}}{=} \varprojlim_n H^i(X_{\mathrm{\acute{e}t}}, (\mathbb{Z}/n\mathbb{Z})(r))$$

(ibid. p. 309). Let

$$\epsilon^{2r} \colon H^{2r}(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r)) \to H^{2r+1}(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r))$$

denote cup-product with the canonical element of $H^1(\Gamma, \widehat{\mathbb{Z}}) \simeq H^1(k_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}})$, and let

$$\chi(X, \widehat{\mathbb{Z}}(r)) \overset{\text{def}}{=} \prod\nolimits_{i \neq 2r, 2r+1} [H^i(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r))]^{(-1)^i} z(\epsilon^{2r})$$

when all terms are defined and finite (ibid. p.298). Theorem 0.1 (ibid. p.298) states that $\chi(X, \widehat{\mathbb{Z}}(r))$ is defined if and only if $S^r(X,l)$ holds for all l, in which case

$$\lim_{t \to q^{-r}} Z(X,t) \cdot (1-q^r t)^{\rho_r} = \pm \chi(X, \widehat{\mathbb{Z}}(r)) \cdot q^{\chi(X, \mathcal{O}_X, r)}. \tag{4}$$

In particular, if $S^r(X,l)$ holds for all l, then the groups $H^i(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r))$ are finite for all $i \neq 2r,\, 2r+1$.

For each $n \geq 1$ and $i \geq 0$, property (A) of $\mathbb{Z}(r)$ gives us an exact sequence

$$0 \to H^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))^{(n)} \to H^i(X_{\mathrm{\acute{e}t}}, (\mathbb{Z}/n\mathbb{Z})(r)) \to H^{i+1}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))_n \to 0.$$

The middle term is finite, and so $H^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))^{(n)}$ is finite for all i and n. On passing to the inverse limit, we obtain an exact sequence

$$0 \to H^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))^\wedge \to H^i(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r)) \to TH^{i+1}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r)) \to 0 \tag{5}$$

in which the middle term is finite for $i \neq 2r, 2r+1$. As $TH^{i+1}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))$ is torsion-free, it must be zero for $i \neq 2r, 2r+1$. In other words, $TH^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r)) = 0$ for $i \neq 2r+1, 2r+2$.

So far we have used only conjecture $S^r(X,l)$ (all l) and property (A) of $\mathbb{Z}(r)$. To continue, we need to use $T^r(X,l)$ (all l) and the property (B) of $\mathbb{Z}(r)$. The l-Tate conjecture $T^r(X,l)$ for all l implies that the cokernel of the map $\mathrm{CH}^r(X) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \to H^{2r}(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r))$ is torsion. As this map factors through $H^{2r}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))^\wedge$, it follows that $TH^{2r+1}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r)) = 0$ and $H^{2r}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))^\wedge \simeq H^{2r}(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r))$. Consider the commutative diagram

$$\begin{array}{ccc}
H^{2r}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))^\wedge & \xrightarrow{\simeq} & H^{2r}(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r)) \\
\downarrow{\scriptstyle \widehat{e^{2r}}} & & \downarrow{\scriptstyle \epsilon^{2r}} \\
H^{2r+1}(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))^\wedge & \longrightarrow & H^{2r+1}(X_{\mathrm{\acute{e}t}}, \widehat{\mathbb{Z}}(r)).
\end{array}$$

As ϵ^{2r} has finite cokernel, so does the bottom arrow, and so $TH^{2r+2}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))=0$. We have now shown that

$$TH^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))=0 \text{ for all } i$$

and so ((5) and Corollary 4)

$$\left\{\begin{array}{l} H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))\hat{}\simeq H^i(X_{\mathrm{\acute{e}t}},\widehat{\mathbb{Z}}(r)) \\ U(H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r)) \text{ is uniquely divisible}\end{array}\right. \quad \text{for all } i.$$

In particular, we have proved the first statement of the theorem except that each group $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ has been replaced by its completion. It remains to prove that $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ is finite for $i\neq 2r,2r+1$ and is finitely generated for $i=2r,2r+1$ (for then $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))\hat{}\simeq H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'\otimes\widehat{\mathbb{Z}}$).

The kernel of $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'\to\left(H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'\right)\hat{}$ is $U(H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))')=0$, and so $H^i(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ is finite for $i\neq 2r,2r+1$.

It remains to show that the groups $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ and $H^{2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ are finitely generated. For this we shall need property (C) of $\mathbb{Z}(r)$. For a fixed prime $l\neq p$, the pairings in (C) give rise to a commutative diagram

$$\begin{array}{ccccc} H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'/\mathrm{tors} & \times & H^{2d-2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(d-r))'/\mathrm{tors} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ H^{2r}(X_{\mathrm{\acute{e}t}},\mathbb{Z}_l(r))/\mathrm{tors} & \times & H^{2d-2r+1}(X_{\mathrm{\acute{e}t}},\mathbb{Z}_l(d-r))/\mathrm{tors} & \longrightarrow & \mathbb{Z}_l \end{array}$$

to which we wish to apply Proposition 6. The bottom pairing is nondegenerate, the group $U(H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))')$ is zero, and the group $H^{2d-2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(d-r))^{(l)}$ is finite, and so the proposition shows that $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'/\mathrm{tors}$ is finitely generated. Because $U(H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))')=0$, the torsion subgroup of $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ injects into the torsion subgroup of $H^{2r}(X_{\mathrm{\acute{e}t}},\widehat{\mathbb{Z}}(r))$, which is finite (Gabber 1983). Hence $H^{2r}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ is finitely generated. The group $H^{2r+1}(X_{\mathrm{W\acute{e}t}},\mathbb{Z}(r))'$ can be treated similarly. □

REMARK 8. In the proof, we didn't use the full force of $T^r(X)$.

We shall need the following standard result.

LEMMA 9. *Let A be a (noncommutative) ring and let $\bar{A}$ be the quotient of A by a nil ideal I (i.e., a two-sided ideal in which every element is nilpotent). Then:*

(a) *an element of A is invertible if it maps to an invertible element of $\bar{A}$;*
(b) *every idempotent in $\bar{A}$ lifts to an idempotent in A, and any two liftings are conjugate by an element of A lying over $1_{\bar{A}}$;*
(c) *let $a\in A$; every decomposition of $\bar{a}$ into a sum of orthogonal idempotents in $\bar{A}$ lifts to a similar decomposition of a in A.*

NOTES. We denote $a+I$ by $\bar{a}$.

(a) It suffices to consider an element a such that $\bar{a}=1_{\bar{A}}$. Then $(1-a)^N=0$ for some $N>0$, and so

$$\overbrace{(1-(1-a))}^{a}\left(1+(1-a)+(1-a)^2+\cdots+(1-a)^{N-1}\right)=1.$$

(b) Let a be an element of A such that $\bar{a}$ is idempotent. Then $(a-a^2)^N=0$ for some $N>0$, and we let $a'=(1-(1-a)^N)^N$. A direct calculation shows that $a'a'=a'$ and that $\bar{a}'=\bar{a}$.

Let e and e' be idempotents in A such that $\bar{e} = \bar{e}'$. Then $a \overset{\text{def}}{=} e'e + (1-e')(1-e)$ lies above $1_{\bar{A}}$ and satisfies $e'a = e'e = ae$.

(c) Follows easily from (b).

PROPOSITION 10. *Let X be a smooth projective variety over a finite field k, and let r be an integer. Assume that for some prime l the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is nil. Then $H^i(X_{\text{et}}, \mathbb{Z}(r))$ is torsion for all $i \neq 2r$, and the Tate conjecture $T^r(X)$ implies that $H^{2r}(X_{\text{et}}, \mathbb{Z}(r))$ is finitely generated modulo torsion.*

PROOF. This is essentially proved in Jannsen 2007, pp. 131–132, and so we only sketch the argument. Set $d = \dim X$ and let $k = \mathbb{F}_q$.

According to Lemma 9, there exist orthogonal idempotents $\pi_0, \ldots, \pi_{2d}$ in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ lifting the Künneth components of the diagonal in the l-adic topology. Let $h^i X = (hX, \pi_i)$ in the category of Chow motives over k. Let $P_i(T)$ denote the characteristic polynomial $\det(T - \varpi_X | H^i(\bar{X}_{\text{et}}, \mathbb{Q}_l)$ of the Frobenius endomorphism ϖ_X of X acting on $H^i(\bar{X}_{\text{et}}, \mathbb{Q}_l)$. Then $P_i(\varpi_X)$ acts as zero on the homological motive $h^i_{\text{hom}} X$, and so $P_i(\varpi_X)^N$ acts as zero on $h^i X$ for some $N \geq 1$ (from the nil hypothesis). We shall need one last property of Bloch's complex, namely, that $H^i(X_{\text{Wét}}, \mathbb{Z}(r))_{\mathbb{Q}} \simeq K_{2r-i}(X)^{(r)}$ where $K_{2r-i}(X)^{(r)}$ is the subspace of $K_{2r-i}(X)_{\mathbb{Q}}$ on which the nth Adams operator acts as n^r for all r.

The qth Adams operator acts as the Frobenius operator, and so ϖ_X acts as multiplication by q^r on $K_{2r-i}(X)^{(r)}$. Therefore $H^i(X_{\text{Wét}}, \mathbb{Z}(r))_{\mathbb{Q}}$ is killed by $P_i(q^r)^N$, which is nonzero for $i \neq 2r$ (by the Weil conjectures), and so $H^i(X_{\text{Wét}}, \mathbb{Z}(r))$ is torsion for $i \neq 2r$.

The Tate conjecture implies that $P_{2r}(T) = Q(T) \cdot (T - q^r)^{\rho_r}$ where $Q(q^r) \neq 0$, and so

$$1 = q(T)Q(T)^N + p(T)(T - q^r)^{N\rho_r}, \quad \text{some } q(T),\, p(T) \in \mathbb{Q}[T].$$

As before, $P_{2r}(\omega_X)^N$ acts as zero on $h^{2r} X$ for some $N \geq 1$. Therefore $q(\varpi_X)Q(\varpi_X)^N$ and $p(\varpi_X)(\varpi_X - q^r)^{N\rho_r}$ are orthogonal idempotents in $\mathrm{End}(h^{2r} X)$ with sum 1, and correspondingly $h^{2r} X = M_1 \oplus M_2$. Now $H^{2r}(M_1, \mathbb{Z}(r))_{\mathbb{Q}} = 0$ because $Q(\varpi_X)^N$ is zero on M_1 and $Q(q^r) \neq 0$. On the other hand, M_2 is isogenous to $(\mathbb{L}^{\otimes r})^{\rho_r}$ where $\mathbb{L}$ is the Lefschetz motive (Jannsen 2007, p. 132), and so $H^{2r}(M_2, \mathbb{Z}(r))$ differs from

$$H^{2r}(\mathbb{L}^{\otimes r}, \mathbb{Z}(r))^{\rho_r} \simeq H^{2r}(\mathbb{P}^d, \mathbb{Z}(r))^{\rho_r} \simeq \mathbb{Z}^{\rho_r}$$

by a torsion group. □

NOTES. When $k = \mathbb{F}_q$, the qth Adams operator acts as ϖ (Hiller 1981, §5; Soulé 1985, 8.1), and so $K_i(X)^{(j)}$ is the subspace on which ϖ acts as q^j (because the m^i-eigenspace of the mth Adams operators is independent of m, Seiler 1988, Theorem 1).

THEOREM 11. *Let X be a smooth projective variety over a finite field such that the Tate conjecture $T^r(X)$ is true for some integer $r \geq 0$. Assume that, for some prime l, the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is nil. Then $\chi(X_{\text{Wét}}, \mathbb{Z}(r))$ is defined, and*

$$\lim_{t \to q^{-r}} Z(X, t) \cdot (1 - q^r t)^{\rho_r} = \pm\chi(X_{\text{Wét}}, \mathbb{Z}(r)) \cdot q^{\chi(X, \mathcal{O}_X, r)}. \tag{6}$$

In particular, the groups $H^i(X_{\text{Wét}}, \mathbb{Z}(r))$ are finite for $i \neq 2r, 2r+1$. For $i = 2r, 2r+1$, they are finitely generated.

PROOF. This will follow from Theorem 7 once we show that the groups $U^i \overset{\text{def}}{=} U(H^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r)))$ are zero. Because $H^i(X_{\mathrm{W\acute{e}t}}, \mathbb{Z}(r))$ is finitely generated modulo torsion (Proposition 10), it does not contain a nonzero $\mathbb{Q}$-vector space, and so $U^i = 0$ (Corollary 4). □

REMARK 12. For a smooth projective algebraic variety X whose Chow motive is finite-dimensional, the ideal of l-homologically trivial correspondences in $\mathrm{CH}^{\dim X}(X \times X)_{\mathbb{Q}}$ is nil for all prime l (Kimura). It is conjectured (Kimura and O'Sullivan) that the Chow motives of algebraic varieties are always finite-dimensional, and this is known for those in the category generated by the motives of abelian varieties. On the other hand, Beilinson has conjectured that, over finite fields, rational equivalence with $\mathbb{Q}$-coefficients coincides with with numerical equivalence, which implies that the ideal in question is always null (not merely nil).

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