Nonhomeomorphic conjugates of connected Shimura varieties

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Abstract

We show that conjugation by an automorphism of $\mathbb{C}$ may change the topological fundamental group of a locally symmetric variety over $\mathbb{C}$. As a consequence, we obtain a large class of algebraic varieties defined over number fields with the property that different embeddings of the number field into $\mathbb{C}$ give complex varieties with nonisomorphic fundamental groups.

Let V be an algebraic variety over $\mathbb{C}$, and let τ be an automorphism of $\mathbb{C}$ (as an abstract field). On applying τ to the coefficients of the polynomials defining V, we obtain a conjugate algebraic variety τV over $\mathbb{C}$. The cohomology groups $H^i(V^{\mathrm{an}}, \mathbb{Q})$ and $H^i((\tau V)^{\mathrm{an}}, \mathbb{Q})$ have the same dimension, and hence are isomorphic, because, when tensored with $\mathbb{Q}_\ell$, they become isomorphic to the étale cohomology groups which are unchanged by conjugation. Similarly, the profinite completions of the fundamental groups of V^{an} and $(\tau V)^{\mathrm{an}}$ are isomorphic because they are isomorphic to the étale fundamental groups. However, Serre [Se] gave an example in which the fundamental groups themselves are *not* isomorphic (see [EV] for a discussion of this and other examples). It seems to have been known (or, at least, expected) for some time that the theory of Shimura varieties provides many more examples, but, as far we know, the details have not been written down anywhere. The purpose of this note is to provide these details.

In the first section, we explain how the semisimple algebraic group attached to a connected Shimura variety changes when one conjugates the variety, and in the second section, we apply Margulis's super-rigidity theorem to deduce that the isomorphism class of the fundamental group changes for large classes of varieties.

We note that all the varieties we consider have canonical models defined over number fields. Thus, they provide examples of algebraic varieties defined over number fields with the property that different embeddings of the number field into $\mathbb{C}$ give complex varieties with nonisomorphic fundamental groups.

We assume that the reader is familiar with the basic theory of Shimura varieties, for example, with the first nine sections of [ISV], and with the standard cohomology theories for algebraic varieties, as explained, for example, in the first section of [De3]. The identity component of a Lie group H is denoted H^+. We use $\approx$ to denote an isomorphism, and $\simeq$ to denote a canonical isomorphism. We let $U_1 = \{z \in \mathbb{C} \mid |z| = 1\}$ regarded as either a real algebraic group or a real Lie group, and we let $\mathbb{P}$ denote the set of prime numbers $\{2, 3, 5, 7, \ldots\}$.

1 Conjugation of connected Shimura varieties

1.1. Throughout this section, F is a totally real field of finite degree over $\mathbb{Q}$ which should *not* be thought of as a subfield of $\mathbb{C}$. For an algebraic group H over F, we let H_* denote the algebraic group over $\mathbb{Q}$ obtained by restriction of scalars. Thus, for any $\mathbb{Q}$-algebra R, $H_*(R) = H(F \otimes_{\mathbb{Q}} R)$ and $H_*(\mathbb{R}) = \prod_{v\colon F\to\mathbb{R}} H_v(\mathbb{R})$ where H_v is the algebraic group over $\mathbb{R}$ obtained by extension of scalars $v\colon F \to \mathbb{R}$. Then $H(F)$ is a subgroup of $H_*(\mathbb{R})$, and we let $H(F)^+ = H(F) \cap H_*(\mathbb{R})^+$.

1.2. By a hermitian symmetric domain, we mean any complex manifold isomorphic to a bounded symmetric domain. Let X be a hermitian symmetric domain, and let $\mathrm{Hol}(X)^+$ be the identity component of the group of holomorphic automorphisms of X. Let H be a semisimple algebraic group over F, and let $H_*(\mathbb{R})^+ \to \mathrm{Hol}(X)^+$ be a surjective homomorphism with compact kernel. For any torsion-free arithmetic subgroup Γ of $H^{\mathrm{ad}}(F)^+$, the quotient manifold $V = \Gamma\backslash X$ has a unique structure of an algebraic variety (Baily-Borel, Borel). When the inverse image of Γ in $H(F)$ is a congruence subgroup, we say that the algebraic variety V is of type (H, X). Note that if V is of type (H, X), then it is also of type (H_1, X) for any isogeny $H_1 \to H$, in particular, for H_1 the simply connected covering group of H.

Theorem 1.3. *Let V be a smooth algebraic variety over $\mathbb{C}$, and let τ be an automorphism of $\mathbb{C}$. If V is of type (H, X) with H simply connected, then τV is of type (H', X') for some semisimple algebraic group H' over F such that*

$$\begin{cases} H'_v \approx H_{\tau\circ v} \text{ for all real primes } v\colon F \to \mathbb{R} \text{ of } F \text{, and} \\ H'_{F_v} \simeq H_{F_v} \text{ for all finite primes } v \text{ of } F. \end{cases} \tag{1}$$

Example 1.4. Let B be a quaternion algebra over F, and let $\mathrm{inv}_v(B) \in \{0, \frac{1}{2}\}$ be the invariant of B at a prime v of F. Recall that B is determined up to isomorphism as an F-algebra by its invariants. We assume that there exists a real prime of F at which B is indefinite, and we let X be a product of copies of the complex upper half-plane indexed by such primes. Let $H = \mathrm{SL}_1(B)$. Then $H_*(\mathbb{R})$ acts in a natural way on X, and for any torsion-free congruence subgroup Γ of $H(F)$, the quotient $V \stackrel{\text{def}}{=} \Gamma\backslash X$ is a smooth algebraic variety over $\mathbb{C}$. The theorem shows that, for any automorphism τ of $\mathbb{C}$, τV arises in a similar fashion from a quaternion algebra B' such that

$$\begin{aligned} &\mathrm{inv}_v(B') = \mathrm{inv}_{\tau\circ v}(B) \quad \text{for all infinite primes } v \text{ of } F\text{, and} \\ &\mathrm{inv}_v(B') = \mathrm{inv}_v(B) \quad \text{for all finite primes } v \text{ of } F. \end{aligned}$$

Remark 1.5. Let $M = \Gamma\backslash X$ be the quotient of a hermitian symmetric domain by a torsion-free discrete subgroup Γ of $\mathrm{Hol}(X)^+$. Then X is the universal covering space of M and Γ is the group of covering transformations of X over M, and so, for any $o \in X$ and its image $\bar{o}$ in M, there is an isomorphism $\Gamma \simeq \pi_1(M, \bar{o})$ which depends only on the choice of o mapping to $\bar{o}$.

Remark 1.6. For V as in the theorem, it is possible to describe the fundamental group of $(\tau V)^{\mathrm{an}}$ in terms of that of V^{an}. By assumption, $V = \Gamma\backslash X$ with Γ a torsion-free arithmetic subgroup of $H^{\mathrm{ad}}(F)^+ \subset \mathrm{Hol}(X)^+$ and the inverse image $\tilde{\Gamma}$ of Γ in $H(F)$ is a congruence subgroup, i.e.,

$\tilde{\Gamma} = H(F) \cap K$ for some compact open subgroup K of $H(\mathbb{A}_F^\infty)$. Similarly, $\tau V = \Gamma' \backslash X'$ and the inverse image $\tilde{\Gamma}'$ of Γ' in $H'(F)$ equals $K' \cap H'(F)$ for a compact open subgroup K' of $H'(\mathbb{A}_F^\infty)$.

It is known that the isomorphisms $H_{F_v} \simeq H'_{F_v}$ in Theorem 1.3 induce an isomorphism $H(\mathbb{A}_F^\infty) \simeq H'(\mathbb{A}_F^\infty)$, and that K maps to K' under this isomorphism. If the map $\tilde{\Gamma} \to \Gamma$ is surjective, then Γ' equals the image of $\tilde{\Gamma}' \stackrel{\text{def}}{=} H'(F) \cap K'$ in $H'^{\text{ad}}(F)$. We can always arrange this by choosing Γ to be the image of a torsion-free congruence subgroup of $H(F)$.

In general, the map[1] $H(F) \to H^{\text{ad}}(F)^+$ is not surjective, and so $\tilde{\Gamma} \to \Gamma$ need not be surjective. However, the isomorphisms $H_{F_v} \simeq H'_{F_v}$ define an isomorphism

$$\widehat{H^{\text{ad}}(F)}^+ \to \widehat{H'^{\text{ad}}(F)}^+$$

where the hat means that we are taking the completion for the topologies defined by the arithmetic subgroups whose inverse images in $H(F)$ or $H'(F)$ are congruence subgroups [MS2, 8.2]. It is known that $\widehat{\Gamma}$ maps to $\widehat{\Gamma'}$ under this isomorphism. As $\Gamma' = \widehat{\Gamma'} \cap H'^{\text{ad}}(F)^+$, this determines the fundamental group of $(\tau V)^{\text{an}}$.

The proof of Theorem 1.3 (and Remark 1.6) will occupy the rest of this section.

Preliminary remarks

1.7. Let H be a simply connected semisimple algebraic group over F, and let $H_*(\mathbb{R})^+ \to \text{Hol}(X)^+$ be a surjective homomorphism with compact kernel as in (1.2). Choose a point $o \in X$. There exists a unique homomorphism $u\colon U_1 \to H_*^{\text{ad}}(\mathbb{R})$ such that

- $u(z)$ fixes o and acts as multiplication by z on the tangent space at o;
- $u(z)$ projects to 1 in each compact factor of $H_*^{\text{ad}}(\mathbb{R})$

[ISV, 1.9]. It is possible to recover X and the homomorphism $H_*(\mathbb{R})^+ \to \text{Hol}(X)^+$ from (H, u) (ibid. 1.21). Moreover, the isomorphism class of the pair $(H_\mathbb{R}, u)$ is determined by the isomorphism class of the pair $(H_\mathbb{C}, u_\mathbb{C})$ (ibid. 1.24). The cocharacter $u_\mathbb{C}$ of $H_\mathbb{C}$ satisfies the following condition (ibid. 1.21):

> (*) in the action of $\mathbb{G}_m$ on $\text{Lie}(G_\mathbb{C})$ defined by $\text{ad} \circ u_\mathbb{C}$, only the characters $z, 1, z^{-1}$ occur.

1.8. Let k be an algebraically closed field of characteristic zero. Consider a simple adjoint group H over k and a cocharacter μ of H satisfying (*). Let T be a maximal torus in H, let R be the set of roots of H relative to T (so its elements are characters of T), and choose a base for R. Among the cocharacters of H conjugate to μ, there is exactly one μ' that factors through T and is such that $\langle \alpha, \mu' \rangle \geq 0$ for all positive $\alpha \in R$, and among the simple roots, there is exactly one α such that $\langle \alpha, \mu' \rangle \neq 0$. The isomorphism class of the pair (H, μ) is determined by the Dynkin diagram of H and the root α [ISV, 1.24 et seq.].

[1] Recall that, for a simply connected semisimple group H, $H(\mathbb{R})$ is connected.

1.9. Let (H, μ) be a pair as in (1.8), and let $\tau\colon k \to k'$ be a homomorphism of fields. Then $\mathrm{Lie}(\tau H) \simeq \mathrm{Lie}(H) \otimes_{k,\tau} k'$, and this isomorphism defines an isomorphism from the Dynkin diagram of $\mathrm{Lie}(H)$ to that of $\mathrm{Lie}(\tau H)$ which sends the special root attached to μ to that attached to $\tau\mu$. Therefore, if $k = k'$, then $(H, \mu) \approx (\tau H, \tau\mu)$.

Conjugation of abelian varieties

Let A be an abelian variety over an algebraically closed field k of characteristic zero. The étale cohomology groups $H^r_{\mathrm{et}}(A, \mathbb{Q}_\ell(s))$ for $\ell \in \mathbb{P}$ are finite-dimensional $\mathbb{Q}_\ell$-vector spaces and the de Rham cohomology groups $H^r_{\mathrm{dR}}(A)(s)$ are finite-dimensional k-vector spaces. For convenience, we denote these groups by $H^r_\ell(A)(s)$ and $H^r_\infty(A)(s)$ respectively. When $k = \mathbb{C}$, we denote the Betti cohomology groups $H^r(A^{\mathrm{an}}, \mathbb{Q}(s))$ by $H^r_B(A)(s)$.

Recall that there are canonical isomorphisms for $l \in \mathbb{P} \cup \{\infty\} \cup \{B\}$,

$$H^1_l(A^n) \simeq nH^1_l(A), \quad H^r_l(A^n) \simeq \bigwedge\nolimits^r H^1_l(A^n), \quad H^r_l(A^n)(s) \simeq H^r_l(A^n) \otimes_{\mathbb{Q}} \mathbb{Q}(s). \tag{2}$$

Recall also that, when $k = \mathbb{C}$, there are canonical comparison isomorphisms

$$\begin{cases} H^r_B(A)(s) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\simeq} H^r_\ell(A)(s), & \ell \in \mathbb{P}, \\ H^r_B(A)(s) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^r_\infty(A)(s). \end{cases} \tag{3}$$

For a homomorphism $\tau\colon k \to k'$ of algebraically closed fields, τA denotes the abelian variety over k' obtained by extension of scalars. The morphism $\tau A \to A$ induces isomorphisms of vector spaces

$$\begin{cases} x \mapsto {}^\tau x\colon H^r_\ell(A)(s) \to H^r_\ell(\tau A)(s), & \ell \in \mathbb{P}, \\ x \otimes 1 \mapsto {}^\tau x\colon H^r_\infty(A)(s) \otimes_{k,\tau} k' \to H^r_\infty(\tau A)(s). \end{cases} \tag{4}$$

A family

$$t = (t_l)_{l \in \mathbb{P} \cup \{\infty\}} \in \prod\nolimits_{l \in \mathbb{P}} H^r_l(A)(s) \times H^r_\infty(A)(s), \quad (r, s) \in \mathbb{N} \times \mathbb{Z},$$

is said to be a Hodge class on A if for some homomorphism $\sigma\colon k \to \mathbb{C}$ there exists a class

$$t_0 \in H^r_B(\sigma A)(s) \cap H^r(\sigma A, \mathbb{C}(s))^{0,0}$$

mapping to ${}^\sigma t_l$ under every comparison isomorphism (3). The main theorem of [De3] says that a Hodge class t on A then has this property for *every* homomorphism $\sigma\colon k \to \mathbb{C}$.

Now fix a homomorphism $i\colon F \to \mathrm{End}^0(A)$. Then $H^1_l(A)$ is a free $F \otimes_{\mathbb{Q}} \mathbb{Q}_l$-module (when $l = \infty$, $\mathbb{Q}_l = k$, and when $l = B$, $\mathbb{Q}_l = \mathbb{Q}$). We let

$$H^r_l(A)' = \bigwedge\nolimits^r_{F \otimes_{\mathbb{Q}} \mathbb{Q}_l} H^1_l(A), \quad l \in \mathbb{P} \cup \{\infty\} \cup \{B\}.$$

Thus $H^r_l(A)'$ is the largest subspace of $H^r_l(A)$ with a natural action of F. By a Hodge class on (A, i), we mean a Hodge class $(t_l)_{l \in \mathbb{P} \cup \{\infty\}}$ on A such that if t_l lies in the space $H^r_l(A)(s)$, then it actually lies in the subspace $H^r_l(A^n)'(s)$. We endow A^n with the diagonal action of F.

Let $t = (t_l)$ be a Hodge class on (A, i). According to Deligne's theorem [De3], for any automorphism τ of k, the family ${}^\tau t \overset{\text{def}}{=} ({}^\tau t_l)$ is a Hodge class on τA. Here ${}^\tau t_l$ is the image of t_l under the isomorphism (4).

We now take $k = \mathbb{C}$.

Study at a finite prime

Let $\ell \in \mathbb{P}$. From the isomorphism $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \prod_{v|\ell} F_v$ and the diagonal action of F on A^n, we get a decomposition $H^r_\ell(A^n)' \simeq \bigoplus_{v|\ell} H^r_\ell(A^n)_v$ in which $H^r_\ell(A^n)_v$ is the subspace on which F acts through $F \to F_v$. The first comparison isomorphism (3) for A^n induces an $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-linear isomorphism $H^r_B(A^n)'(s) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq H^r_\ell(A^n)'(s)$, and hence an F_v-linear isomorphism

$$H^r_B(A^n)'(s) \otimes_F F_v \xrightarrow{\simeq} H^r_\ell(A^n)_v(s). \tag{5}$$

The first isomorphism (4) for A^n induces an $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-linear isomorphism $H^r_\ell(A^n)'(s) \to H^r_\ell(\tau A^n)'(s)$, and hence an F_v-linear isomorphism

$$H^r_\ell(A^n)_v(s) \xrightarrow{\simeq} H^r_\ell(\tau A^n)_v(s). \tag{6}$$

Study at an infinite prime

We identify the infinite primes of F with homomorphisms $v\colon F \to \mathbb{R} \subset \mathbb{C}$. Note that $F \otimes_{\mathbb{Q}} \mathbb{R}$ and $F \otimes_{\mathbb{Q}} \mathbb{C}$ are products of copies of $\mathbb{R}$ and $\mathbb{C}$ respectively indexed by the real primes of F. The second comparison isomorphism (3) for A^n induces an $F \otimes_{\mathbb{Q}} \mathbb{C}$-linear isomorphism

$$H^r_B(A^n)'(s) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^r_\infty(A^n)'(s),$$

and hence a $\mathbb{C}$-linear isomorphism

$$H^r_B(A^n)'(s) \otimes_{F,v} \mathbb{C} \xrightarrow{\simeq} H^r_\infty(A^n)'_v(s) \tag{7}$$

for each real prime v of F.

The second isomorphism (4) for A^n induces an $F \otimes_{\mathbb{Q}} \mathbb{C}$-isomorphism

$$H^r_\infty(A^n)'(s) \otimes_{\mathbb{C},\tau} \mathbb{C} \to H^r_\infty(\tau A^n)'(s),$$

and hence a τ-linear isomorphism

$$H^r_\infty(A^n)'_v(s) \to H^r_\infty(\tau A^n)'_{\tau\circ v}(s) \tag{8}$$

for each real prime v of F.

We now assume that A admits complex multiplication, so that there exists a CM algebra E containing F and a homomorphism $E \to \mathrm{End}^0(A)$ making $H_1(A, \mathbb{Q})$ into a free E-module of rank 1. Let $T = (\mathbb{G}_m)_{E/F}$ (restriction of scalars, so T is a torus over F), and let $T_v = T \otimes_{F,v} \mathbb{R}$.

The action of E on $H_1(A,\mathbb{Q})$ defines an action of T on $H_1(A,\mathbb{Q})$ and on its dual $H^1_B(A)$. Recall that

$$H^1_B(A)\otimes_{\mathbb{Q}}\mathbb{C}\simeq H^{1,0}(A)\oplus H^{0,1}(A),\quad H^{1,0}\overset{\text{def}}{=}H^0(A,\Omega^1_A),\quad H^{0,1}\overset{\text{def}}{=}H^1(A,\mathcal{O}_A),\tag{9}$$

from which we deduce a decomposition

$$H^1_B(A)\otimes_{F,v}\mathbb{C}\simeq H^{1,0}_v\oplus H^{0,1}_v.$$

There is a cocharacter μ_A of T_* such that, for $z\in\mathbb{C}^\times$, $\mu_A(z)$ acts on $H^{1,0}$ as z^{-1} and on $H^{0,1}$ as 1. This decomposes into cocharacters μ_v of $T_{v\mathbb{C}}$ such, for $z\in\mathbb{C}^\times$, $\mu_v(z)$ acts on $H^{1,0}_v$ as z^{-1} and on $H^{0,1}_v$ as 1.

The decomposition (9) is algebraic:

$$H^1_\infty(A)\simeq H^0(A,\Omega^1_A)\oplus H^1(A,\mathcal{O}_A)$$

(Zariski cohomology of algebraic sheaves). The action of E on A defines an action of E on τA, and hence an action of T on its cohomology groups. There is a commutative diagram

$$\begin{tikzcd} H^1_\infty(A) \arrow[r, "\simeq"] \arrow[d, "\simeq"', "\tau\text{-linear}"] & H^0(A,\Omega^1_A)\oplus H^1(A,\mathcal{O}_A) \arrow[d, "\simeq"', "\tau\text{-linear}"] \\ H^1_\infty(\tau A) \arrow[r] & H^0(\tau A,\Omega^1_A)\oplus H^1(\tau A,\mathcal{O}_A) \end{tikzcd}$$

which is compatible with the actions of T, from which it follows that

$$\mu_{\tau A}=\tau\mu_A\tag{10}$$

($\tau\mu_A$ makes sense because both $\mathbb{G}_m$ and T_* are $\mathbb{Q}$-tori). Let μ'_v be the cocharacter of T_v arising from the action of T on the cohomology of τA. Then (10) shows that

$$\mu'_{\tau\circ v}=\tau\mu_v.\tag{11}$$

Proof of Theorem 1.3

Let V be an algebraic variety over $\mathbb{C}$ of type (H,X) in the sense of (1.2) with H simply connected. Theorem 1.1 of [Mi] shows that, for any automorphism τ of $\mathbb{C}$, the conjugate τV of V is of type (H',X') where the pair (H',X') has a precise description, which we now explain.

Let $\mathbb{Q}^{\mathrm{al}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and let M be the category of motives for absolute Hodge classes generated by the Artin motives over $\mathbb{Q}$ and the abelian varieties over $\mathbb{Q}$ that become of CM-type over $\mathbb{Q}^{\mathrm{al}}$ [DM, §6]. The functor sending a variety W over $\mathbb{Q}$ to the Betti cohomology of $W_{\mathbb{C}}$ defines a fibre functor ω_B on M. There is an exact sequence

$$1\longrightarrow S^\circ\longrightarrow S\xrightarrow{\pi}\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})\longrightarrow 1\tag{12}$$

in which S is the pro-algebraic group attached to the Tannakian category M and its fibre functor ω_B, $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ is the constant pro-algebraic group attached to ω_B and the Tannakian subcategory of Artin motives over $\mathbb{Q}$, and S° is the protorus attached to ω_B and the Tannakian category of CM motives over $\mathbb{Q}^{\mathrm{al}}$ (equivalently, over $\mathbb{C}$) (ibid. 6.28). The homomorphisms in the sequence are defined by the obvious tensor functors. The groups S° and S are respectively the connected Serre group and the Serre group.

Now let M_F be the category of motives in M with coefficients in F. Thus, an object of M_F is an object X of M together with a homomorphism $F \to \mathrm{End}(X)$ (see [De2, 2.1] for an alternative interpretation). Betti cohomology defines a fibre functor ω_B on M_F to F-vector spaces, and the exact sequence

$$1 \longrightarrow S_F^\circ \longrightarrow S_F \xrightarrow{\pi} \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})_F \longrightarrow 1$$

attached to M_F and ω_B is obtained from (12) by extension of scalars $\mathbb{Q} \to F$ [DM, 3.12].

With (H, X) as in the first paragraph, there exists a $o \in X$ that is fixed by $T(\mathbb{R})^+$ for some maximal torus $T \subset H^{\mathrm{ad}}$. Let $u\colon U_1 \to T_{*\mathbb{R}} \subset H^{\mathrm{ad}}_{*\mathbb{R}}$ be the homomorphism corresponding to (H, X) as in (1.7), and let μ be the cocharacter $u_{\mathbb{C}}$ of $T_{*\mathbb{C}}$. There is a unique homomorphism $\rho\colon S_F^\circ \to T$ sending the universal cocharacter of S_F° to μ [MS1, §1]. For $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, the inverse image $P(\tau)$ of τ in S_F is an S_F°-torsor, and the pair (H', μ') is obtained from (H, μ) by twisting by $P(\tau)$:

$$(H', \mu') = (H, \mu) \wedge^{S_F^\circ} P(\tau). \tag{13}$$

When $F = \mathbb{Q}$, this is proved in [Mi, Theorem 1.1].[2] The statement with F is, in fact, weaker than the statement with $\mathbb{Q}$, and follows from it. Note that a variety V of type (H, X) with H a semisimple algebraic group over F is also of type (H_*, X), where H_* is now a semisimple algebraic group over $\mathbb{Q}$, and that the ρ for (H, X) is related to the ρ for (H_*, X) by the canonical isomorphism

$$\mathrm{Hom}(S_F^\circ, T) \simeq \mathrm{Hom}(S^\circ, T_*).$$

Using this, it is easy to deduce the statement for F from that for $\mathbb{Q}$.

We now prove Theorem 1.3 for a variety V of type (H, X). The homomorphism $\rho\colon S_F^\circ \to T \subset H$ factors through an algebraic quotient of S_F°. This allows us to replace M_F in the above discussion with the subcategory generated by the Artin motives and a single (very large) abelian variety A equipped with an F-action $i\colon F \to \mathrm{End}^0(A)$. Then S_F° is replaced with the subgroup $\mathrm{MT}(A, i)$ of $\mathrm{GL}_F(H^1_B(A)) \times \mathrm{GL}_1(F(1))$ of elements fixing all Hodge classes on $(A_{\mathbb{Q}^{\mathrm{al}}}, i)$ and its powers. Moreover, $P(\tau)$ is the $\mathrm{MT}(A, i)$-torsor such that, for any F-algebra R, $P(\tau)(R)$ consists of the isomorphisms

$$a\colon H^1_B(A_{\mathbb{C}}) \otimes_F R \to H^1_B(\tau A_{\mathbb{C}}) \otimes_F R$$

sending t to ${}^\tau t$ for all Hodge classes t on $A_{\mathbb{Q}^{\mathrm{al}}}$ [DM, 6.29].

[2] In fact, the results of [Mi] are stated in terms of Langlands's Taniyama group rather than the Serre group. However, effectively they are proved for the Serre group, and then Deligne's theorem [De4] is used to replace the Serre group with the Taniyama group.

Let v be a finite prime of F dividing ℓ. Define a_v to be the isomorphism making the following diagram commute:

$$\begin{array}{ccc} H^1_B(A)\otimes_F F_v & \xrightarrow{a_v} & H^1_B(\tau A)\otimes_F F_v \\ \simeq\downarrow(5) & & \simeq\downarrow(5) \\ H^1_\ell(A)_v & \underset{(6)}{\xrightarrow{\simeq}} & H^1_\ell(\tau A)_v \end{array}$$

Then there are commutative diagrams

$$\begin{array}{ccc} H^r_B(A^n)'(s)\otimes_F F_v & \xrightarrow{\text{“}a_v\text{”}} & H^r_B(\tau A^n)'(s)\otimes_F F_v \\ \simeq\downarrow(5) & & \simeq\downarrow(5) \\ H^r_\ell(A^n)_v(s) & \underset{(6)}{\xrightarrow{\simeq}} & H^r_\ell(\tau A^n)_v(s) \end{array}$$

for all $(r,n,s)\in\mathbb{N}\times\mathbb{N}\times\mathbb{Z}$, where “$a_v$” is the map deduced from a_v by means of the isomorphisms (2). This shows that $a_v\in P(\tau)(F_v)$ and so it defines an isomorphism $H_{F_v}\simeq H'_{F_v}$ (see (13)).

Before considering the real primes, we make another observation. Let T be a Tannakian category, let ω be a fibre functor on T with values in the category of vector spaces over some field k. The group of tensor automorphisms $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ of ω is an affine group scheme G over k. A homomorphism $\tau\colon k\to k'$ defines a fibre functor

$$^{\tau}\omega\colon X\rightsquigarrow\omega(X)\otimes_{k,\tau}k'$$

on T, and the group G' of tensor automorphisms of $^{\tau}\omega$ is canonically isomorphic to τG [DM, 3.12]. Now suppose $k=k'$. Then the functor $P=\underline{\mathrm{Hom}}^{\otimes}(\omega,{}^{\tau}\omega)$ of tensor homomorphisms from ω to $^{\tau}\omega$ is a torsor for G, and $^{\tau}G$ is the twist of G by this torsor:

$$^{\tau}G=G\wedge^{G}P.$$

We now consider a real prime $v\colon F\to\mathbb{R}\subset\mathbb{C}$. In this case, we define a_v to be the τ-linear isomorphism making the following diagram commute:

$$\begin{array}{ccc} H^1_B(A)\otimes_{F,v}\mathbb{C} & \xrightarrow{a_v} & H^1_B(\tau A)\otimes_{F,\tau\circ v}\mathbb{C} \\ \simeq\downarrow(7) & & \simeq\downarrow(7) \\ H^1_\infty(A)_v & \underset{(8)}{\xrightarrow{\simeq}} & H^1_\infty(\tau A)_{\tau\circ v}. \end{array}$$

Again, it gives commutative diagrams

$$\begin{array}{ccc} H^r_B(A^n)'(s)\otimes_{F,v}\mathbb{C} & \xrightarrow{\text{“}a_v\text{”}} & H^r_B(\tau A^n)'(s)\otimes_{F,\tau\circ v}\mathbb{C} \\ \simeq\downarrow(7) & & \simeq\downarrow(7) \\ H^r_\infty(A^n)_v(s) & \underset{(8)}{\xrightarrow{\simeq}} & H^r_\infty(\tau A^n)_{\tau\circ v}(s) \end{array}$$

Thus a_v gives an isomorphism $(\tau H_v)_{\mathbb{C}} \xrightarrow{\simeq} (H'_{\tau\circ v})_{\mathbb{C}}$. Because of (11), this isomorphism sends $\tau\mu_v$ to $\mu'_{\tau\circ v}$. In other words,

$$((H'_{\tau\circ v})_{\mathbb{C}}, \mu'_{\tau\circ v}) \simeq ((\tau H_v)_{\mathbb{C}}, \tau\mu) \overset{(1.9)}{\simeq} ((H_v)_{\mathbb{C}}, \mu).$$

We now apply (1.8) to deduce that $H'_{\tau\circ v} \approx H_v$.

This completes the proof of Theorem 1.3, but it remains to explain Remark 1.6. The statement in (1.6) concerning K and K' follows immediately from Theorem 1.1 of [Mi]. For the stronger statement concerning $\widehat{\Gamma}$ and $\widehat{\Gamma'}$, we need to appeal to Proposition 6.1 of [Mi].

Remark 1.10. The proof we have given of Theorem 1.3 makes use of the main theorem of [Mi], and hence of the main theorem of [Ka]. For Shimura varieties of abelian type, it is possible to give a more direct proof of the theorem that avoids using these results.

2 The examples

In this section, we find examples of connected Shimura varieties S over $\mathbb{C}$ and field automorphisms τ of $\mathbb{C}$ such that S^{an} and $(\tau S)^{\mathrm{an}}$ have nonisomorphic fundamental groups. Such S will be attached to a pair (H, X) as in the first section, where H is an absolutely simple algebraic group over a totally real number field $F \neq \mathbb{Q}$. As H_* will be $\mathbb{Q}$-simple with $\dim_{\mathbb{C}} X > 1$, (i) when S is compact, it will be of general type because the canonical bundle will be ample, while (ii) when S is noncompact, it will be an open subvariety (with complement of codimension ≥ 2) of a normal projective variety (the Baily-Borel compactification) with ample canonical bundle.

We prepare necessary tools in the first two subsections, spell out the algorithm for construction in the third, and then deal with some examples.

Super-rigidity

The crucial ingredient in our construction of nonhomeomorphic conjugate Shimura varieties is the super-rigidity of Margulis:

Theorem 2.1 (Margulis)**.** *Let F be a totally real number field and let H and H' be two absolutely simple, simply connected algebraic groups over F with the (real) rank of $H_* \otimes_{\mathbb{Q}} \mathbb{R}$ at least 2. Suppose that Γ (resp. Γ') is a lattice of $H_*(\mathbb{Q}) = H(F)$ (resp. $H'_*(\mathbb{Q})$). If Γ and Γ' are isomorphic, then there exists a field automorphism σ of F such that H' is isomorphic to σH.*

Proof. This follows from [Ma, Theorem C, p. 259], together with the fact that an epimorphism between H' and a conjugate of H is necessarily an isomorphism, because both are assumed to be simply connected. □

Corollary 2.2. *Let F, H and H' be as in Theorem 2.1 and suppose there is an isomorphism*

$$\rho : H \otimes_F \mathbb{A}_F^{\infty} \cong H' \otimes_F \mathbb{A}_F^{\infty}. \tag{14}$$

Let K be an open compact subgroup of $H\left(\mathbb{A}_F^\infty\right)$ such that the intersections

$$\Gamma := K \cap H(F) \text{ and } \Gamma' := \rho(K) \cap H'(F)$$

are both torsion-free. If H' is not isomorphic to σH for any $\sigma \in \mathrm{Aut}(F)$ as an F-group, then Γ and Γ' are not isomorphic. □

Automorphism-free number fields and marking finite places

In applying Corollary 2.2, it will be convenient if the only field automorphism of F is the identity. We recall how to make such number fields.

Proposition 2.3. *For any integer $d \geq 3$, there exist totally real number fields F of degree d over $\mathbb{Q}$ not admitting any field automorphism other than the identity.*

Proof. Let K be a Galois extension of $\mathbb{Q}$ with Galois group S_d, and let F be the fixed field of the subgroup S_{d-1} of S_d of elements fixing 1. The automorphism group of F is the normalizer of S_{d-1} in S_d modulo S_{d-1}, which is trivial. Thus, to prove the proposition, it suffices to find a polynomial g of degree d in $\mathbb{Q}[X]$ whose Galois group is the full group S_d of permutations of its roots and which splits over $\mathbb{R}$, because then the field F generated by any root of g will have the required properties. Such a polynomial can be constructed by a standard argument, which we recall.

Let p_1, p_2, p_3 be distinct primes, and let $f_1, f_2, f_3 \in \mathbb{Z}[X]$ be monic polynomials of degree d such that f_1 is irreducible modulo p_1, f_2 is the product of a linear polynomial and an irreducible polynomial modulo p_2, and f_3 is the product of an irreducible quadratic polynomial and distinct irreducible polynomials of odd degree modulo p_3. Finally, let $f_\infty \in \mathbb{R}[X]$ be monic of degree d with d distinct real roots. It follows from the weak approximation theorem applied to the coefficients of the polynomials that there exists a monic polynomial $g \in \mathbb{Q}[X]$ such that $g - f_i \in p_i\mathbb{Z}_{p_i}[X]$ for $i = 1, 2, 3$ and g is sufficiently close to f_∞ in the real topology that it also has d distinct real roots. The Galois group of g is a transitive subgroup of S_d containing a transposition and a $(d-1)$ cycle, and so equals S_d. □

When F admits nontrivial automorphisms, the following proposition will be useful.

Proposition 2.4. *Let F be a number field and let p be a rational prime number which splits completely in F. If a field automorphism σ of F fixes one p-adic place of F, then $\sigma = 1$.*

Proof. Choose a Galois closure $K/\mathbb{Q}$ of $F/\mathbb{Q}$ with group G. Denote by X the set of p-adic places of K; as p also splits completely in K, X can be identified with the set

$$\mathrm{Hom}\left(K, \mathbb{Q}_p\right)$$

of ring homomorphisms of K into $\mathbb{Q}_p$, and G acts simply transitively on X by composition.

Let H be the Galois group of K/F. Then the set of p-adic places of F is identified with the set X/H of H-orbits in X, and $\mathrm{Aut}(F)$ with $N_G(H)/H$, where $N_G(H)$ denotes the normalizer of H in G. In this identification, the action of $\mathrm{Aut}(F)$ on the p-adic places of F corresponds to the induced action of $N_G(H)/H$ on X/H. Thus the problem reduces to the following trivial statement: If H is a subgroup of a group G and if X is a principal homogeneous space for G, then an element of $N_G(H)$ leaving stable one H-orbit in X necessarily belongs to H. □

Construction

Let F be a totally real field. We explain how to construct

- pairs (H, X) as in (1.2) with H satisfying the hypotheses of Theorem 2.1, and
- automorphisms τ of $\mathbb{C}$ such that any group H' satisfying (3) of Theorem 1.3 will not be isomorphic to σH for any automorphism σ of F.

Corollary 2.2 then implies that, for any sufficiently small congruence subgroup Γ of $H(F)$, $S \overset{\text{def}}{=} \Gamma\backslash X$ is an algebraic variety over $\mathbb{C}$ whose fundamental group is not isomorphic to that of τS.

Let T be one of A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_6 or E_7.

For each real prime v of F, choose a simply connected simple algebraic group $H(v)$ over $\mathbb{R}$ of type T that is either compact or is associated with a hermitian symmetric domain $X(v)$ (i.e. $H(v)^{\mathrm{ad}}(\mathbb{R})^{+} = \mathrm{Hol}(X(v))^{+}$). Recall that, up to isomorphism, there is exactly one compact possibility for $H(v)$ and at least one hermitian possibility (and more than one unless $T = A_1$, A_2, B_n, C_n, D_4, E_6 or E_7); see [He, p. 518] or [De1, 1.3.9]. We choose the $H(v)$ so that

(i) for some $v \neq v'$, $H(v) \not\approx H(v')$, and

(ii) the sum of the ranks $\sum_{v|\infty} \mathrm{rk}_{\mathbb{R}} H(v)$ is at least 2.

Let $H(\infty)$ (resp. X) be the product of $H(v)$'s (resp. $X(v)$'s) as v runs over all the real primes of F (resp. over the real primes of F at which $H(v)$ is noncompact). There is a surjective homomorphism $H(\infty)(\mathbb{R})^{+} \to \mathrm{Hol}(X)^{+}$ with compact kernel.

When F admits nontrivial automorphisms, we also need to mark finite places in some way (cf. Remark 2.5 below). In this case, choose a rational finite prime p that splits completely in F (such primes abound by the Chebotarev density theorem) and a prime v_0 of F lying above p. For each prime v of F lying over p, choose an absolutely simple, simply connected $\mathbb{Q}_p$-group $H(v)$ such that $H(v) \not\approx H(v_0)$ if $v \neq v_0$.

When F admits no nontrivial automorphism, we take S to be the set of real primes of F, and otherwise we take it to be the set of real or p-adic primes of F. According to a theorem of Borel and Harder [BH, Theorem B], there exists an absolutely simple, simply connected group H over F such that for each $v \in S$, the localization H_v of H at v is isomorphic to $H(v)$. Then H satisfies the hypotheses of Theorem 2.1 and there is a surjective homomorphism $H(\mathbb{R})^{+} \to \mathrm{Hol}(X)^{+}$ with compact kernel.

Let τ be an automorphism of $\mathbb{C}$ such that, for at least one real place v_1, $H(v_1)$ and $H(\tau \circ v_1)$ are not isomorphic.[3] Given (H, X), such a τ always exists because of the condition (i) and the fact that $\mathrm{Aut}(\mathbb{C})$ acts transitively on the set of real primes of F. Conversely, given a $\tau \in \mathrm{Aut}(\mathbb{C})$ acting non-trivially on the real places of F and a type T, we can always choose a pair (H, X) of type T satisfying this hypothesis.

[3] The choice of the $H(v)$ defines a partition of set of real primes of F, in which two primes lie in the same set if the groups are isomorphic. The hypothesis on τ is that it not preserve this partition.

It remains to prove, with this choice for (H, X), a group H' satisfying the isomorphisms (1) of Theorem 1.3 can not be isomorphic to σH for any automorphism σ of F. If $\mathrm{Aut}(F) = 1$, then σH is not isomorphic to H' because, σ being 1, that would imply that $H(v_1) \approx H'(v_1) \approx H(\tau \circ v_1)$, contrary to hypothesis. When $\mathrm{Aut}(F) \neq 1$, the condition at the p-adic primes force $\sigma = 1$ in view of Proposition 2.4, and the same argument applies.

The dimension of the resulting Shimura varieties is equal to $\dim_{\mathbb{C}} X = \sum_v \dim_{\mathbb{C}} X(v)$, where v runs over the infinite places of F with $H(v)$ noncompact; the table on [He, p. 518] gives the real dimension of $X(v)$.

As noted above, there is only one noncompact real form associated with a hermitian symmetric domain for the types A_1, A_2, B_n, C_n, D_4, E_6 and E_7, and $H(v)$ is compact for at least one real place v. This means that the resulting Shimura varieties are necessarily projective ([ISV, 3.3(b)]). For the types $A_n (n \geq 3)$ and $D_n (n \geq 5)$, however, we can construct noncompact examples. Furthermore, for A_n, we can find these examples (compact or otherwise) among PEL moduli spaces of abelian varieties. These will be explained in the subsections below, after we deal with the quaternionic examples a bit more concretely.

Quaternionic Shimura varieties

We start with quaternionic Shimura varieties as in Example 1.4; so we denote by B a quaternion algebra over a totally real number field F of degree $d \geq 3$ over $\mathbb{Q}$ and by $\Sigma(B)$ the set of places v of F at which B is ramified (i.e., $\mathrm{inv}_v(B) = 1/2$). The last set decomposes into $\Sigma(B) = \Sigma(B)_f \amalg \Sigma(B)_\infty$ of finite and infinite places of ramification. We let $H := \mathrm{SL}_1(B)$; the real rank of H_* is equal to $d - |\Sigma(B)_\infty|$.

Automorphism-free F

Suppose F admits only the identity as automorphism. Choose B/F and $\tau \in \mathrm{Aut}(\mathbb{C})$ such that (i) $\tau\Sigma(B)_\infty \neq \Sigma(B)_\infty$ and (ii) $d - |\Sigma(B)_\infty| \geq 2$; we can either first choose B/F that satisfies (ii) and then choose a suitable τ, or choose $\tau \in \mathrm{Aut}(\mathbb{C})$ acting non-trivially on the real places of F and then choose a suitable B/F.

Then H and the group H' attached to (H, X) and τ satisfy the conditions in Corollary 2.2.

General $F/\mathbb{Q}$

If F does admit an automorphism other than the identity, we need to "mark" some finite places. Choose a rational prime p over which F splits completely, and then B/F and $\tau \in \mathrm{Aut}(\mathbb{C})$ such that in addition to the two conditions (i) and (ii) as above, we have (iii) there is exactly one (out of d) p-adic place v_p in $\Sigma(B)$.

Then, again, H and H' for this choice of B and τ satisfy the conditions in Corollary 2.2.

Remark 2.5. To see why one does need some extra argument marking finite places, let F be Galois over $\mathbb{Q}$, and suppose that for all the rational prime numbers p, any two p-adic places v and w are simultaneously contained in $\Sigma(B)$ (i.e., either both are in or both are out); for example, suppose that B is unramified at every finite place. Then, even if B satisfies the two conditions (i) and (ii), for *any* choice of $\tau \in \mathrm{Aut}(\mathbb{C})$, B and B' will be isomorphic as $\mathbb{Q}$-algebras (though not as F-algebras). Consequently, H_* and H'_* will be isomorphic as $\mathbb{Q}$-groups, and as such will admit isomorphic lattices.

The dimension of the resulting quaternionic Shimura varieties is equal to $d - |\Sigma(B)_\infty|$, which can be any integer ≥ 2. As $\Sigma(B)_\infty$ contains at least one, but misses at least two, of infinite places, it follows that all these varieties (a) are necessarily compact and (b) cannot have rational weights, and a fortiori cannot be a classifying space of rational Hodge structures [ISV, 5.24 and p. 332].

Unitary PEL Shimura varieties

Now we turn to certain unitary Shimura varieties, which are moduli spaces of polarized abelian varieties with endomorphism and level structures. We will keep the following notation throughout: F is a totally real number field of degree $d \geq 2$, E/F is a totally imaginary quadratic extension, and $n \geq 2$ is an integer.

We consider hermitian matrices $A \in M_n(E)$, that is, those with $A^* = A$. Recall that over the real numbers, there are $\lfloor n/2 \rfloor + 1$ distinct isomorphism classes of special unitary groups $SU(p,q)$ with $p \geq q \geq 0$ and $p+q = n$, while over any p-adic local field, there are one or two isomorphism classes of special unitary groups, according as n is odd or even. Moreover, all of these can be defined by *diagonal* hermitian matrices. By the approximation theorem applied to the diagonal entries (in F) of A, for any finite subset Σ of places of F and any prescribed type of special unitary group $H(v)$ for each $v \in \Sigma$ (there is only one choice if v is finite and n is odd), there is a diagonal matrix $A \in M_n(F)$ such that the localization at each $v \in \Sigma$ of the special unitary group $SU(A)$ over F attached to A is isomorphic to $H(v)$.

Automorphism-free F

Suppose $\mathrm{Aut}(F) = 1$ (hence $d = [F : \mathbb{Q}] \geq 3$). For each infinite place v of F, choose a pair (p_v, q_v) with $p_v \geq q_v \geq 0$ and $p_v + q_v = n$, subject to the conditions that (i) not all of the pairs should be the same and (ii) $\sum_{v|\infty} q_v \geq 2$. As $\mathrm{Aut}(\mathbb{C})$ acts transitively on the infinite places, we can choose $\tau \in \mathrm{Aut}(\mathbb{C})$ such that $p_{v_0} \neq p_{\tau\circ v_0}$ for some infinite place v_0. When $n \geq 4$, we can even make $q_v > 0$ for all $v|\infty$. Choose a diagonal matrix $A \in M_n(F)$ such that the associated special unitary group $H := SU(A)$ over F has the chosen type $SU(p_v, q_v)$ at each infinite place v of F.

Then there is a PEL Shimura datum (G, X) such that $G^{\mathrm{der}} \cong H_*$, and the groups H and H' satisfy the conditions in Corollary 2.2. The resulting varieties have dimension equal to

$$\sum_{v|\infty} p_v q_v.$$

They are compact if $q_v = 0$ for at least one $v|\infty$. When $q_v > 0$ for all $v|\infty$, one can make noncompact examples by letting A contain each of 1 and -1 as diagonal entries at least

$$q_0 := \min_{v|\infty} q_v > 0$$

times; then the associated special unitary group contains the q_0-fold product of SL_2 over F, hence is isotropic, and S will be noncompact (cf. [ISV, 3.3(b)]). The smallest dimension of noncompact examples obtained this way is $10 = 3 \cdot 1 + 3 \cdot 1 + 2 \cdot 2$.

General $F/\mathbb{Q}$, with n even

Now drop the condition on $\mathrm{Aut}(F)$, and assume that p is a rational prime number which (a) splits completely in F and (b) every p-adic place v of F remains inert in E (given F, such an E/F can be obtained by taking the compositum of F with a suitable imaginary quadratic field). In case n is even, we can mark p-adic places, since there are two distinct types of special unitary groups for the extension $E_v/\mathbb{Q}_p$.

More precisely, choose the diagonal matrix $A \in M_n(F)$ which satisfies, in addition to the conditions (i) and (ii) as above, (iii) there is one p-adic place v_0 of F such that $SU(A)_{v_0}$ is not isomorphic to $SU(A)_v$ for every other p-adic place $v \neq v_0$. Put $H := SU(A)$ and choose $\tau \in \mathrm{Aut}(\mathbb{C})$ that doesn't preserve the signatures at infinity. Then the resulting groups H and H' satisfy the conditions.

The sufficient condition for compactness and the formula for dimension as above still hold in this case. The smallest dimension of noncompact examples in this case is $7 = 3 \cdot 1 + 2 \cdot 2$ over real quadratic fields.

Remark 2.6. Using division algebras (instead of the matrix algebra $M_n(E)$) with involution of the second kind extending the complex conjugation on E/F, one can make inner forms of unitary groups H and choose $\tau \in \mathrm{Aut}(\mathbb{C})$ so that the resulting groups H and H' satisfy the conditions in Corollary 2.2, regardless of the parity of n (see [Cl, §2] for details). In this case, H and H' will be anisotropic and the corresponding Shimura varieties will be projective. They still have interpretation as PEL moduli varieties.

Noncompact examples of type D

Consider the case of type D_n, with $n \geq 4$. There are two series (called $BD\ I(q = 2)$ and $D\ III$ [He, p. 518]) of hermitian symmetric domains, coinciding exactly when $n = 4$. Thus, there are two choices of noncompact forms when $n \geq 5$, and one can ask if this extra choice allows noncompact examples. In this section, we construct such examples; these will *not* be of abelian type (see [ISV, p. 331]).

Let F be a number field and let B be a quaternion division algebra over F, with the main involution $* = *_B$. Then the matrix algebra $M := M_n(B)$ also has an involution $*_M$, mapping an element $(a_{ij})_{1 \leq i,j \leq n}$ to $\left(a_{ji}^*\right)_{1 \leq i,j \leq n}$. An invertible skew-hermitian matrix A defines another involution $*_A$ on M by conjugation:

$$g^{*_A} := A \cdot g^{*_M} \cdot A^{-1},$$

and one can form the special unitary group over F, given by

$$SU(A)(R) := \left\{ g \in M_n(B) \otimes_F R : g^{*_A} \cdot g = 1 \text{ and } \mathrm{Nrd}_F^M(g) = 1 \right\}$$

for an F-algebra R. Over an algebraic closure $\overline{F}$, $SU(A)$ becomes isomorphic to SO_{2n}.

If v is a place of F at which B is unramified, there is an isomorphism

$$\varphi_v : B \otimes_F F_v \xrightarrow{\sim} M_2(F_v) \tag{15}$$

of F_v-algebras, under which $*_B$ corresponds to the involution

$$\dagger : x \mapsto J \cdot {}^t x \cdot J^{-1}, \tag{16}$$

where t stands for the transpose of a 2×2 matrix and $J \in M_2(F_v)$ is t-skew-symmetric (i.e., ${}^tJ = -J$). As any two non-degenerate alternating forms on $F_v^{\oplus 2}$ are equivalent, by composing φ_v with an inner automorphism, we may assume that $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For an integer $n \geq 1$, put $J(n) := \operatorname{diag}(J, \cdots, J) \in M_n(M_2(F_v))$. Then under the isomorphism (that we still call φ_v) of F_v-algebras induced by φ_v:

$$M \otimes_F F_v = M_n(B) \otimes_F F_v \cong M_n(B \otimes_F F_v) \xrightarrow{\sim} M_n(M_2(F_v)),$$

the involution $*_M \otimes 1$ corresponds to the involution on the target

$$\dagger : (a_{ij})_{1 \leq i,j \leq n} \mapsto \left(J \cdot {}^t a_{ji} \cdot J^{-1}\right)_{1 \leq i,j \leq n}.$$

Let $I_{1,1} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(F_v)$. The matrix $J(n)$ (resp. the element $I_{1,1}$) is $\dagger$-skew-symmetric, and so defines an involution and the special unitary group over F_v as above, such that

$$SU(J(n))(F_v) \cong \left\{ g \in M_{2n}(F_v) : {}^t g \cdot g = 1 \text{ and } \det(g) = 1 \right\},$$
$$SU(I_{1,1})(F_v) \cong \left\{ g \in M_2(F_v) : {}^t g \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \det(g) = 1 \right\}.$$

Now we assume that $n \geq 5$, that F is a totally real number field of degree $d \geq 3$ with $\operatorname{Aut}(F) = 1$, and that the sets $S(B)_{\mathbb{H}}$ and $S(B)_{\mathbb{R}}$ of definite and indefinite infinite places of B/F, respectively, are both non-empty, so that there is a $\tau \in \operatorname{Aut}(\mathbb{C})$ not leaving $S(B)_{\mathbb{H}}$ stable. (See below for removing the assumption $\operatorname{Aut}(F) = 1$.)

For each $v \in S(B)_{\mathbb{R}}$, fix isomorphisms φ_v as in (15) satisfying the condition (16) with the standard J as above. By using the approximation theorem, we can choose a $*$-skew-hermitian element $a_1 \in B$ whose image under φ_v is very close to $I_{1,1} \in M_2(F_v) \cong M_2(\mathbb{R})$ for each $v \in S(B)_{\mathbb{R}}$. Put $a_2 := -a_1$. Then choose $*$-skew-hermitian elements $a_3, \cdots, a_n \in B$ such that $\varphi_v(a_i)$ is very close to J for each $v \in S(B)_{\mathbb{R}}$ and every $i = 3, \cdots, n$. Then the matrix $A = \operatorname{diag}(a_1, \cdots, a_n) \in M_n(B)$ is skew-hermitian and invertible.

Let H be the universal covering of $SU(A)$. The matrix A was crafted so that H_v is of type $BD\ I(q = 2)$ (resp. of type $D\ III$) for $v \in S(B)_{\mathbb{R}}$ (resp. $v \in S(B)_{\mathbb{H}}$). As τ does not leave $S(B)_{\mathbb{H}}$ stable and as we assumed $\operatorname{Aut}(F) = 1$, the group H' attached to H and τ satisfy the conditions in Corollary 2.2 (the real rank of the form $BD\ I(q = 2)$ (resp. $D\ III$) is 2 (resp. $\lfloor n/2 \rfloor \geq 2$), and the rank condition (ii) in the construction is automatically met).

The group H is isotropic over F—hence the resulting Shimura varieties are noncompact—because $SU(A)$ contains $SU(a_1, -a_1)$, which in turn contains $SO(1, -1)$ over F (since a_1 commutes with F), which is isomorphic to the split torus over F. The Shimura varieties attached to H have dimension

$$|S(B)_{\mathbb{R}}| \cdot (2n-2) + |S(B)_{\mathbb{H}}| \cdot n(n-1)/2.$$

Again, when F (of degree ≥ 2 over $\mathbb{Q}$) has non-trivial automorphisms, we can mark p-adic places, where p is a rational prime such that $F/\mathbb{Q}$ splits completely above p and B/F is unramified at every p-adic place of F: Choose a_1 and a_2 just as above, but this time include the p-adic places into consideration when applying the approximation theorem to $a_3, \cdots, a_n$, so as to distinguish the special orthogonal (and hence the spin) groups at p-adic places; then use Proposition 2.4.

Varieties defined over number fields

As noted in the introduction, all of the varieties we consider have canonical models defined over number fields. Briefly, given (H, X), we need to choose a (nonconnected) Shimura datum (G, Y) such that G is a reductive group with derived group H_* and Y is disjoint union of hermitian symmetric domains, one component of which is X. Then the Shimura variety defined by (G, Y) has a canonical model over its reflex field $E(G, Y)$, which is a number field, and each of the varieties $\Gamma \backslash X$ has a canonical model over an abelian extension of $E(G, Y)$ which can be described by class field theory. See [ISV]. We explain this in the case of quaternionic Shimura varieties.

Thus, let B be a quaternion algebra over a totally real field F. Then B defines a partition of the real primes of F

$$\mathrm{Hom}(F, \mathbb{R}) = \Sigma_\infty \sqcup \Sigma'_\infty \tag{17}$$

with Σ_∞ the set of primes where B ramifies. Let G be the reductive group over $\mathbb{Q}$ with $G(\mathbb{Q}) = B^\times$, and let Y be a product of copies of $\mathbb{C} \smallsetminus \mathbb{R}$ indexed by Σ'_∞ (assumed to be nonempty). Then (G, Y) is a Shimura datum. The Galois group of $\mathbb{Q}^{\mathrm{al}}$ acts on $\mathrm{Hom}(F, \mathbb{Q}^{\mathrm{al}}) \simeq \mathrm{Hom}(F, \mathbb{R})$, and its reflex field $E = E(G, Y)$ is the fixed field of the set of automorphisms of $\mathbb{Q}^{\mathrm{al}}$ stabilizing the sets Σ_∞ and Σ'_∞. See [ISV, 12.4(d)]

The norm map defines a homomorphism $\mathrm{Nm}\colon G \to (\mathbb{G}_m)_{F/\mathbb{Q}}$. The theory of Shimura varieties shows that the Shimura variety attached to (G, Y) has a canonical model over $E(G, Y)$. Moreover, its set of connected components is $(\mathbb{A}_F^\infty)^\times / \overline{F^\times}$, and the theory provides a specific (reciprocity) homomorphism $r\colon (\mathbb{A}_E^\infty)^\times / E^\times \to (\mathbb{A}_F^\infty)^\times / \overline{F^\times}$. Let K be a compact open subgroup of $G(\mathbb{A}^\infty)$ sufficiently small that the congruence subgroup $\Gamma = K \cap H_*(\mathbb{Q})$ of $H_*(\mathbb{Q})$ is torsion free. Use the norm to map K to a subgroup of $(\mathbb{A}_F^\infty)^\times / \overline{F^\times}$, and let K' be the inverse image of this subgroup in $(\mathbb{A}_E^\infty)^\times / E^\times$. The complex algebraic variety $\Gamma \backslash X$ has a canonical model V over the abelian extension E' of E corresponding by class field theory to K'.

Now assume that $|\Sigma'_\infty| \geq 2$, and let σ_1 and σ_2 be homomorphisms $E' \to \mathbb{C}$. If F has no nontrivial automorphisms, then $\sigma_1 V$ and $\sigma_2 V$ have nonisomorphic fundamental group whenever $\sigma_1 | E \neq \sigma_2 | E$ (because then there exists a real prime v of F such that $\sigma_1 \circ v$ and $\sigma_2 \circ v$ lie in different subsets of the partition (17)). If F has nontrivial automorphisms, then, as before, we need to impose a condition on a prime p in order to be able to draw the same conclusion.

Remark 2.7. Finally we remark that we have proved that two arithmetic groups Γ and Γ' are not isomorphic by showing that they live in different algebraic groups so that we can apply Margulis super-rigidity. However, of course, lattices Γ and Γ' may be nonisomorphic even when they lie in the same algebraic group. Since the theory of Shimura varieties gives an explicit description of Γ' in terms of Γ and τ (see 1.6), this suggests that it may be possible to enlarge our class of examples even further.

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