Points on Shimura varieties over finite fields: the conjecture of Langlands and Rapoport

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Abstract

We state an improved version of the conjecture of Langlands and Rapoport, and we prove the conjecture for a large class of Shimura varieties. In particular, we obtain the first proof of the (original) conjecture for Shimura varieties of PEL-type.

Contents

Introduction

Das Problem der Fortsetzbarkeit der Hasse-Weil-Zeta-Funktionen und allgemeiner der motivischen L-Funktionen ist nach wie vor ein zentrales Problem der Zahlentheorie. Es wird oft in zwei Probleme aufgeteilt .... Es ist erstens zu zeigen, dass jede motivische L-Funktion gleich einer automorphen L-Funktion ist, und zweitens, dass jede automorphe L-Funktion fortsetzbar ist. Beide Probleme sind in herzlich wenigen Fällen gelöst und dann nur dank der Bemühungen vieler Mathematiker über lange Zeit. Nach den abelschen Varietäten sind in arithmetischer Hinsicht die Shimuravarietäten wohl die zugänglichsten, und diese Arbeit soll ein Beitrag zum ersten Problem für die ihnen zugeordneten motivschen L-Funktionen sein.[1]

Langlands and Rapoport 1987, p113.

Shimura varieties arose out of the study of automorphic functions, and are defined by a reductive group G and additional data X. In order to show that the zeta function of a Shimura variety is an automorphic L-function, one must find a group-theoretic description of the points of the variety with values in a finite field, and then apply a combinatorial argument involving the stabilized Arthur-Selberg trace formula and the fundamental lemma. Langlands (1976) gives a general conjectural description of the points. For the Shimura varieties attached to totally indefinite quaternion algebras I showed that this conjecture can be proved using only the results of Weil, Tate, and Honda on abelian varieties over finite fields (Milne 1979b,a). From this, it follows that the zeta functions of these varieties are automorphic (see Casselman 1979). Similar results were obtained using Honda-Tate theory for other quaternionic Shimura varieties by Reimann (1997), and for simple Shimura varieties of PEL-types A and C by Kottwitz (1992).

Although these results have important consequences, as Langlands himself pointed out, his original conjecture is inadequate. Typically in the theory of Shimura varieties, one proves a statement for some small class of Shimura varieties and extends it to a much larger class through the intermediary of connected Shimura varieties. Langlands's conjecture is too imprecise for this approach to work. Moreover, it groups together objects that are only locally isomorphic whereas one should have a finer statement in which globally nonisomorphic objects are distinguished. Finally, Langlands states his conjecture in terms of an embedding of a group into $G(\mathbb{A}_f^p)$, and this embedding is not sufficiently precisely defined to permit passage to the combinatorial part of the argument in the general case.

In their fundamental paper, Langlands and Rapoport (1987) use a "Galois-gerbe", which is conjecturally the groupoid attached to a fibre functor on the category of motives over a finite field, to give a precise conjectural description of the points on the reduced variety (ibid. 5.e). This conjecture removes the inadequacies of Langlands's original conjecture, and is a much deeper statement. In particular, it is not susceptible to proof, even for Shimura varieties of PEL-type,[2] by using only Honda-Tate theory.[3] Recall that this theory provides a list of the isogeny classes of abelian varieties

[1]The problem of analytically continuing Hasse-Weil zeta functions, and more generally motivic L-functions, is one of the central problems in number theory. It is often divided into two problems.... The first is to show that every motivic L-function is an automorphic L-function, and the second is to show that every automorphic L-function can be analytically continued. Both problems have been solved in mightily few cases, and then only thanks to the efforts of many mathematicians over a long period. After abelian varieties, Shimura varieties are, from the arithmetical point of view, the most approachable, and this work is a contribution to the first problem for their associated motivic L-series.

[2]Although Shimura varieties of PEL-type are very important, they are very special; see Deligne 1971, p123.

[3]Except for some very special Shimura varieties — loosely speaking, those for which there is no L-indistinguishability — for example, those defined by quaternion algebras; see Reimann 1997, pp52-59.

over a finite field and determines the isomorphism class of the endomorphism algebra attached to each class. In Section 6 of their paper, Langlands and Rapoport proved their conjecture for simple Shimura varieties of PEL-types A and C assuming

(c1) the Hodge conjecture for CM abelian varieties (over $\mathbb{C}$),
(c2) the Tate conjecture for abelian varieties over a finite field, and
(c3) Grothendieck's standard conjectures for abelian varieties over finite fields.

These statements allowed them to obtain a precise description of the *category* of abelian motives over a finite field (and therefore of the category of abelian varieties) as a polarized tannakian category with the standard fibre functors.

After their paper, two problems remained:

- remove the three assumptions (c1,c2,c3) from their proof;
- extend the proof to all Shimura varieties.

Concerning the first problem, in Milne 1999, 2002, I proved that (c1) implies both (c2) and (c3). Thus, the Hodge conjecture for CM abelian varieties alone suffices for the result of Langlands and Rapoport. More recently (Milne 2000, Milne 2009) I showed that a much weaker statement, namely, the rationality conjecture for CM abelian varieties, has many of the same consequences as the Hodge conjecture for CM abelian varieties; in particular, it suffices for the above proof of Langlands and Rapoport.

Concerning the second problem, in Milne 1994b I gave a partial heuristic derivation of the conjecture of Langlands and Rapoport assuming the existence of a sufficiently good theory of motives in mixed characteristic, and in the original version of this article (Milne 1995), I examined what was needed to turn the heuristic argument into a proof. I was led to state two conjectures, one concerning lifts of special points and one concerning a comparison of integral cohomologies. Vasiu (2003a) has announced a results on the first conjecture, and both Vasiu (2003b) and Kisin (2007, 2009) have announced results on the second conjecture.

This progress has encouraged me to rewrite my 1995 article. Let $\mathbb{F}$ be an algebraic closure of the field $\mathbb{F}_p$ of p elements. After some preliminaries on tannakian categories in Section 1, I construct in Section 2 a category $\mathsf{Mot}(\mathbb{F})$ of abelian motives over $\mathbb{F}$ having most of the properties that the category of Grothendieck motives over $\mathbb{F}$ would have if the three conjectures (c1,c2,c3) were known. In more detail:

- The category $\mathsf{Mot}(\mathbb{F})$ is constructed as a quotient of the category $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ of CM-motives over $\mathbb{Q}^{\mathrm{al}}$. According to the theory of quotient tannakian categories in Milne 2007, to construct such a quotient, we need a fibre functor ω_0 on a certain subcategory of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$. The proof of the existence of ω_0 makes use of, among other things, the main result of Wintenberger 1991 (which gives an explicit description of the functor sending a CM-motive to its associated filtered Dieudonné module).
- The proof that $\mathsf{Mot}(\mathbb{F})$ has the correct fundamental group uses the main ideas of Milne 1999 (which proves that (c1) implies (c2)).
- The proof that the polarization on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ descends to a polarization on $\mathsf{Mot}(\mathbb{F})$ uses the main ideas of Milne 2002 (which proves that (c1) implies (c3)).

In Section 3, I use the category $\mathsf{Mot}(\mathbb{F})$ to give what I believe to be the "right" statement of the conjecture of Langlands and Rapoport (henceforth called the Conjecture LR+; see below for a discussion of the various forms of the conjecture). It attaches to each Shimura p-datum (G, X) a set $\mathcal{L}(G, X)$ with operators, and the conjecture states that there should be a functorial isomorphism $\mathcal{L}(G, X) \to \mathrm{Sh}_p(G, X)(\mathbb{F})$.

In the fourth section, I explain (following Milne 1994b) how to realize many Shimura varieties in characteristic zero as moduli varieties of abelian *motives*. One problem in tackling the Langlands-Rapoport conjecture is that the level structure at p in characteristic zero is stated in terms of étale cohomology, whereas the level structure at p over the finite field is stated in terms of the crystalline cohomology. In order to pass from one to the other, we need the integral comparison conjecture (first stated in Milne 1995) which says that a Hodge class on an abelian variety with good reduction is integral for the de Rham cohomology if it is integral for the p-adic étale cohomology. Since proofs of enough of this conjecture for our purposes have been announced by both Vasiu (2003b) and Kisin (2007, 2009), I shall assume it for the remainder of this introduction.

Another obstacle is that the conjecture of Langlands and Rapoport implicitly implies that points of the Shimura variety with coordinates in a finite field lifts to special points in characteristic zero (up to isogeny). I call this statement the special-points conjecture. For simple Shimura varieties of PEL type, it was proved by Zink (1983), and more general results been announced by Vasiu (2003a).

In the final two sections of the paper, I prove that, for any Shimura p-datum of Hodge type (that is, embeddable in a Siegel p-datum), there is a canonical equivariant map $\mathcal{L}(G,X) \to \mathrm{Sh}_p(\mathbb{F})$. For an appropriate integral model the map is injective, and it is surjective if (and only if) the special-points conjecture is true. In particular, the Conjecture LR+ (a fortiori, the original conjecture of Langlands and Rapoport) is proved for Shimura varieties of PEL-type (by Zink's result). I also discuss how to extend the proofs of the Conjecture LR+ to other Shimura varieties, including many that are not moduli varieties, not even conjecturally; cf. Pfau 1993 1996a,b. Moreover, I discuss what is needed to extend the proof to all Shimura varieties of abelian type, and perhaps to all Shimura varieties.

The various forms of the conjecture of Langlands and Rapoport (good reduction case)

In an attempt to reduce the confusion surrounding the statement of the conjecture, I list its main forms. To avoid overloading the exposition, I state the conjectures only for Shimura varieties whose weight is defined over $\mathbb{Q}$.

LRo The original statement is (5.e), p169, of Langlands and Rapoport 1987. In the first three sections of the paper, the authors define a groupoid (die pseudomotivische Galoisgruppe), and use it to attach to a Shimura p-datum a set $\mathcal{L}(G,X)$ with a Frobenius operator and an action of $G(\mathbb{A}_f)$. When G^{der} is simply connected, they conjecture that this set with operators is isomorphic to the set $\mathrm{Sh}_p(\mathbb{F})$ defined by some integral model of the Shimura variety $\mathrm{Sh}_p(G,X)$.[4]

LRm In Milne 1992 I made some improvements to the original conjecture (ibid. 4.8).

- ◇ Langlands and Rapoport (1987, §7) show that their statement of the conjecture can not be true when G^{der} is not simply connected. I modified the statement of the conjecture so that it applies to all Shimura varieties (when the derived group is simply connected, the modified statement becomes the original statement because of Langlands and Rapoport 1987, Satz 5.3, p173).
- ◇ I defined the notion of a canonical integral model for the Shimura variety, which is uniquely characterized by having a certain extension property, and added the requirement that the conjecture hold for that particular integral model.
- ◇ I added the condition that the isomorphism $\mathcal{L}(G,X) \to \mathrm{Sh}_p(G,X)(\mathbb{F})$ commutes with

[4] Langlands and Rapoport inadvertently omitted the condition that the map be $G(\mathbb{A}_f)$-equivariant — Langlands has assured me that this should be considered part of the conjecture. Langlands and Rapoport also state a conjecture when the Shimura variety has mild bad reduction, but we are not concerned with that.

the actions of $Z(\mathbb{Q}_p)$ where Z is the centre of G. With the addition of this condition, I showed that the conjecture for Shimura varieties with simply connected derived group implies the conjecture for all Shimura varieties.

LRp Pfau (1993, 1996b,a) pointed out that neither LRo nor LRm is sufficiently strong to pass from Shimura varieties of Hodge type to Shimura varieties of abelian type. Specifically, if one assumes that Conjecture LRo (or LRm) holds for all Shimura varieties of Hodge type, then it is not possible to deduce that it holds for all Shimura varieties of abelian type. For that, one needs a "refined" conjecture in which the isomorphism $\mathcal{L}(G,X) \to \mathrm{Sh}_p(G,X)(\mathbb{F})$ is required to respect the maps to the sets of connected components.

LR+ As Deligne pointed out to me, Langlands and Rapoport define only the isomorphism class of their groupoid. In fact, the groupoid is not well-defined, even conjecturally (at a minimum it requires the choice of a fibre functor). In §3, I restate the conjecture in terms of the category $\mathsf{Mot}(\mathbb{F})$ defined in §2. At present, the construction of this category also requires a choice, but the possibly-provable rationality conjecture for CM-abelian varieties (weaker than the Hodge conjecture for CM abelian varieties) implies that there is a unique preferred choice; moreover, the choice doesn't affect the construction of $\mathcal{L}(G,X)$. Now that both objects are well defined, it is possible to require that the isomorphism $\mathcal{L}(G,X) \to \mathrm{Sh}_p(G,X)(\mathbb{F})$ be functorial in (G,X) — this is the Conjecture LR+. With the choice of a fibre functor for $\mathsf{Mot}(\mathbb{F})$, Conjecture LR+ implies Conjecture LRp, and so it is strictly stronger than both LRo and LRm. When one assumes LR+ for all Shimura varieties of Hodge type, then it is possible to deduce it for all Shimura varieties of abelian type.

The conjecture of Langlands and Rapoport is of interest to everyone working on Shimura varieties. For those interested only in the zeta functions of Shimura varieties, all that is needed is the integral formula,

$$T(j,f) = \left|\mathrm{Ker}^1(\mathbb{Q},G)\right| \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0;\gamma,\delta)\cdot O_\gamma(f^p)\cdot TO_\delta(\phi_r)\cdot \mathrm{Tr}\,\xi(\gamma_0) \tag{1}$$

first conjectured in a preliminary form by Langlands, and then by Kottwitz (1990, 3.1)[5]. It is proved in Milne 1992 that Conjecture LRm implies this formula (the converse, of course, is false).

Some history

The original conjecture of Langlands and Rapoport (LRo) predicted that two objects, not well-defined, are isomorphic. It is not possible to prove such a statement without first defining the objects. This led me (in Milne 1992) to introduce the notion of a canonical integral model, which ensured that the set-with-operators $\mathrm{Sh}_p(G,X)(\mathbb{F})$ is well-defined.

Although Honda-Tate theory suffices to prove Langlands's original conjecture for some PEL Shimura varieties, it soon became clear (to me and others) that it was insufficient to prove the Conjecture LRo. Hence there was a need to obtain a description of the category of abelian varieties over $\mathbb{F}$, or, more generally, of abelian motives, and not just the set of its isomorphism classes. In 1995, at the time I proved the results in Milne 1999, I thought these results could be used

> to construct a canonical category of "motives" over $\mathbb{F}$ that has the "correct" fundamental group, equals the true category of motives if the Tate conjecture holds for abelian

[5] But only when G^{der} is simply connected. There may be some interest in writing the formula also in the case that the derived group is not simply connected; that is, in extending §5–§7 of Milne 1992 to that case.

> varieties over $\mathbb{F}$, and canonically contains the category of abelian varieties up to isogeny as a polarized subcategory (ibid, p47).

I applied this statement to investigate the conjecture of Langlands and Rapoport with the goal of determining what more was needed to prove the conjecture. I determined (I believe correctly) that the two key technical results needed are the special points conjecture and integral comparison conjecture (see below). I wrote this work up as Milne 1995 for my own personal use, but I gave the manuscript to a few people because I wanted to encourage the experts to work on the two conjectures.

Alas, when I tried to write out the proof of the quoted statement, I found a gap in my argument (the rationality conjecture!). However my later work has enabled me to construct a category of motives with the required properties (but not to prove that it is canonical or that it contains the category of abelian varieties). This, together with the work of Kisin and Vasiu on the two conjectures, has encouraged me to return to the topic.

ASIDE. A problem one has in working on the conjecture of Langlands and Rapoport is the misperceptions that exist about the conjecture in the mathematical community. One factor contributing to this is that the original paper is in German, and hence inaccessible to most mathematicians, who may also be deterred by its length (108 pages). Another has been the misstatements in the literature, most egregiously in Clozel's Bourbaki talk (Clozel 1993), where he writes (Introduction):

> ...En collaboration avec Rapoport, [Langlands] formula ensuite une conjecture précise (LR1987), qui était démontrée modulo une partie des "conjectures standard" de géometrie algébrique.
> L'objet de cet exposé est le travail de Kottwitz (K1992) qui démontre inconditionnellement, pour les variétés de Shimura qui décrivent des problèmes de modules de variétés abéliennes munies de quelque structures ..., une reformulation de la conjecture de Langlands-Rapoport.
>
> (...In collaboration with Rapoport, [Langlands] next formulated a precise conjecture (LR1987), which was proved modulo part of the "standard conjectures" in algebraic geometry.
> The object of this exposition is the work of Kottwitz (K1992) which proves unconditionally, for those Shimura varieties describing moduli problems for abelian varieties endowed with some structures..., a reformulation of the conjecture of Langlands-Rapoport.)

If you believe this, as many mathematicians seem to judging by their writings, then you will think that the conjecture of Langlands and Rapoport was proved for (at least) all Shimura varieties of PEL type by Kottwitz in 1992, and that the general case was proved by Langlands and Rapoport in 1987 assuming only a part of the standard conjectures. In fact, Kottwitz (1992) proves only the integral formula (1) (see p442 of his paper) and only for simple Shimura varieties of PEL types A and C, while Langlands and Rapoport (1987) prove their conjecture only for simple Shimura varieties of PEL types A and C, and only assuming the Hodge conjecture, the Tate conjecture, and the standard conjectures.[6] □

Notations and conventions

Throughout $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $\mathbb{Q}^{\mathrm{cm}}$ is the union of the CM-subfields of $\mathbb{Q}^{\mathrm{al}}$. For a Galois extension K/k, we let $\Gamma_{K/k} = \mathrm{Gal}(K/k)$. When K is an algebraic closure of k, we omit it from the notation. Complex conjugation on $\mathbb{C}$ and its subfields is denoted by ι or $z \mapsto \bar{z}$.

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[6] In diesem Abschnitt wollen wir zeigen, dass für gewisse Shimuravarietäten die Vermutung im §5 eine Folge der Identifizierung der pseudomotivischen Galoisgruppe und der motivischen Galoisgruppe ist, die im §4 unter Annahme der Standardvermutungen und der Tate-Vermutung sowie der Hodge-Vermutung vorgenommen wurde (ibid. p198).

(In this section we shall show that, for certain Shimura varieties, the conjecture in §5 is a consequence of the identification of the pseudomotivic Galois group with the motivic Galois group, which was proved in §4 under the assumption of the standard conjectures and the Tate conjecture as well as the Hodge conjecture.)

We fix a prime v of $\mathbb{Q}^{\mathrm{al}}$ dividing p, and we write v_K (or just v) for the prime it induces on a subfield K of $\mathbb{Q}^{\mathrm{al}}$. The completion $(\mathbb{Q}^{\mathrm{al}})_v$ of $\mathbb{Q}^{\mathrm{al}}$ at v is algebraically closed, and we let $\mathbb{Q}_p^{\mathrm{al}}$ denote the algebraic closure of $\mathbb{Q}_p$ in $(\mathbb{Q}^{\mathrm{al}})_v$. We let $\mathbb{Q}_p^{\mathrm{un}}$ denote the largest unramified extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^{\mathrm{al}}$. Its residue field, which we denote $\mathbb{F}$, is an algebraic closure of $\mathbb{F}_p$. We let $B(\mathbb{F})$ denote the closure of $\mathbb{Q}_p^{\mathrm{un}}$ in $(\mathbb{Q}^{\mathrm{al}})_v$, and we let $W(\mathbb{F})$ denote its ring of integers (equal to the ring of Witt vectors with coefficients in $\mathbb{F}$). The Frobenius map $x \mapsto x^p$ on $\mathbb{F}$ and its lift to $W(\mathbb{F})$ are both denoted by σ.

$$
\begin{tikzcd}
\mathbb{C} \arrow[dd, dash] & & & & & & \\
 & & (\mathbb{Q}^{\mathrm{al}})_v \arrow[ddr, dash] & & & & \\
v\ \mathbb{Q}^{\mathrm{al}} \arrow[urr, "\text{complete}"] \arrow[r, dash] \arrow[dd, dash] & \mathbb{Q}_p^{\mathrm{al}} \arrow[ur, dash] \arrow[d, dash] & & & & & \\
 & \mathbb{Q}_p^{\mathrm{un}} \arrow[rrr, "\text{complete}"] \arrow[dr, dash] & & B(\mathbb{F}) \arrow[dl, dash] & \supset & W(\mathbb{F}) \arrow[r, two heads] \arrow[d, dash] & \mathbb{F} \arrow[d, dash] \\
p\ \mathbb{Q} \arrow[rr, "\text{complete}"] & & \mathbb{Q}_p & & & \mathbb{Z}_p \arrow[r, two heads] & \mathbb{F}_p
\end{tikzcd}
$$

A reductive group is a smooth affine group scheme whose geometric fibres are connected reductive algebraic groups. For such a group G over a field, G^{der} denotes the derived group of G, $Z(G)$ the centre of G, $G^{\mathrm{ad}} \overset{\mathrm{def}}{=} G/Z(G)$ the adjoint group of G, and $G^{\mathrm{ab}} \overset{\mathrm{def}}{=} G/G^{\mathrm{der}}$ the largest commutative quotient of G.

For a (pro)torus T over a field k, $X^*(T)$ and $X_*(T)$ denote the character group of T and its cocharacter group (characters and cocharacters defined over some algebraic closure of k). The pairing

$$\langle\, ,\rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}$$

is defined by the formula

$$(\chi \circ \mu)(t) = t^{\langle \chi, \mu \rangle},\ t \in T(k^{\mathrm{al}}).$$

For an affine group scheme G over a ring R and an R-algebra S, $\mathsf{Rep}_S(G)$ denotes the category of representations of G on *finitely generated projective* S-modules (equivalently, flat S-modules of finite presentation). A representation will be denoted $\xi \colon G \to \mathrm{GL}(V(\xi))$ or $\xi \colon G \to \mathrm{GL}(\Lambda(\xi))$ depending on whether R is a field or not. Thus, $V \colon \xi \rightsquigarrow V(\xi)$ and $\Lambda \colon \xi \rightsquigarrow \Lambda(\xi)$ are the forgetful fibre functors $\mathsf{Rep}_S(G) \to \mathsf{Mod}_S$. A tensor functor $\mathsf{Rep}_R(G) \to \mathsf{Mod}_S$ is said to be *exact* if (a) it maps sequences in $\mathsf{Rep}_R(G)$ that are exact as sequences of R-modules to exact sequences in Mod_S and (b) every homomorphism in $\mathsf{Rep}_R(G)$ whose image in Mod_S is an isomorphism is itself an isomorphism.

For a finite extension of fields $k \supset k_0$ and an algebraic group G over k, $\mathrm{Res}_{k/k_0} G$ and $(G)_{k/k_0}$ both denote the algebraic group over k_0 obtained from G by restriction of scalars. For an infinite extension k/k_0, we let $(\mathbb{G}_m)_{k/k_0}$ denote the protorus $\varprojlim (\mathbb{G}_m)_{K/k_0}$ where K runs over the finite extensions of k_0 contained in k.

By a *Shimura p-datum*, we mean a reductive group[7] G over $\mathbb{Z}_{(p)}$ together with a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ such that $(G_{\mathbb{Q}}, X)$ satisfies the conditions (SV1), (SV2), and (SV3) of Milne 2005 (equal to the conditions (2.1.1.1), (2.1.1.2), and (2.1.1.3) of Deligne 1979). Except for the last two subsections of §6, we shall assume that $(G_{\mathbb{Q}}, X)$ satisfies (SV4) (the weight is defined over $\mathbb{Q}$) and (SV6) (the connected centre splits over a CM-field).

In general, an object defined over $\mathbb{Q}_l$ is denoted by $?_l$, whereas an object over $\mathbb{Q}_l$ that comes

[7] To give a reductive group over $\mathbb{Z}_{(p)} \overset{\mathrm{def}}{=} \mathbb{Q} \cap \mathbb{Z}_p$ amounts to giving a reductive group G_0 over $\mathbb{Q}$, a reductive group G_p over $\mathbb{Z}_p$, and an isomorphism $(G_0)_{\mathbb{Q}_p} \to (G_p)_{\mathbb{Q}_p}$ (apply Bosch et al. 1990, 6.2, Proposition D.4, p147).

from an object ? over $\mathbb{Q}$ by extension of scalars is denoted by ?(l). We sometimes abbreviate $S \otimes_R ?$ to $?_S$.

We sometimes use $[x]$ to denote the equivalence class of an element x.

We use $\approx$ to denote an isomorphism, and $\simeq$ to denote a canonical (or given) isomorphism.

We use l to denote a prime of $\mathbb{Q}$, i.e., $l \in \{2, 3, \ldots, p, \ldots \infty\}$, and ℓ to denote a prime $\neq p, \infty$.

We let $\mathbb{A}_f$ denote the ring of finite adèles $(\varprojlim_m \mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{A}_f^p$ the ring of finite adèles with the p-component omitted.

A diagram of functors and categories is said to commute if it commutes up to a canonical natural isomorphism.

1 Tannakian preliminaries

By a tensor category over a ring R we mean an additive symmetric monoidal category such that $R = \mathrm{End}(\mathbb{1})$ for any identity object $\mathbb{1}$ (cf. Deligne and Milne 1982). Tensor functors are required to be linear for the relevant rings. A tensor category over a field is tannakian if it is abelian, rigid, and admits an R-valued fibre functor for some nonzero k-algebra R. When the fundamental group of a tannakian category is commutative, we identify it with an affine group scheme in the usual sense (cf. Deligne 1989, §6). For a subgroup H of the fundamental group of a tannakian category C, we let C^H denote the full subcategory of objects fixed by H (that is, on which the action of H is trivial).

Fibre functors

Recall that a field k is said to have dimension ≤ 1 if the Brauer group of every field algebraic over it is zero (Serre 1964, II §3). For example, a finite field has dimension ≤ 1 and the field $B(\mathbb{F})$ has dimension ≤ 1 (ibid.). For a connected algebraic group G over a perfect field of dimension ≤ 1, $H^1(k, G) = 0$ (Steinberg 1965, 1.9).

PROPOSITION 1.1. *Let G be a reductive group over a henselian discrete valuation ring R whose residue field has dimension ≤ 1. Every exact tensor functor $\omega\colon \mathsf{Rep}_R(G) \to \mathsf{Mod}_R$ is isomorphic to the forgetful functor ω^G; moreover,* $\mathrm{Hom}^{\otimes}(\omega^G, \omega)$ *is a principal homogenous space for $G(R)$.*

PROOF. The tensor isomorphisms from ω to the forgetful functor form a G-torsor (e.g., Milne 1995, 1.1), which determines an element of $H^1(R, G)$. Because G is of finite type, this fpqc cohomology group can be interpreted as an fppf group (Saavedra Rivano 1972, III 3.1.1.1), and because G is smooth

$$H^1(R, G) = H^1(k, G_k)$$

where k is the residue field of R (e.g., Milne 1980, III 3.1). But G_k is connected, and so $H^1(k, G_k) = 0$ by Steinberg's theorem, from which the statement follows. □

Let G be an affine group scheme, flat and of finite type over a ring R, and let ξ be a representation of G on a finitely generated projective R-module $\Lambda(\xi)$. By a tensor on ξ we mean an element of $\Lambda(\xi)^{\otimes r} \otimes \Lambda(\xi)^{\vee \otimes s}$ for some r, s that is fixed under the action of G. Note that a tensor t can regarded as a homomorphism $R \to \Lambda(\xi)^{\otimes r} \otimes \Lambda(\xi)^{\vee \otimes s}$ of G-modules, and so it defines a tensor $\omega(t)$ on $\omega(\Lambda(\xi))$ for any tensor functor ω on $\mathsf{Rep}_R(G)$. A representation ξ_0 of G together with a family $(t_i)_{i \in I}$ of tensors is said to be *defining* if, for all flat R-algebras S,

$$G(S) = \{g \in \mathrm{Aut}(S \otimes_R \Lambda(\xi_0)) \mid g t_i = t_i, \text{ for all } i \in I\}.$$

In particular, this implies that ξ_0 is faithful. For conditions under which a defining representation and tensors exist, see Saavedra Rivano 1972, p151. For example, defining tensors exist if R is a discrete valuation ring and G is a closed flat subgroup of $\mathrm{GL}(\Lambda)$ whose generic fibre is reductive (by a standard argument, cf. Deligne 1982, 3.1).

PROPOSITION 1.2. *Let G be an affine group scheme, flat over a henselian discrete valuation ring R whose residue field has dimension ≤ 1, and assume that $(\xi_0,(t_i)_{i\in I})$ is defining for G.*

(a) *Consider a finitely generated projective R-module Λ and a family $(s_i)_{i\in I}$ of tensors for Λ. There exists an exact tensor functor $\omega\colon \mathsf{Rep}_R(G)\to\mathsf{Mod}_R$ such that*

$$(\omega(\xi_0),(\omega(t_i))_{i\in I})=(\Lambda,(s_i)_{i\in I}) \tag{2}$$

if and only if there exists an isomorphism $\Lambda(\xi_0)\to\Lambda$ of R-modules mapping each t_i to s_i.

(b) *For any exact tensor functor $\omega\colon\mathsf{Rep}_R(G)\to\mathsf{Mod}_R$, the map $\alpha\mapsto\alpha(\xi_0)$ identifies $\mathrm{Hom}^{\otimes}(\omega^G,\omega)$ with the set of isomorphisms $\omega^G(\xi_0)\to\omega(\xi_0)$ mapping each $\omega(t_i)$ to t_i. Here ω^G denotes the forgetful functor on $\mathsf{Rep}_R(G)$.*

PROOF. (a) If ω exists, then according to Proposition 1.1 there exists an isomorphism $\omega^G\to\omega$, and so the condition is necessary. For the converse, let S be an R-algebra, and define $P(S)$ to be the set of isomorphisms $S\otimes_R\Lambda(\xi_0)\to S\otimes_R\Lambda$ mapping each t_i to s_i. Then $S\rightsquigarrow P(S)$ is a G-torsor which, by assumption, is trivial. The twist of ω^G by P is an R-valued fibre functor that satisfies (2).

(b) Both sets are principal homogeneous space for $G(R)$, and so any $G(R)$-equivariant map from one to the other is a bijection. □

Objects with G-structure

DEFINITION 1.3. Let R be a Q-algebra, and let C be an R-linear rigid abelian tensor category. Let G be a reductive group over R. An *object in* C *with a G-structure*[8] (or, more briefly, a *G-object* in C) is an exact faithful tensor functor $M\colon\mathsf{Rep}(G)\to\mathsf{C}$. We say that two G-objects are *equivalent* if they are equivalent as tensor functors.

Let $(\xi_0,(t_i)_{i\in I})$ be defining for G. A G-object M in C defines an object $M(\xi_0)$ together with a family of tensors $(M(t_i)_{i\in I})$. Loosely speaking, one can think of a G-object in C as an object with a family of tensors satisfying some condition.[9]

When the representations of G can be described explicitly, so can the G-objects in C (cf. Rapoport and Richartz 1996, 3.3).

EXAMPLE 1.4. If $G=\mathrm{GL}(V)$, then to give a G-object in C amounts to giving an object X in C of dimension $\dim V$. To see this, note that because exact tensor functors preserve traces, a G-object $M\colon\mathsf{Rep}(G)\to\mathsf{C}$ will map V to an object $M(V)$ of dimension $\dim V$ in C. Conversely, let X be an object of dimension $\dim V$ in C. For each $n\in\mathbb{N}$, choose an object $\mathrm{Sym}^n V$ in $\mathsf{Rep}(G)$, and for each partition λ of n, let $S_\lambda V$ be the image of the Schur operator in $\mathrm{Sym}^n V$. Then the representations $S_\lambda V$ form a set of representatives for the isomorphism classes of simple representations

[8]The concept was used in the original version of this article (Milne 1995). For this version, I have borrowed the name from Simpson 1992, p86, and Rapoport and Richartz 1996, 3.3.

[9]One change from the first version of the article is that I have chosen to work directly with G-objects rather than choosing a defining representation and tensors. This avoids making choices and reveals the basic constructions to be more obviously canonical.

of G (Fulton and Harris 1991, 15.47). There exists a skeleton $\mathsf{Rep}(G)'$ whose objects are direct sums of $S_\lambda V$s. Now repeat the process in C with V replaced by X. There is an exact tensor functor $\mathsf{Rep}(G)' \to \mathsf{C}$ sending each $S_\lambda V$ to $S_\lambda X$ and each chosen direct sum in $\mathsf{Rep}(G)'$ to the corresponding direct sum in C. The functor $\mathsf{Rep}(G)' \to \mathsf{Rep}(G)$ is a tensor equivalence, and so has a tensor inverse (Saavedra Rivano 1972, I 4.4). Therefore, we obtain a G-object M of C with $M(V) = X$. Moreover, M is uniquely determined by X up to a unique isomorphism.

EXAMPLE 1.5. If $G = \mathrm{Sp}_n$, then to give a G-object in C amounts to giving an object X of dimension n together with a non-degenerate alternating pairing

$$X \otimes X \to \mathbb{1}.$$

The proof is similar to the last example.

Objects with G-structure in a Tate triple Recall that a *Tate triple* T over a field Q is a tannakian category C over Q equipped with a (weight) $\mathbb{Z}$-gradation $w\colon \mathbb{G}_m \to \underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{C}})$ and an invertible (Tate) object T of weight -2. A tensor functor of Tate triples is a tensor functor η of tannakian categories preserving the gradation together with an isomorphism $\eta(T) \to T'$. A fibre functor on T is a fibre functor ω on C together with an isomorphism $\omega(T) \to \omega(T^{\otimes 2})$ (equivalently, an isomorphism $R \to \omega(T)$).

EXAMPLE 1.6. The category $\mathsf{Hdg}_{\mathbb{Q}}$ of rational Hodge structures becomes a Tate triple with the weight gradation and the Tate object $\mathbb{Q}(1)$ (equal to $2\pi i\mathbb{Q} \subset \mathbb{C}$ with the Hodge structure of weight -2).

EXAMPLE 1.7. To give a Tate triple structure on $\mathsf{Rep}(G)$ is the same as giving a central homomorphism $w\colon \mathbb{G}_m \to G$ and a homomorphism $t\colon G \to \mathbb{G}_m$ such that $t \circ w = -2$. The Tate object is any one-dimensional space on which G acting through t. We shall call (t, w) a *Tate triple structure* on G.

Consider (G, w, t) and a Tate triple $\mathsf{T} = (\mathsf{C}, w, T)$. An object in T with a (G, w, t)-structure (or, more briefly, a (G, w, t)-object in T or C) is an exact tensor functor $M\colon \mathsf{Rep}(G) \to \mathsf{T}$ of Tate triples.

EXAMPLE 1.8. Let ψ be a nondegenerate alternating pairing $V \times V \to Q$ on the finite dimensional vector space V, and let $G = \mathrm{GSp}(\psi)$. Thus

$$G(Q) = \{g \in \mathrm{GL}(V) \mid \psi(gv, gv') = t(g)\psi(v, v') \text{ some } t(g) \in Q^\times\}.$$

Let $w\colon \mathbb{G}_m \to G$ be the homomorphism such that $w(c)$ acts on V as multiplication by c^{-1} for $c \in Q^\times$. Then (w, t) is a Tate triple structure on G, and to give a (G, w, t)-object in a Tate triple (C, w, T) amounts to giving an object X of C of dimension $\dim V$ together with a nondegenerate alternating pairing

$$X \otimes X \to T.$$

2 The category of motives over $\mathbb{F}$

In this section, we define a category of motives $\mathsf{Mot}(\mathbb{F})$ over $\mathbb{F}$ with the Weil protorus as its fundamental group, standard fibre functors, and a canonical polarization; moreover, there is a reduction functor from the category $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ of CM-motives to $\mathsf{Mot}(\mathbb{F})$. At present, the category depends on the choice of a fibre functor ω_0 on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ with certain properties. However, the rationality conjecture for CM abelian varieties (Milne 2009, 4.1) implies that there is a unique preferred ω_0, and the Hodge conjecture for abelian varieties of CM-type implies that, with this ω_0, $\mathsf{Mot}(\mathbb{F})$ is indeed the category of abelian motives over $\mathbb{F}$ defined using algebraic cycles modulo numerical equivalence.

The realization categories

The realization category at infinity. Let R_∞ be the category of pairs (V, F) consisting of a $\mathbb{Z}$-graded finite-dimensional complex vector space $V = \bigoplus_{m\in\mathbb{Z}} V^m$ and an ι-semilinear endomorphism F such that $F^2 = (-1)^m$ on V^m. With the obvious tensor structure, R_∞ becomes a tannakian category over $\mathbb{R}$ with fundamental group $\mathbb{G}_m$. The objects fixed by $\mathbb{G}_m$ are those of weight zero. If (V, F) is of weight zero, then

$$V^F \stackrel{\text{def}}{=} \{v \in V \mid Fv = v\}$$

is an $\mathbb{R}$-structure on V. The functor $V \rightsquigarrow V^F$ is an $\mathbb{R}$-valued fibre functor on $\mathsf{R}_\infty^{\mathbb{G}_m}$.

The realization category at $\ell \neq p, \infty$. Let R_ℓ be the category of finite-dimensional $\mathbb{Q}_\ell$-vector spaces. It is a tannakian category with trivial fundamental group. The forgetful functor is a $\mathbb{Q}_\ell$-valued fibre functor on R_ℓ.

The crystalline realization category. Let R_p be the category of F-isocrystals over $\mathbb{F}$. Thus, an object of R_p is a pair (V, F) consisting of a finite-dimensional vector space V over $B(\mathbb{F})$ and a σ-semilinear isomorphism $F\colon V \to V$. With the obvious tensor structure, R_p becomes a tannakian category over $\mathbb{Q}_p$ whose fundamental group is the universal covering group $\mathbb{G}$ of $\mathbb{G}_m$ (so $X^*(\mathbb{G}) = \mathbb{Q}$). The objects fixed by $\mathbb{G}$ are those of slope zero. If (V, F) is of slope zero, then

$$V^F \stackrel{\text{def}}{=} \{v \in V \mid Fv = v\}$$

is a $\mathbb{Q}_p$-structure on V. The functor $V \rightsquigarrow V^F$ is a $\mathbb{Q}_p$-valued fibre functor on $\mathsf{R}_p^{\mathbb{G}}$.

The category of CM-motives over $\mathbb{Q}^{\mathrm{al}}$

By a Hodge class on an abelian variety A over a field k of characteristic zero, we mean an absolute Hodge cycle in the sense of Deligne 1982, and we let $\mathcal{B}(A)$ denote the $\mathbb{Q}$-algebra of such classes on A.

Let $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ be the category of CM-motives over $\mathbb{Q}^{\mathrm{al}}$. Thus, an object of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ is a triple (A, e, m) with A an abelian variety of CM-type over $\mathbb{Q}^{\mathrm{al}}$, e an idempotent in the ring $\mathcal{B}^{\dim A}(A \times A)$, and m an integer; the morphisms are given by the rule,

$$\mathrm{Hom}((A, e, m), (B, f, n)) = f \cdot \mathcal{B}^{\dim A - m + n}(A \times B) \cdot e.$$

With the usual tensor structure, $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ becomes a semisimple tannakian category over $\mathbb{Q}$. Its fundamental group is the Serre group S. Recall that S has a canonical (weight) cocharacter $w = w^S$ defined over $\mathbb{Q}$, and a canonical cocharacter $\mu = \mu^S$ such that $w = -(1+\iota)\mu$; moreover, the pair (S, μ^S) is universal.

The local realization at ∞. Let (V,h) be a real Hodge structure, and let C act on V as $h(i)$. Then the square of the operator $v \mapsto C\bar{v}$ acts as $(-1)^m$ on V^m. Therefore, $\mathbb{C}\otimes_{\mathbb{R}} V$ endowed with its weight gradation and this operator is an object of R_∞. We let

$$\xi_\infty\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{R}_\infty, \quad X \rightsquigarrow (\omega_B(X)_{\mathbb{R}}, C),$$

denote the functor sending X to the object of R_∞ defined by the real Hodge structure $\omega_B(X)_{\mathbb{R}}$. Then ξ_∞ is an exact tensor functor, and the cocharacter $x_\infty\colon \mathbb{G}_m \to S_{\mathbb{R}}$ it defines is equal to $w_{\mathbb{R}}$. We obtain an $\mathbb{R}$-valued fibre functor ω_∞ on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\mathbb{G}_m}$ as follows:

$$\begin{tikzcd} \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\mathbb{G}_m} \arrow[r, "\xi_\infty"'] \arrow[rr, bend left, "\omega_\infty"] & \mathsf{R}_\infty^{\mathbb{G}_m} \arrow[r, "V \rightsquigarrow V^F"'] & \mathsf{Vec}(\mathbb{R}). \end{tikzcd}$$

The local realization at ℓ. For each $\ell \neq p,\infty$, we let ω_ℓ denote the fibre functor on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ defined by ℓ-adic étale cohomology.

The local realization at p. A CM abelian variety A over $\mathbb{Q}^{\mathrm{al}}$ has good reduction at the prime v to an abelian variety A_0 over $\mathbb{F}$. The map

$$(A,e,m) \mapsto e \cdot H^*_{\mathrm{crys}}(A_0)(m)$$

extends to an exact tensor functor

$$\xi_p\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{R}_p.$$

Let x_p denote the homomorphism $\mathbb{G} \to S_{\mathbb{Q}_p}$ defined by ξ_p. We obtain a $\mathbb{Q}_p$-valued fibre functor ω_p on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\mathbb{G}}$ as follows:

$$\begin{tikzcd} \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\mathbb{G}} \arrow[r, "\xi_p"'] \arrow[rr, bend left, "\omega_p"] & \mathsf{R}_p^{\mathbb{G}} \arrow[r, "V \rightsquigarrow V^F"'] & \mathsf{Vec}(\mathbb{Q}_p). \end{tikzcd}$$

The Shimura-Taniyama homomorphism

Let P be the Weil-number protorus (see, for example, Milne 1994a, §2). Thus, P is a protorus over $\mathbb{Q}$, and every element of $W \stackrel{\mathrm{def}}{=} X^*(P)$ is represented by a Weil p^n-number π; two pairs (π,n) and (π',n') represent the same element of P if and only if $\pi^{n'N} = \pi^{nN}$ for some integer $N \geq 1$. Define homomorphisms

$$\begin{aligned} z_\infty\colon \mathbb{G}_m \to P, &\quad \langle[\pi,n], z_\infty\rangle = m \text{ if } |\pi|_\infty = (p^{n/2})^m, \\ z_p\colon \mathbb{G} \to P, &\quad \langle[\pi,n], z_p\rangle = -\frac{\mathrm{ord}_v(\pi)}{\mathrm{ord}_v(p^n)}. \end{aligned}$$

There is a unique homomorphism $r^{\mathrm{ST}}\colon P \to S$, which I call the *Shimura-Taniyama homomorphism,* sending z_∞ to x_∞ and z_p to x_p. The homomorphism r^{ST} is injective, which allows us to identify P with a subgroup of S.

ASIDE. Let K be a CM field. A CM-type on K is a cocharacter f of S^K of weight -1 taking values in $\{0,1\}$. Let A_f be an abelian variety of CM-type f over $\mathbb{Q}^{\mathrm{al}}$. Then A_f has good reduction to an abelian variety B_f over $\mathbb{F}$, which defines an element $[\pi(f)]$ of W^K. The Shimura-Taniyama formula expresses $[\pi(f)]$ in terms of f. The Shimura-Taniyama homomorphism is the unique homomorphism $r\colon P \to S$ such that $X^*(r)$ maps every CM-type f to $[\pi(f)]$. This explains the choice of name. □

The category of abelian motives over $\mathbb{F}$

For a $\mathbb{Q}$-valued fibre functor ω on a tannakian category, let $\omega(l)$ denote the $\mathbb{Q}_l$-valued fibre functor $X \rightsquigarrow \mathbb{Q}_l \otimes_{\mathbb{Q}} \omega(X)$.

The fibre functors ω_l on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ constructed above restrict to fibre functors $\omega_l|$ on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^P$.

THEOREM 2.2. *There exists a $\mathbb{Q}$-valued fibre functor ω_0 on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^P$ such that*

$$\omega_0(l) \approx \omega_l|$$

for all l (including p and ∞).

PROOF. Let ω_B be the Betti fibre functor on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$. For a $\mathbb{Q}$-valued fibre functor ω on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^P$, let $\wp(\omega) = \underline{\mathrm{Hom}}^{\otimes}(\omega_B, \omega)$. This is an S/P-torsor, and the theory of tannakian categories shows that every S/P-torsor is isomorphic to $\wp(\omega)$ for some fibre functor ω. The fibre functor ω will satisfy the condition in the proposition if and only if the class of $\wp(\omega)$ in $H^1(\mathbb{Q}, S/P)$ maps to the class of $\wp(\omega_l)$ in $H^1(\mathbb{Q}_l, S/P)$ for all l. That such a class exists is proved in Milne 2003, Theorem 4.1.[10] □

The isomorphism class of the restriction of ω_0 to any algebraic subcategory of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^P$ is uniquely determined, but this is not true for ω_0 itself (Milne 2003, 1.11). The rationality conjecture (Milne 2009, 4.1) for CM abelian varieties implies that there is a unique preferred ω_0 satisfying the condition of the proposition.[11]

Choose a fibre functor ω_0 as in the proposition, and define $\mathsf{Mot}_{\omega_0}(\mathbb{F})$ to be the corresponding quotient category

$$R\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}_{\omega_0}(\mathbb{F})$$

in the sense of Milne 2007. Thus $\mathsf{Mot}_{\omega_0}(\mathbb{F})$ is a semisimple tannakian category over $\mathbb{Q}$ with fundamental group P, which can be described as follows. For a CM abelian variety A over $\mathbb{Q}^{\mathrm{al}}$, let $\mathcal{B}_{\omega_0}(A) = \omega_0(h(A)^P)$ where $h(A)$ is the object $(A,1,0)$ of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$. The objects of $\mathsf{Mot}_{\omega_0}(\mathbb{F})$ are the triples (A,e,m) with A a CM abelian variety over $\mathbb{Q}^{\mathrm{al}}$, e an idempotent in the ring $\mathcal{B}_{\omega_0}^{\dim A}(A \times A)$, and $m \in \mathbb{Z}$. We sometimes write $\hbar(A,e,m)$ for the object (A,e,m) of $\mathsf{Mot}_{\omega_0}(\mathbb{F})$. Note that the maps

$$\mathcal{B}(A) \simeq \mathrm{Hom}(\mathbb{1}, h(A)) \simeq \mathrm{Hom}(\mathbb{1}, h(A)^P) \xrightarrow{\omega_0} \mathrm{Hom}(\mathbb{Q}, \mathcal{B}_{\omega_0}(A)) \simeq \mathcal{B}_{\omega_0}(A)$$

realize $\mathcal{B}(A)$ as a subspace of $\mathcal{B}_{\omega_0}(A)$. The functor R is

$$(A,e,m) \rightsquigarrow \hbar(A,e,m)$$

(on the right e is to be regarded as an element of $\mathcal{B}_{\omega_0}(A)$). For any objects X and Y of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$,

$$\mathrm{Hom}(RX, RY) = \omega_0(\underline{\mathrm{Hom}}(X,Y)^P).$$

[10] It should be noted that the proof of Milne 2003, Lemma 4.2, requires the main result of Wintenberger 1991 in order to replace the cohomology class c_p^K with an elementarily defined class (cf. ibid. p31).

[11] In more detail, the rationality conjecture for CM abelian varieties implies that there exists a (unique!) good theory of rational Tate classes on abelian varieties over finite fields and that Hodge classes on CM abelian varieties reduce to rational Tate classes. We can use the rational Tate classes to construct a category $\mathsf{Mot}(\mathbb{F})$ of abelian motives over $\mathbb{F}$, and there will be a canonical reduction functor $R\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}(\mathbb{F})$. The functor $X \rightsquigarrow \mathrm{Hom}(\mathbb{1}, R(X))$ is a $\mathbb{Q}$-valued fibre functor ω_0 on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^P$, and R defines an equivalence $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})/\omega_0 \to \mathsf{Mot}(\mathbb{F})$. If the rationality conjecture holds for all abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ with good reduction, then this gives a functor from the category of motives generated by such abelian varieties to $\mathsf{Mot}(\mathbb{F})$, which would simplify the proof of the Langlands-Rapoport conjecture.

The choice of an isomorphism $\omega_0(l) \to \omega_l|$ determines an exact tensor functor

$$\zeta_l \colon \mathsf{Mot}_{\omega_0}(\mathbb{F}) \to \mathsf{R}_l$$

such $\zeta_l \circ R = \xi_l$ (see Milne 2007, end of §2). Therefore, such a choice determines a commutative diagram of tannakian categories and exact tensor functors as at right:

$$\begin{tikzcd}
\mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \arrow[dd, "R"'] \arrow[dr, "\xi_l"] & & \\
& \mathsf{R}_l \arrow[r, "\text{forget}"] & \left\{\begin{array}{l}\mathsf{Vec}_{\mathbb{C}}\\ \mathsf{Vec}_{\mathbb{Q}_\ell}\\ \mathsf{Vec}_{B(\mathbb{F})}\end{array}\right. \\
\mathsf{Mot}_{\omega_0}(\mathbb{F}) \arrow[ur, "\zeta_l"] \arrow[urr, "\omega_l=\omega_l^{\mathbb{F}}"'] & &
\end{tikzcd}$$

The canonical polarization on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ (regarded as a Tate triple) passes to the quotient and defines a polarization on $\mathsf{Mot}(\mathbb{F})$ (see Milne 2002).

Note that, for a motive M over $\mathbb{F}$, we have defined $\omega_l^{\mathbb{F}}(M)$ to be the vector space underlying $\zeta_l(M)$, and so for $l = p, \infty$ it has a Frobenius operator F and for $l \neq p, \infty$ it has a germ of a Frobenius operator (Milne 1994a, p422).

We now fix a fibre functor ω_0 and isomorphisms $\omega_0(l) \to \omega_l|$ as above, and we write $\mathsf{Mot}(\mathbb{F})$ for $\mathsf{Mot}_{\omega_0}(\mathbb{F})$.

The category of motives over $\mathbb{F}$ with $\mathbb{Z}_{(p)}$-coefficients

Define a *motive M over $\mathbb{F}$ with coefficients in $\mathbb{Z}_{(p)}$* to be a triple (M_p, M_0, m) consisting of

(a) a finitely generated $W(\mathbb{F})$-module M_p and a σ-linear map $F \colon M_p \to M_p$ whose kernel is torsion,

(b) an object M_0 of $\mathsf{Mot}(\mathbb{F})$, and

(c) an isomorphism $m \colon (M_p)_{\mathbb{Q}} \to \omega_p^{\mathbb{F}}(M_0)$.

A morphism of motives $\alpha \colon M \to N$ with coefficients in $\mathbb{Z}_{(p)}$ is a pair of morphisms

$$(\alpha_p \colon M_p \to N_p, \alpha_0 \colon M_0 \to N_0)$$

such that $n \circ \alpha_p = \omega_p^{\mathbb{F}}(\alpha_0) \circ m$. Let $\mathsf{Mot}_{(p)}(\mathbb{F})$ be the category of motives over $\mathbb{F}$ with coefficients in $\mathbb{Z}_{(p)}$, and let $\mathsf{Mot}'_{(p)}(\mathbb{F})$ be the full subcategory of objects M such that M_p is torsion-free. The objects of $\mathsf{Mot}'_{(p)}(\mathbb{F})$ will be called torsion-free.

PROPOSITION 2.3. *With the obvious tensor structure,* $\mathsf{Mot}_{(p)}(\mathbb{F})$ *is an abelian tensor category over $\mathbb{Z}_{(p)}$, and* $\mathsf{Mot}'_{(p)}(\mathbb{F})$ *is a rigid pseudo-abelian tensor subcategory of* $\mathsf{Mot}_{(p)}(\mathbb{F})$. *Moreover:*

(a) the tensor functor $M \rightsquigarrow M_p \colon \mathsf{Mot}_{(p)}(\mathbb{F}) \to \mathsf{Mod}_{W(\mathbb{F})}$ is exact;

(b) there are canonical equivalences of categories

$$\mathsf{Mot}'_{(p)}(\mathbb{F})_{(\mathbb{Q})} \to \mathsf{Mot}_{(p)}(\mathbb{F})_{(\mathbb{Q})} \to \mathsf{Mot}(\mathbb{F});$$

(c) for all torsion-free motives M, N, the cokernel of

$$\mathrm{Hom}(M,N) \otimes_{\mathbb{Z}_{(p)}} W(\mathbb{F}) \to \mathrm{Hom}(M_p, N_p)$$

is torsion-free.

PROOF. These can be proved in the same way as the similar statements in Milne and Ramachandran 2004. □

3 Conjecture LR+

In this section we state a conjecture that both strengthens and simplifies the original conjecture of Langlands and Rapoport.

Throughout this section, (G, X) is a Shimura (p)-datum satisfying (SV1-4,6) — see p8.

Motives with G-structure

By a motive over $\mathbb{F}$ with *G-structure*, or, more briefly, a *G-motive* over $\mathbb{F}$, we mean an exact tensor functor $M\colon \mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Mot}(\mathbb{F})$. We say that two G-motives are *equivalent* if they are isomorphic as tensor functors.

We shall be especially interested in the G-motives for which

$$\omega_{\infty}^{\mathbb{F}} \circ M \approx \mathbb{C} \otimes_{\mathbb{Q}} V, \quad \omega_{\ell}^{\mathbb{F}} \circ M \approx \mathbb{Q}_{\ell} \otimes_{\mathbb{Q}} V, \quad \omega_{p}^{\mathbb{F}} \circ M \approx B(\mathbb{F}) \otimes_{\mathbb{Q}} V, \tag{3}$$

(isomorphisms of fibre functors on $\mathsf{Rep}(G)$). Here V denotes the forgetful fibre functor $\xi \rightsquigarrow V(\xi)$ on $\mathsf{Rep}_{\mathbb{Q}}(G)$, and $R \otimes_{\mathbb{Q}} V$ denotes the fibre functor $\xi \rightsquigarrow R \otimes_{\mathbb{Q}} V(\xi)$.

Integral structures at p

3.1. Let (G, X) be a Shimura p-datum. Recall that the reflex field $E = E(G, X)$ is the field of definition of the $G(\mathbb{C})$-conjugacy class $\mathcal{C}$ of cocharacters of $G_{\mathbb{C}}$ containing μ_x for all $x \in X$. Because $G_{\mathbb{Q}_p}$ admits a hyperspecial subgroup, the prime v in E is unramified, and so the closure E_v of E in $(\mathbb{Q}^{\mathrm{al}})_v$ is a subfield of $B(\mathbb{F})$. Let S be a maximal split subtorus of $G_{B(\mathbb{F})}$ whose apartment in the Bruhat-Tits building of $G_{B(\mathbb{F})}$ contains the hyperspecial vertex stabilized by $G(W(\mathbb{F}))$. Then $\mathcal{C}$ is represented by a cocharacter μ_0 of S defined over $B(\mathbb{F})$ whose orbit under the action of the Weyl group of S is uniquely determined; moreover, the elements of the Weyl group of S are represented by elements in $G(W(\mathbb{F}))$. (For more details and references, see Milne 1994b, pp503–4.)

Recall that $\mathsf{Rep}_{\mathbb{Q}}(G)_{(\mathbb{Q}_l)} \simeq \mathsf{Rep}_{\mathbb{Q}_l}(G)$. Therefore, a G-motive M defines by extension of scalars a functor

$$M(p)\colon \mathsf{Rep}_{\mathbb{Q}_l}(G) \to \mathsf{Mot}(\mathbb{F})_{(\mathbb{Q}_l)}.$$

We sometimes write $M_{(l)}$ for $\omega_l^{\mathbb{F}} \circ M$ (or its extension $\omega_l^{\mathbb{F}} \circ M(l)$ to $\mathsf{Rep}_{\mathbb{Q}_l}(G)$).

Let M be a G-motive such that $M_{(p)} \approx B(\mathbb{F}) \otimes_{\mathbb{Q}} V$.

DEFINITION 3.2. A *p-integral structure* on M is an exact tensor functor $\underline{\Lambda}\colon \mathsf{Rep}_{\mathbb{Z}_p}(G) \to \mathsf{Mod}_{W(\mathbb{F})}$ such that

(a) for all ξ in $\mathsf{Rep}_{\mathbb{Z}_p}(G)$, we have that $\underline{\Lambda}(\xi)$ is a $W(\mathbb{F})$-lattice in $M_{(p)}(\xi_{\mathbb{Q}_p})$, and
(b) there exists an isomorphism $\eta\colon W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}$ of tensor functors such that $\eta_{B(\mathbb{F})}$ maps $\mu_0(p^{-1}) \cdot \Lambda(\xi)_W$ onto $F\underline{\Lambda}(\xi)$ for all ξ in $\mathsf{Rep}_{\mathbb{Z}_p}(G)$.

In (b), Λ denotes the forgetful functor $(\Lambda, \xi) \rightsquigarrow \Lambda$ on $\mathsf{Rep}_{\mathbb{Z}_p}(G)$.

REMARK 3.3. (a) Condition (a) means that $\underline{\Lambda}(\xi) \subset M_{(p)}(\xi_{\mathbb{Q}_p})$ for all ξ, and that there is a commutative diagram

$$\begin{tikzcd}
\mathsf{Rep}_{\mathbb{Q}_p}(G) \arrow[r, "M(p)"] & \mathsf{Mot}(\mathbb{F})_{(\mathbb{Q}_p)} \arrow[r, "\omega_p^{\mathbb{F}}"] & \mathsf{Vec}_{B(\mathbb{F})} \\
\mathsf{Rep}_{\mathbb{Z}_p}(G) \arrow[u, "\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -"] \arrow[rr, "\underline{\Lambda}"] & & \mathsf{Mod}_{W(\mathbb{F})}. \arrow[u, "B \otimes_W -"']
\end{tikzcd}$$

(b) We define a *filtered* $B(\mathbb{F})$*-module* to be an F-isocrystal (N, F) over $\mathbb{F}$ together with a finite filtration

$$N = \mathrm{Filt}^{i_0}(N) \supset \cdots \supset \mathrm{Filt}^{i}(N) \supset \mathrm{Filt}^{i+1}(N) \supset \cdots \supset \mathrm{Filt}^{i_1}(N) = 0$$

on N. In Fontaine's terminology, a W-lattice Λ in N is *strongly divisible* if

$$\sum\nolimits_i p^{-i} F(\mathrm{Filt}^i N \cap \Lambda) = \Lambda,$$

and a filtered $B(\mathbb{F})$-module admitting a strongly divisible lattice is said to be *weakly admissible* (Fontaine 1983, p90). If $\mu\colon \mathbb{G}_m \to \mathrm{GL}(\Lambda)$ splits the filtration on Λ, i.e.,

$$\mathrm{Filt}^j \Lambda = \bigoplus\nolimits_{i \geq j} \Lambda^i, \quad \Lambda^i \stackrel{\text{def}}{=} \{m \in \Lambda \mid \mu(x)m = x^i m, \text{ all } x \in B(\mathbb{F})^\times\},$$

then the condition to be strongly divisible is that $F\Lambda = \mu(p)\Lambda$. The cocharacter μ_0^{-1} of $G_{B(\mathbb{F})}$ constructed in (3.1) defines a filtration on $\mathbb{Q} \otimes \Lambda(\xi)$ for all ξ, and μ_0 has been chosen so that μ_0^{-1} splits the filtration on $\Lambda(\xi)$ for all ξ. Thus condition (b) for $\underline{\Lambda}$ to be a p-integral structure on M can be restated as:

> there exists an isomorphism $\eta\colon W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}$ of tensor functors such that, for all $\xi \in \mathsf{Rep}_{\mathbb{Z}_p}(G)$, $\underline{\Lambda}(\xi)$ is strongly divisible for the filtration on $B(\mathbb{F}) \otimes_W \underline{\Lambda}(\xi)$ defined (via η) by μ_0^{-1}.

The set $\mathcal{L}(M)$

Let M be a G-motive satisfying the condition (3), p16. For a p-structure $\underline{\Lambda}$ on M, define $\Phi\underline{\Lambda}$ to be the p-structure such that

$$(\Phi\underline{\Lambda})(\xi) = F^{[k(v):\mathbb{F}_p]} \cdot \underline{\Lambda}(\xi), \quad \text{for all } \xi \in \mathsf{Rep}_{\mathbb{Z}_p}(G).$$

Here $k(v)$ is the residue field at the prime v of E (so $[k(v):\mathbb{F}_p] = [E_v:\mathbb{Q}_p]$).

Define

$$\begin{aligned}
I(M) &= \mathrm{Aut}^{\otimes}(M),\\
X^p(M) &= \mathrm{Isom}^{\otimes}(\mathbb{A}_f^p \otimes_{\mathbb{Q}} V, \omega_f^p \circ M),\\
X_p(M) &= \{p\text{-integral structures on } M\}.
\end{aligned}$$

The group $I(M)$ acts on both $X^p(M)$ and $X_p(M)$ on the left, and so we can define

$$\mathcal{L}(M) = I(M)\backslash\big(X^p(M)/Z^p\big) \times X_p(M)\,.$$

Here Z^p is the closure of $Z(\mathbb{Z}_{(p)})$ in $Z(\mathbb{A}_f^p)$ where $Z = Z(G)$. The group $G(\mathbb{A}_f^p)$ acts on $X^p(M)$ through its action on $\mathbb{A}_f^p \otimes_{\mathbb{Q}} V$, and we let it act on $\mathcal{L}(M)$ through its action on $X^p(M)$. We let $Z(\mathbb{Q}_p)$ and Φ act $\mathcal{L}(M)$ through their actions on $X_p(M)$.

An isomorphism $M \to M'$ of G-motives defines an equivariant bijection $a_{M',M}\colon\mathcal{L}(M) \to \mathcal{L}(M')$ which is independent of the choice of the isomorphism. For an equivalence class m of G-motives, we define

$$\mathcal{L}(m) = \varprojlim_{M \in m} \mathcal{L}(M).$$

This is a set with actions of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$ and of a Frobenius operator Φ; it is equipped with equivariant isomorphisms $a_M\colon\mathcal{L}(m) \to \mathcal{L}(M)$ such that $a_{M',M} \circ a_M = a_{M'}$ for all M, $M' \in m$.

Special G-motives

Recall that $\mathsf{Hdg}_{\mathbb{Q}}$ is the category of polarizable Hodge structures over $\mathbb{Q}$. A point x of X defines an exact tensor functor

$$H_x\colon \mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Hdg}_{\mathbb{Q}}, \quad \xi \rightsquigarrow (V(\xi), \xi_{\mathbb{R}} \circ h_x).$$

When x is special, H_x takes values in the full subcategory of $\mathsf{Hdg}_{\mathbb{Q}}$ whose objects are the rational Hodge structures of CM-type (because we are assuming (SV4) and (SV6) — see the notations). This subcategory is equivalent (via ω_B) to $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$. Fix a tensor inverse $\mathsf{Hdg}_{\mathbb{Q}} \to \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ to ω_B. On composing H_x with it, we obtain a tensor functor $M_x\colon \mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ together with an isomorphism $\omega_B \circ M_x \simeq H_x$. Any G-motive equivalent to $R \circ M_x$ for some special $x \in X$ will be called *special.*

LEMMA 3.4. *Every special G-motive $M\colon \mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Mot}(\mathbb{F})$ satisfies the condition (3), p16.*

PROOF. We consider only the case $l = p$ since the other cases are easier. As the statement depends only on the isomorphism class of M, we may assume that $M = R \circ M_x$ with x a special point of X.

The reduction functor $R\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}(\mathbb{F})$ has the property that

$$(\mathbb{Q}^{\mathrm{al}})_v \otimes_{B(\mathbb{F})} (\omega_p^{\mathbb{F}} \circ R) \simeq (\mathbb{Q}^{\mathrm{al}})_v \otimes_{\mathbb{Q}^{\mathrm{al}}} \omega_{\mathrm{dR}}$$

where ω_{dR} denotes the de Rham fibre functor on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$. On composing both sides with M_x, we find that

$$(\mathbb{Q}^{\mathrm{al}})_v \otimes_{B(\mathbb{F})} (\omega_p^{\mathbb{F}} \circ M) \simeq (\mathbb{Q}^{\mathrm{al}})_v \otimes_{\mathbb{Q}^{\mathrm{al}}} (\omega_{\mathrm{dR}} \circ M_x).$$

There is a comparison isomorphism

$$\omega_{\mathrm{dR}} \to \mathbb{Q}^{\mathrm{al}} \otimes_{\mathbb{Q}} \omega_B,$$

and the definition of M_x gives an isomorphism

$$\omega_B \circ M_x \to V.$$

On combining these isomorphisms, we obtain an isomorphism of tensor functors

$$(\mathbb{Q}^{\mathrm{al}})_v \otimes_{B(\mathbb{F})} M_{(p)} \to (\mathbb{Q}^{\mathrm{al}})_v \otimes_{\mathbb{Q}} V.$$

It remains to show that we can replace $(\mathbb{Q}^{\mathrm{al}})_v$ with $B(\mathbb{F})$ in this statement.

Consider the functor of $B(\mathbb{F})$-algebras

$$R \rightsquigarrow \underline{\mathrm{Isom}}^{\otimes}(R \otimes_{\mathbb{Q}} V, R \otimes_{B(\mathbb{F})} M_{(p)}).$$

This is a pseudo-torsor for $\underline{\mathrm{Aut}}^{\otimes}(B(\mathbb{F}) \otimes_{\mathbb{Q}} V) = G_{B(\mathbb{F})}$ and, in fact, a torsor because it has a $(\mathbb{Q}^{\mathrm{al}})_v$-point. It therefore defines an element of $H^1(B(\mathbb{F}), G)$. As the field $B(\mathbb{F})$ has dimension ≤ 1 and G is connected, $H^1(B(\mathbb{F}), G) = 0$ (Steinberg 1965, 1.9), and so the torsor is trivial. □

In particular, when M is special, the set $\mathcal{L}(M)$ is defined.

Shimura varieties of dimension zero

By a *zero-dimensional Shimura p-datum*, we mean a pair (T, X) in which T is a torus over $\mathbb{Z}_{(p)}$ and X is a finite set of homomorphisms $\mathbb{S} \to T_{\mathbb{R}}$ on which $T(\mathbb{R})/T(\mathbb{R})^+$ acts transitively. Consistent with our standing assumptions for Shimura data, we require (except in §6) that the weights of the elements of X are defined over $\mathbb{Q}$ and that T splits over $\mathbb{Q}^{\mathrm{cm}}$. Then, as in the preceding subsection, each element x of X defines a T-motive M_x satisfying the condition (3), p16, to which we can attach a set $\mathcal{L}(M_x)$ with an action of $T(\mathbb{A}_f)$ and of Φ.

The zero-dimensional Shimura variety attached to (T, X) is as defined in Milne 2005, §5. Every Shimura p-datum (G, X) defines a zero-dimensional Shimura p-datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, and $\mathrm{Sh}_p(G^{\mathrm{ad}}, X^{\mathrm{ad}})(\mathbb{C}) = \pi_0(\mathrm{Sh}_p(G, X))$ when G^{der} is simply connected (ibid.).

From now on "Shimura p-datum" will mean either a Shimura p-datum, as defined in the Introduction, or a zero-dimensional Shimura p-datum as just defined.

Statement of Conjecture LR+

Define

$$\mathcal{L}(G, X) = \bigsqcup_m \mathcal{L}(m) \tag{4}$$

where m runs over the set of equivalence classes of special G-motives. Then $\mathcal{L}(G, X)$ is a set with an action of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p) \times \{\Phi\}$, and

$$(G, X) \rightsquigarrow \mathcal{L}(G, X)$$

is a functor from the category of Shimura p-data to the category of sets with a Frobenius operator.

Let E_v be the closure of E in $(\mathbb{Q}^{\mathrm{al}})_v$ and let $\mathcal{O}_v$ be its ring of integers. Let $\mathrm{Sh}_p(G, X)$ be the canonical integral model over $\mathcal{O}_v$ (in the sense of Milne 1992)[12] of the Shimura variety with complex points

$$G(\mathbb{Q})\backslash \left(X \times G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \right).$$

From its definition, Sh_p is uniquely determined, and it is known to exist except possibly for $p = 2$ (Vasiu 1999, 2008a,b, Kisin 2007, 2009). Write $\mathrm{Sh}_p(\mathbb{F})$ for the functor

$$(G, X) \rightsquigarrow \mathrm{Sh}_p(G, X)(\mathbb{F})$$

from the category of Shimura p-data to the category of sets with a Frobenius operator.

[12] Because of an error in Faltings and Chai 1990, V 6.8, the definition in Milne 1992 needs to be slightly modified — see Milne 1994b, p513, and Moonen 1998, §3.

CONJECTURE LR+ 3.5. *There exists a canonical isomorphism of functors*

$$\mathcal{L} \to \mathrm{Sh}_p(\mathbb{F})$$

such that, for each Shimura p-datum (G,X), the isomorphism

$$\mathcal{L}(G,X) \to \mathrm{Sh}_p(G,X)(\mathbb{F})$$

is equivariant for the actions of $G(\mathbb{A}_f^p)$ and $Z(\mathbb{Q}_p)$.

In symbols:

$$\begin{array}{ccccc} (G,X) & & \mathcal{L}(G,X) & \overset{1:1}{\leftrightarrow} & \mathrm{Sh}_p(G,X)(\mathbb{F}) \\ \downarrow & \rightsquigarrow & \downarrow & \square & \downarrow \\ (G',X') & & \mathcal{L}(G',X') & \overset{1:1}{\leftrightarrow} & \mathrm{Sh}_p(G',X')(\mathbb{F}) \end{array}$$

In fact, one can show that, for Shimura varieties of abelian type, there exists at most one isomorphism of functors $\mathcal{L} \to \mathrm{Sh}_p(\mathbb{F})$ taking a specific value on Siegel varieties and zero-dimensional Shimura varieties.

Since $\mathcal{L}(G,X)$ includes terms corresponding only to special homomorphisms, Conjecture 3.5 forces the following conjecture.

SPECIAL-POINTS CONJECTURE 3.6. *Up to isogeny, every point on $\mathrm{Sh}_p(G,X)$ with coordinates in $\mathbb{F}$ lifts to a special point on $\mathrm{Sh}_p(G,X)$ with coordinates in a finite extension of $B(\mathbb{F})$.*

Recall that, from the definition of the canonical integral model and Hensel's lemma,

$$\mathrm{Sh}_p(B(\mathbb{F})) \overset{1:1}{\leftrightarrow} \mathrm{Sh}_p(W(\mathbb{F})) \xrightarrow{\text{onto}} \mathrm{Sh}_p(\mathbb{F}).$$

The special-points conjecture is proved in Zink 1983, 2.7, for simple abelian varieties of PEL-type;[13] [14] more general results have been announced by Vasiu (2003a). It should be noted that the special-points conjecture is false in the case of bad reduction (Langlands and Rapoport 1987).

[13] Let L be a simple finite-dimensional algebra over $\mathbb{Q}$ with a positive involution $x \mapsto x^*$, and let p be a prime number such that

- $L \otimes \mathbb{Q}_p$ is a product of matrix algebras,
- p is unramified in the centre Z of L, and
- $(\mathcal{O}_L \otimes \mathbb{Z}_p)^* = (\mathcal{O}_L \otimes \mathbb{Z}_p)$.

Let k be a field of characteristic p, and let (A,λ) be a polarized abelian L-variety over k such that

$$\mathrm{Tr}_{\mathbb{C}}(l \mid \mathrm{Lie}(A)) = \sum_{\rho: Z \to \mathbb{C}} r_\rho \cdot \rho(\mathrm{Trd}(l)), \quad l \in L, \qquad (*)$$

for some fixed integers r_ρ. Assume that the degree of λ is prime to p. Then Zink proves the following:

> Let R be a product of CM-fields and let $\theta_A\colon R \to \mathrm{End}_L^0(A)$ be a homomorphism such that $\theta_A(R)$ is stable under the Rosati involution and
>
> $$\dim_{\mathbb{Q}}(R) = 2\dim A/[L{:}Z]^{1/2}$$
>
> (that is, A admits complex multiplication (R,θ_A) relative to L). Then there exists a discrete valuation ring $\mathcal{O}$ that is a finite extension of $W(k)$ and a polarized abelian $\mathcal{O}_L$-variety $(\tilde{A},\tilde{\lambda})$ with complex multiplication $(R,\theta_{\tilde{A}})$ over $\mathcal{O}$ satisfying (*) whose reduction is isogenous to (A,λ,R,θ_A).

[14] According to Zink (1983, p103), in this case the result was originally stated in a letter from Langlands to Rapoport, but the proof there, which is based on the methods of Grothendieck and Messing, is incorrect. (Dieses Resultat wurde in einem Brief von Langlands an Rapoport behauptet. Der Beweis dort, der auf Methoden von Grothendieck und Messing beruht ist aber fehlerhaft.)

Example: the Shimura variety of $\mathbb{G}_m$

In order to check our signs, we prove the conjecture for a Shimura p-datum (G,X) with G a one-dimensional split torus over $\mathbb{Z}_{(p)}$. Any such Shimura p-datum is of the form $(\mathbb{G}_m,\{h_n\})$ where h_n is the nth power of the norm map $\mathbb{S}\to\mathbb{G}_{m\mathbb{R}}$; thus $h_n(z)=(z\bar{z})^n$ for $z\in\mathbb{C}$. The cocharacter μ_n of $\mathbb{G}_m$ attached to h_n is $z\mapsto z^n$.

The maps

$$\mathrm{Sh}_p(\mathbb{C})\leftarrow\mathrm{Sh}_p(\mathbb{Q}^{\mathrm{al}})\to\mathrm{Sh}_p((\mathbb{Q}^{\mathrm{al}})_v)\leftarrow\mathrm{Sh}_p(B(\mathbb{F}))\leftarrow\mathrm{Sh}_p(W(\mathbb{F}))\to\mathrm{Sh}_p(\mathbb{F})$$

are bijective because Sh_p is of dimension zero and pro-étale over $\mathbb{Z}_p$. Therefore

$$\mathrm{Sh}_p(\mathbb{F})\simeq\mathbb{Q}^\times\backslash\mathbb{A}_f^p\times\mathbb{Q}_p^\times/\mathbb{Z}_p^\times.$$

The Frobenius automorphism $x\mapsto x^p$ of $\mathbb{F}$ acts on this as multiplication by p^{-n} on $\mathbb{Q}_p^\times$ (see Milne 1992, §1).

Let ξ denote a one-dimensional representation of $\mathbb{G}_m$ with character $x\mapsto x$. Up to equivalence, the only special $\mathbb{G}_m$-motive M over $\mathbb{F}$ is that with $M(\xi)=\mathbb{1}(n)$, the nth tensor power of the Tate motive. We have

$$\begin{aligned}I(M)&=\mathrm{Aut}(\mathbb{1}(n))\simeq\mathbb{Q}^\times,\\ X^p(M)&=\mathrm{Hom}(\mathbb{A}_f^p\otimes V(\xi),\mathbb{A}_f^p(n)),\end{aligned}$$

and $X_p(M)$ is equal to the set of lattices Λ in $\omega_p(\mathbb{1}(n))$ that are strongly divisible for the filtration defined by μ_n^{-1}. The Frobenius operator Φ acts by sending Λ to $F\Lambda$.

As an $\mathbb{A}_f^p$-module, $\mathbb{A}_f^p(n)=\mathbb{A}_f^p$, and so the choice of a basis element for $V(\xi)$ determines a bijection $X^p(M)\to\mathbb{A}_f^p$.

The isocrystal $\omega_p(\mathbb{1}(n))$ is $B(\mathbb{F})$ with F acting as $p^{-n}\sigma$. Every strongly divisible lattice Λ in $\omega_p(\mathbb{1}(n))$ arises from a lattice Λ_0 in $\mathbb{Q}_p$ (cf. Wintenberger 1984, 4.2.5(1)). Specifically, to say that Λ is strongly divisible means that $\mu_n(p^{-1})\Lambda=F\Lambda$, i.e., $p^{-n}\Lambda=p^{-n}\sigma\Lambda$; therefore $\Lambda=\sigma\Lambda$, and so $\Lambda^{\sigma=1}$ is a lattice in $\mathbb{Q}_p$. Let $\Lambda^{\sigma=1}=a(\Lambda)\cdot\mathbb{Z}_p$ with $a(\Lambda)\in\mathbb{Q}_p$; then $\Lambda\mapsto a(\Lambda)$ defines a bijection $X_p(M)\to\mathbb{Q}_p/\mathbb{Z}_p$ under which the Frobenius maps correspond.

This completes the proof of Conjecture 3.5 for $(\mathbb{G}_m,\{h_n\})$.

A criterion for a G-motive to be special

We begin by reviewing some constructions.

3.7. A point x of X defines an exact tensor functor

$$\xi\rightsquigarrow(V(\xi)_{\mathbb{R}},\xi_{\mathbb{R}}\circ h_x)\colon\mathsf{Rep}_{\mathbb{Q}}(G)\to\mathsf{Hdg}_{\mathbb{R}},$$

and hence a G-object N_x of R_∞. If $x'=gx$ with $g\in G(\mathbb{R})$, then g defines an isomorphism $N_x\to N_{x'}$, and so the equivalence class of N_x depends only on X.

3.8. For any torus T split by $\mathbb{Q}^{\mathrm{cm}}$ and cocharacter μ with weight $-(\iota+1)\mu$ defined over $\mathbb{Q}$, there is a unique homomorphism $S\to T$ sending μ^S to μ (universal property of (S,μ^S)). This homomorphism defines a functor $\mathsf{Rep}(T)\to\mathsf{Rep}(S)\simeq\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$,i.e., a T-object $N(T,\mu)$ of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$. For example, from (G,X), we get a G^{ab}-object $N(G^{\mathrm{ab}},\mu_X)$ in $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ where μ_X is the common composite of μ_x with $G\to G^{\mathrm{ab}}$ for $x\in X$.

PROPOSITION 3.9. *When G^{der} is simply connected, a G-motive $M\colon \mathsf{Rep}_{\mathbb{Q}}(G)\to \mathsf{Mot}(\mathbb{F})$ is special if and only if it satisfies the following conditions:*

(a) the $G_{\mathbb{R}}$-object $M_{(\infty)}$ is equivalent to N_x ($x\in X$);

(b) for all $\ell\neq p,\infty$, the $G_{\mathbb{Q}_\ell}$-object $M_{(\ell)}$ is equivalent to $\mathbb{Q}_\ell\otimes_{\mathbb{Q}} V$ (here V is the forgetful functor $\xi\mapsto V(\xi)$);

(c) the tensor functors $M_{(p)}$ and $B(\mathbb{F})\otimes_{\mathbb{Q}} V$ are equivalent, and there exists a p-integral structure on M;

(d) the G^{ab} object in $\mathsf{Mot}(\mathbb{F})$ obtained from M by restriction is equivalent to $N(G^{\mathrm{ab}},\mu_X)$.

PROOF. The proof is similar to that of Langlands and Rapoport 1987, 5.3, p173. □

Re-statement of the conjecture in terms of motives with $\mathbb{Z}_{(p)}$-coefficients

By a *G-motive over $\mathbb{F}$ with coefficients in $\mathbb{Z}_{(p)}$*, we mean an exact tensor functor

$$M\colon \mathsf{Rep}(G)\to \mathsf{Mot}_{(p)}(\mathbb{F}),\quad \xi \rightsquigarrow (M_p(\xi),M_0(\xi),m(\xi)).$$

We say that M is *admissible* if the image of $m(\xi)$ is a p-structure on the G-motive $\xi\rightsquigarrow M_0(\xi)$. For an admissible M, let

$$\begin{aligned} I(M) &= \mathrm{Aut}^{\otimes}(M) \qquad (=\mathrm{Aut}^{\otimes}(M_0)\cap \mathrm{Aut}^{\otimes}(M_p)),\\ X^p(M) &= X^p(M_0). \end{aligned}$$

The group $I(M)$ acts on $X^p(M)$, and so we can define

$$\mathcal{L}(M)=I(M)\backslash X^p(M)/Z^p.$$

An admissible G-motive M over $\mathbb{F}$ with coefficients in $\mathbb{Z}_{(p)}$ is special if M_0 is special. For an equivalence class m of special admissible G-motives with coefficients in $\mathbb{Z}_{(p)}$, define

$$\mathcal{L}(m)=\varprojlim_{M\in m}\mathcal{L}(M).$$

PROPOSITION 3.10. *We have*

$$\mathcal{L}(G,X)\simeq \bigsqcup\nolimits_m \mathcal{L}(m) \tag{5}$$

where m runs over the set of equivalence classes of special admissible G-motives with coefficients in $\mathbb{Z}_{(p)}$.

More precisely, let $M=(M_p,M_0,m)$ be a special $\mathbb{Z}_{(p)}$-motive over $\mathbb{F}$ with G-structure. There is an obvious map $\mathcal{L}(M)\to\mathcal{L}(M_0)$, and these maps induce an isomorphism (5). The proof of this is straightforward using that $G(\mathbb{Q}_p)=G(\mathbb{Q})\cdot G(\mathbb{Z}_p)$ (Milne 1994b, 4.9).

4 The functor of points defined by a Shimura variety

Throughout this section (G,X) is a Shimura p-datum such that $(G_{\mathbb{Q}},X)$ is of abelian type in the sense of Milne and Shih 1982, §1.[15] Moreover, we assume that $\mathrm{ad}\,h_x(i)$ is a Cartan involution on

[15]Recall that a Shimura datum (G,X) is said to be of *abelian type* if there exists an isogeny $H\to G^{\mathrm{der}}$ with H a product of almost-simple groups H_i such that either

(a) H_i is simply connected of type A, B, C, or $D^{\mathbb{R}}$, or,

(b) H_i is of type $D_n^{\mathbb{H}}$ ($n\geq 5$) and equals $\mathrm{Res}_{F/\mathbb{Q}}H'$ for F a totally real field and H' is a form of $\mathrm{SO}(2n)$ (double covering of the adjoint group).

$G_{\mathbb{R}}/w_X(\mathbb{G}_m)$ for one (hence all) $x \in X$, and we fix a homomorphism $t\colon G_{\mathbb{Q}} \to \mathbb{G}_m$ (assumed to exist) such that $t \circ w_X = -2$. Recall that the condition on $\operatorname{ad} h_x(i)$ implies that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$ and that $Z(\mathbb{Z}_{(p)})$ is discrete in $Z(\mathbb{A}_f^p)$ (e.g., Milne 2005, 5.26).

As before, Sh_p denotes the canonical integral model of $\mathrm{Sh}(G,X)$ over $\mathcal{O}_v$ (see p19).

For a field k of characteristic zero, $\mathsf{Mot}(k)$ denotes the category of motives over k based on abelian varieties and using the Hodge classes as correspondences. It is a semisimple tannakian category over $\mathbb{Q}$ whose objects are the abelian motives over k. We shall simply call them motives. A G-motive over k is an exact tensor functor $\mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Mot}(k)$.

When $k = \mathbb{C}$, Betti cohomology defines a $\mathbb{Q}$-valued fibre functor ω_B. Etale cohomology defines fibre functors $\omega_l\colon \mathsf{Mot}(k) \to \mathsf{Vec}_{\mathbb{Q}_l}$ for all primes $l \neq \infty$, and an exact tensor functor $\omega_f^p\colon \mathsf{Mot}(k) \to \mathsf{Mod}_{\mathbb{A}_f^p}$. Strictly speaking, these depend on the choice of an algebraic closure k^{al} of k, and $\omega_l(M)$ is a $\mathrm{Gal}(k^{\mathrm{al}}/k)$-module for each M.

Definition of the functor $\mathcal{M}$

Admissible G-motives. Let E be the reflex field $E(G,X)$ of (G,X), and let k be a field containing E. As before, a point x of X defines an exact tensor functor

$$H_x\colon \mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Hdg}_{\mathbb{Q}}, \quad \xi \rightsquigarrow (V(\xi), \xi_{\mathbb{R}} \circ h_x).$$

An E-homomorphism $\tau\colon k \to \mathbb{C}$ defines an exact tensor functor

$$\omega_\tau\colon \mathsf{Mot}(k) \to \mathsf{Hdg}_{\mathbb{Q}}, \quad X \mapsto \omega_B(\tau X).$$

We say that a G-motive M over k is *admissible with respect to* τ if $\omega_\tau \circ M$ is isomorphic to H_x for some $x \in X$.

PROPOSITION 4.1. *If M is admissible with respect to one E-homomorphism $k \to \mathbb{C}$, then it is admissible with respect to every E-homomorphism $k \to \mathbb{C}$.*

PROOF. The proof is the same as that of Milne 1994b, 3.29. □

We say that a G-motive M over a field k containing E is *admissible* if there exists a G-motive M_0 over a subfield k_0 of k containing E such that

- ◇ M_0 gives rise to M by extension of scalars and
- ◇ M_0 is admissible with respect to some homomorphism $k_0 \to \mathbb{C}$.

$$\begin{tikzcd}
M \arrow[r, dash] \arrow[d, dash] & M_0 \arrow[r, dash] \arrow[d, dash] & H_x \arrow[d, dash] \\
k & k_0 \arrow[l] \arrow[r] & \mathbb{C} \\
& E \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Etale p-integral structures. Let k be a field containing E, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$ for some algebraic closure k^{al} of k. For a G-motive M over k, let $M(p)\colon \mathsf{Rep}_{\mathbb{Q}_p}(G) \to \mathsf{Mot}(k)_{(\mathbb{Q}_p)}$ be the exact tensor functor obtained by extension of scalars $\mathbb{Q} \to \mathbb{Q}_p$.

DEFINITION 4.2. An *étale p-integral structure* on a G-motive $M\colon \mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Mot}(k)$ is an exact tensor functor $\underline{\Lambda}_p\colon \mathsf{Rep}_{\mathbb{Z}_p}(G_p) \to \mathsf{Rep}_{\mathbb{Z}_p}(\Gamma)$ such that, for all ξ in $\mathsf{Rep}_{\mathbb{Z}_p}(G)$, $\underline{\Lambda}_p(\xi)$ is a $\mathbb{Z}_p$-lattice in $(\omega_p \circ M(p))(\xi_{\mathbb{Q}_p})$.

The condition means that $\underline{\Lambda}_p(\xi) \subset (\omega_p \circ M(p))(\xi_{\mathbb{Q}_p})$ for all ξ, and there is a commutative diagram

$$\begin{tikzcd}
\mathsf{Rep}_{\mathbb{Q}_p}(G) \arrow[r, "M(p)"] & \mathsf{Mot}(\mathbb{F})_{(\mathbb{Q}_p)} \arrow[r, "\omega_p"] & \mathsf{Rep}_{\mathbb{Q}_p}(\Gamma) \\
\mathsf{Rep}_{\mathbb{Z}_p}(G) \arrow[u, "\mathbb{Q}_p\otimes_{\mathbb{Z}_p}-"'] \arrow[rr, "\underline{\Lambda}_p"] & & \mathsf{Rep}_{\mathbb{Z}_p}(\Gamma). \arrow[u, "\mathbb{Q}_p\otimes_{\mathbb{Z}_p}-"']
\end{tikzcd}$$

LEMMA 4.3. *For every exact tensor functor $\underline{\Lambda}_p\colon \mathsf{Rep}_{\mathbb{Z}_p}(G) \to \mathsf{Rep}_{\mathbb{Z}_p}(\Gamma)$ there exists an isomorphism $\Lambda \to \omega_{\text{forget}} \circ \underline{\Lambda}_p$ of tensor functors. Here Λ denotes the forgetful functor $\xi \mapsto \Lambda(\xi)\colon \mathsf{Rep}_{\mathbb{Z}_p}(G) \to \mathsf{Mod}_{\mathbb{Z}_p}$.*

PROOF. Apply (1.1). □

The functor $\mathcal{M}$. Let k be a field containing E. For an admissible G-motive M over k, define

$$\begin{aligned}
I(M) &= \mathrm{Aut}^{\otimes}(M), \\
X^p(M) &= \mathrm{Isom}^{\otimes}(\mathbb{A}_f^p \otimes V, \omega_f^p \circ M), \\
X_p(M) &= \{\text{étale } p\text{-integral structures on } M\}.
\end{aligned}$$

Then $I(M)$ acts on $X^p(M)$ and $X_p(M)$ on the left, $G(\mathbb{A}_f^p)$ acts on $X^p(M)$ on the right, and $Z_p(\mathbb{Q}_p)$ acts on $X_p(M)$. We define

$$\mathcal{M}(M) = I(M)\backslash X^p(M) \times X_p(M)$$

regarded as a $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$-set. An isomorphism $M \to M'$ defines an equivariant bijection $\mathcal{M}(M) \to \mathcal{M}(M')$ which is independent of the isomorphism. For an equivalence class m of G-motives over k, we define

$$\mathcal{M}(m) = \varprojlim_{M \in m} \mathcal{M}(M).$$

Define

$$\mathcal{M}(G,X)(k) = \bigsqcup\nolimits_m \mathcal{M}(m)$$

where m runs over the equivalence classes of admissible G-motives. This is a set with an action of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$. For a fixed k, it is a functor from the category of Shimura p-data to the category of sets, and, for a fixed (G,X), it is a functor from the category of E-fields to the category of sets endowed with an action of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$.

The points of Sh_p with coordinates in the complex numbers

With our assumptions

$$\begin{aligned}
\mathrm{Sh}_p(\mathbb{C}) &\simeq G(\mathbb{Z}_{(p)})\backslash X \times G(\mathbb{A}_f^p) \\
&\simeq G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f^p) \times (G(\mathbb{Q}_p)/G(\mathbb{Z}_p))
\end{aligned}$$

(Milne 1994b, 4.11, 4.12).

Let M be an admissible G-motive over $\mathbb{C}$, and let $(\eta, \underline{\Lambda}_p) \in X^p(M) \times X_p(M)$. Because M is admissible, there exists an isomorphism of tensor functors $\beta: \omega_B \circ M \to V$ sending h_M to h_x for some $x \in X$. When we tensor this with $\mathbb{A}_f^p$ and compose with η,

$$\begin{tikzcd} \mathbb{A}_f^p \otimes_{\mathbb{Q}} V \arrow[r, "\eta"'] \arrow[rrr, bend left, "g(\eta)"] & \omega_f^p \circ M \arrow[r, "\simeq"] & \mathbb{A}_f^p \otimes_{\mathbb{Q}} (\omega_B \circ M) \arrow[r, "\mathbb{A}_f^p \otimes_{\mathbb{Q}} \beta"'] & \mathbb{A}_f^p \otimes_{\mathbb{Q}} V, \end{tikzcd},$$

we get an element of $\underline{\mathrm{Aut}}^{\otimes}(\mathbb{A}_f^p \otimes_{\mathbb{Q}} V) \simeq G(\mathbb{A}_f^p)$. The p-integral structure $\underline{\Lambda}_p$ is transformed by β into a p-integral structure on the forgetful functor $V: \mathsf{Rep}_{\mathbb{Q}_p}(G) \to \mathsf{Vec}_{\mathbb{Q}_p}$, and it follows from (1.1) there exists a $g = g(\underline{\Lambda}_p) \in G(\mathbb{Q}_p)$ such that the isomorphism

$$\beta(\xi): (\omega_p \circ M(p))(\xi_{\mathbb{Q}_p}) \longrightarrow V(\xi_{\mathbb{Q}_p})$$

maps $\underline{\Lambda}_p(\xi)$ onto $g \cdot \Lambda(\xi)$ for all ξ in $\mathsf{Rep}_{\mathbb{Z}_p}(G)$. Since β is uniquely determined up to an element of $G(\mathbb{Q})$, we get a well-defined map

$$[\eta, \underline{\Lambda}_p] \mapsto [x, \beta \circ \eta, g(\underline{\Lambda}_p)]: \mathcal{M}(M) \to \mathrm{Sh}_p(\mathbb{C}). \tag{6}$$

One checks the following statement as in Milne 1994b, 4.14: for each equivalence class m of admissible G-motives, the maps (6) define an injective map $\mathcal{M}(m) \stackrel{\mathrm{def}}{=} \varprojlim_{m \in M} \mathcal{M}(M) \to \mathrm{Sh}_p(\mathbb{C})$, and the images of the maps $\mathcal{M}(m) \to \mathrm{Sh}_p(\mathbb{C})$ for distinct m are disjoint and cover $\mathrm{Sh}_p(\mathbb{C})$. In other words, the following is true.

PROPOSITION 4.4. *The above maps define a $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$-equivariant bijection*

$$\alpha(\mathbb{C}): \mathcal{M}(G, X)(\mathbb{C}) \to \mathrm{Sh}_p(G, X)(\mathbb{C}).$$

The points of Sh_p with coordinates in a field of characteristic zero

The next result is a restatement of Milne 1994b, 3.13.

THEOREM 4.5. *Let k be a field containing E. For any E-homomorphism $\tau: k \to \mathbb{C}$, the restriction of $\alpha(\mathbb{C})$ to $\mathcal{M}(k)$ factors through $\mathrm{Sh}_p(k)$:*

$$\begin{array}{ccc} \mathcal{M}(\mathbb{C}) & \xrightarrow{\alpha(\mathbb{C})} & \mathrm{Sh}_p(\mathbb{C}) \\ \uparrow \tau & & \cup \uparrow \tau \\ \mathcal{M}(k) & \xrightarrow{\alpha(k)} & \mathrm{Sh}_p(k) \end{array}$$

The map $\alpha(k)$ is a bijection which is independent of τ, and it is equivariant for the actions of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$ and $\mathrm{Gal}(k^{\mathrm{al}}/k)$.

For a fixed k, $\alpha(k)$ is an isomorphism of functors from the category of Shimura p-data to sets, and, for a fixed (G, X), it is an isomorphism of functors from the category of E-fields to the category of sets with an action of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$.

The category of motives over a field of characteristic zero with $\mathbb{Z}_{(p)}$-cofficients

It is, perhaps, more natural to state Theorem 4.5 in terms of motives with $\mathbb{Z}_{(p)}$-coefficients. Let k be a field of characteristic zero.

A *motive M over k with coefficients in* $\mathbb{Z}_{(p)}$ is a triple (M_p, M_0, m) consisting of

(a) a finitely generated $\mathbb{Z}_p$-module M_p equipped with a continuous action of $\mathrm{Gal}(k^{\mathrm{al}}/k)$,
(b) an object M_0 of $\mathsf{Mot}(\mathbb{F})$, and
(c) an isomorphism $m\colon (M_p)_{\mathbb{Q}} \to \omega_p^{\mathbb{F}}(M_0)$.

A morphism $\alpha\colon M \to N$ of motives with $\mathbb{Z}_{(p)}$-coefficients is a pair of morphisms

$$(\alpha_p\colon M_p \to N_p, \alpha_0\colon M_0 \to N_0)$$

such that $n \circ \alpha_p = \omega_p^{\mathbb{F}}(\alpha_0) \circ m$. Let $\mathsf{Mot}_{(p)}(k)$ be the category of motives over k with coefficients in $\mathbb{Z}_{(p)}$, and let $\mathsf{Mot}'_{(p)}(k)$ be the full subcategory of objects M such that M_p is torsion-free. The objects of $\mathsf{Mot}'_{(p)}(k)$ will be called torsion-free.

PROPOSITION 4.6. *With the obvious tensor structure,* $\mathsf{Mot}_{(p)}(k)$ *is an abelian tensor category over* $\mathbb{Z}_{(p)}$*, and* $\mathsf{Mot}'_{(p)}(k)$ *is a rigid pseudo-abelian tensor subcategory of* $\mathsf{Mot}_{(p)}(k)$*. Moreover:*

(a) the tensor functor $M \rightsquigarrow M_p\colon \mathsf{Mot}_{(p)}(\mathbb{F}) \to \mathsf{Mod}_{\mathbb{Z}_p}$ *is exact;*
(b) there are canonical equivalences of categories

$$\mathsf{Mot}'_{(p)}(k)_{\mathbb{Q}} \to \mathsf{Mot}_{(p)}(k)_{\mathbb{Q}} \to \mathsf{Mot}(k);$$

(c) for all torsion-free motives M, N, the cokernel of

$$\mathrm{Hom}(M, N) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \to \mathrm{Hom}(M_p, N_p)$$

is torsion-free.

PROOF. Routine. □

By a *G-motive over k with coefficients in* $\mathbb{Z}_{(p)}$, we mean an exact tensor functor

$$M\colon \mathsf{Rep}(G) \to \mathsf{Mot}_{(p)}(\mathbb{F}), \qquad \xi \rightsquigarrow (M_p(\xi), M_0(\xi), m(\xi)).$$

We say that M is *admissible* if $\xi \rightsquigarrow M_0(\xi)$ is admissible and the image of $m(\xi)$ is an étale p-integral structure on M_0. For an admissible M, let

$$I(M) = \mathrm{Aut}^{\otimes}(M), \text{ and}$$
$$X^p(M) = X^p(M_0).$$

The group $I(M)$ acts on $X^p(M)$, and so we can define

$$\mathcal{L}(M) = I(M)\backslash X^p(M).$$

For an equivalence class m of admissible G-motives over k with coefficients in $\mathbb{Z}_{(p)}$, define

$$\mathcal{L}(m) = \varprojlim_{M \in m} \mathcal{L}(M).$$

Then

$$\mathcal{L}(G, X) \simeq \bigsqcup\nolimits_m \mathcal{L}(m)$$

where m runs over the equivalence classes of admissible G-motives with coefficients in $\mathbb{Z}_{(p)}$.

5 The map $\mathcal{M}(G,X)(W(\mathbb{F})) \to \mathcal{L}(G,X)$

Let (G,X) be a Shimura p-datum, and $\mathrm{Sh}_p(G,X)$ be a canonical integral model. We often write ? for ?(G,X). Conjecturally, there should be a map

$$\mathcal{M}(B(\mathbb{F})) \to \mathcal{L}(\mathbb{F})$$

corresponding to the map

$$\mathrm{Sh}_p(B(\mathbb{F})) \simeq \mathrm{Sh}_p(W(\mathbb{F})) \twoheadrightarrow \mathrm{Sh}_p(\mathbb{F}),$$

but (at present) we are able to define such a map only on a subset of $\mathcal{M}(B(\mathbb{F}))$.[16]

The points of Sh_p with coordinates in $B(\mathbb{F})$

Theorem 4.5 gives, in particular, a motivic description of the points of Sh_p with coordinates in $B(\mathbb{F})$. However, as we explained in Milne 1994b, p509, because étale p-integral structures do not reduce well, to pass from the points on a Shimura variety with coordinates in $B(\mathbb{F})$ to the points with coordinates in $\mathbb{F}$, we need to replace the étale p-integral structures with crystalline p-integral structures.

LEMMA 5.1. *For any admissible G-motive M over $B(\mathbb{F})$, there exists an isomorphism*

$$B(\mathbb{F}) \otimes_{\mathbb{Q}} V \to \omega_{\mathrm{dR}} \circ M$$

of tensor functors $\mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Vec}_{B(\mathbb{F})}$ carrying the filtration defined by μ_0^{-1} into the de Rham filtration (here μ_0 is as in 3.1 and V is the forgetful functor).

PROOF. For each $B(\mathbb{F})$-algebra R, let $\mathcal{F}(R)$ be the set of isomorphisms of R-linear tensor functors

$$R \otimes_{\mathbb{Q}} V \to R \otimes_{B(\mathbb{F})} (\omega_{\mathrm{dR}} \circ M)$$

carrying $\mathrm{Filt}(\mu_0^{-1})$ into the de Rham filtration. Then $\mathcal{F}$ is a pseudo-torsor for the subgroup P of $G_{B(\mathbb{F})}$ respecting the filtration defined by μ_0^{-1} on each representation of G. This is a parabolic subgroup of G (Saavedra Rivano 1972, IV 2.2.5, p223), and hence is connected by a theorem of Chevalley (Borel 1991, 11.16, p154). Once we show $\mathcal{F}(\mathbb{C}) \neq \emptyset$, so that $\mathcal{F}$ is a torsor, it will follow from Steinberg 1965, 1.9, that $\mathcal{F}(B(\mathbb{F})) \neq \emptyset$.

Choose an E-homomorphism $\tau\colon B(\mathbb{F}) \to \mathbb{C}$. There is a canonical comparison isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} (\omega_B \circ \tau M) \to \mathbb{C} \otimes_{B(\mathbb{F})} (\omega_{\mathrm{dR}} \circ M)$$

which carries the Hodge filtration on the left to the de Rham filtration on the right. By assumption, for some $x \in X$, there exists an isomorphism

$$H_x \to \omega_B \circ \tau M$$

[16] The problem is that we defined $\mathsf{Mot}(\mathbb{F})$ as a quotient of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$, but we would like to realize it as a quotient of the category of abelian motives over $\mathbb{Q}^{\mathrm{al}}$ having good reduction at v, or, at least, of the category generated by abelian varieties over $\mathbb{Q}^{\mathrm{al}}$ having good reduction at v. The latter would be possible if we knew the rationality conjecture for all abelian varieties with good reduction at v.

preserving Hodge structures. On combining these isomorphisms, we obtain an isomorphism

$$\mathbb{C} \otimes_{\mathbb{Q}} H_x \to \mathbb{C} \otimes_{B(\mathbb{F})} (\omega_{\mathrm{dR}} \circ M)$$

carrying $\mathrm{Filt}(\mu_x^{-1})$ to the Hodge filtration. By its very definition, μ_0 lies in the same $G(\mathbb{C})$-conjugacy class as μ_x, and so there exists an isomorphism of tensor functors

$$\mathbb{C} \otimes V \to \mathbb{C} \otimes H_x$$

carrying $\mathrm{Filt}(\mu_0^{-1})$ to $\mathrm{Filt}(\mu_x^{-1})$. The composite of the last two isomorphisms is an element of $\mathcal{F}(\mathbb{C})$. □

Let $\mathsf{MF}_{B(\mathbb{F})}$ denote the category of weakly admissible filtered $B(\mathbb{F})$-modules (see 3.3),[17] and let $\mathsf{MF}_{W(\mathbb{F})}$ denote the category $\underline{\mathrm{MF}}_{\mathrm{tf}}$ of Fontaine 1983, 2.1. Thus an object of $\mathsf{MF}_{W(\mathbb{F})}$ is a finitely generated $W(\mathbb{F})$-module Λ together with

(a) a finite exhaustive separated decreasing filtration

$$\cdots \supset \mathrm{Filt}^i \Lambda \supset \mathrm{Filt}^{i+1} \Lambda \supset \cdots$$

by submodules that are direct summands of Λ, and

(b) a family of σ-linear maps $\varphi^i_\Lambda : \mathrm{Filt}^i \Lambda \to \Lambda$ such that $\varphi^i_\Lambda(x) = p\varphi^{i+1}_\Lambda(x)$ for $x \in \mathrm{Filt}^{i+1}\Lambda$ and $\sum_i \mathrm{Im}\,\varphi^i_\Lambda = \Lambda$.

Let Λ be an object of $\mathsf{MF}_{W(\mathbb{F})}$. Then $\Lambda_{B(\mathbb{F})}$ is an object of $\mathsf{MF}_{B(\mathbb{F})}$, and the image of Λ in $\Lambda_{B(\mathbb{F})}$ is a strongly divisible lattice such that $\mathrm{Filt}^i \Lambda \overset{\mathrm{def}}{=} \Lambda \cap \mathrm{Filt}^i N$ is a direct summand of Λ for all i. To give an object of $\mathsf{MF}_{W(\mathbb{F})}$ that is torsion-free as a W-module is the same as giving an object (N, F, Filt^i) of $\mathsf{MF}_{B(\mathbb{F})}$ together with a strongly divisible lattice Λ such that $\Lambda \cap \mathrm{Filt}^i N$ is a direct summand of Λ for all i. The category $\mathsf{MF}_{W(\mathbb{F})}$ is a $\mathbb{Z}_p$-linear abelian category.

The functor $\omega_{\mathrm{dR}} : \mathsf{Mot}(B(\mathbb{F})) \to \mathsf{Mod}_{B(\mathbb{F})}$ has a canonical factorization into

$$\mathsf{Mot}(B(\mathbb{F})) \xrightarrow{\omega_{\mathrm{crys}}} \mathsf{R}_p \xrightarrow{\text{forget}} \mathsf{Mod}_{B(\mathbb{F})};$$

and it even factors through $\mathsf{MF}_{B(\mathbb{F})}$ on a large subcategory of $\mathsf{Mot}(B(\mathbb{F}))$.

DEFINITION 5.2. A *crystalline p-integral structure* on a G-motive M over $B(\mathbb{F})$ is an exact tensor functor $\underline{\Lambda}_{\mathrm{crys}} : \mathsf{Rep}_{\mathbb{Z}_p}(G) \to \mathsf{MF}_{W(\mathbb{F})}$ such that $\underline{\Lambda}_{\mathrm{crys}}(\xi)$ is a $W(\mathbb{F})$-lattice in $(\omega_{\mathrm{dR}} \circ M(p))(\xi_{\mathbb{Q}_p})$ for all ξ in $\mathsf{Rep}_{\mathbb{Z}_p}(G_p)$.

The condition means that $\underline{\Lambda}_{\mathrm{crys}}(\xi) \subset (\omega_{\mathrm{dR}} \circ M(p))(\xi_{\mathbb{Q}_p})$ for all ξ, and there is a commutative diagram

$$\begin{tikzcd}
\mathsf{Rep}_{B(\mathbb{F})}(G) \arrow[r, "B(\mathbb{F})\otimes M"] & B(\mathbb{F}) \otimes \mathsf{Mot}(B(\mathbb{F})) \arrow[r, "\omega_{\mathrm{crys}}"] & \mathsf{MF}_{B(\mathbb{F})} \\
\mathsf{Rep}_{\mathbb{Z}_p}(G) \arrow[u] \arrow[rr, "\underline{\Lambda}_{\mathrm{crys}}"] & & \mathsf{MF}_W \arrow[u]
\end{tikzcd}$$

Recall that for any filtered module N, there is a canonical splitting μ_W of the filtration on N, and that μ_W splits the filtration on any strongly divisible submodule of N (Wintenberger 1984).

[17] Thus an object of $\mathsf{MF}_{B(\mathbb{F})}$ is an F-isocrystal over $\mathbb{F}$ together with a finite exhaustive separated decreasing filtration that admits a strongly divisible W-lattice.

LEMMA 5.3. *Let $\underline{\Lambda}_{\mathrm{crys}}$ be a p-integral crystalline structure on an admissible G-motive M over $B(\mathbb{F})$. Then there exists an isomorphism of tensor functors $W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}_{\mathrm{crys}}$ carrying μ_0^{-1} into μ_W.*

PROOF. It follows from (1.1) that there exists an isomorphism $\alpha\colon W \otimes_{\mathbb{Z}_p} \Lambda \to \underline{\Lambda}_{\mathrm{crys}}$, uniquely determined up to composition with an element of $G(W)$. Let μ' be the cocharacter of G_W mapped by α to μ_W. We have to show that μ' is $G(W)$-conjugate to μ_0^{-1}.

According to Lemma 5.1, there exists an isomorphism $\beta\colon B(\mathbb{F}) \otimes_{\mathbb{Z}_p} \Lambda \to B(\mathbb{F}) \otimes_W \underline{\Lambda}_{\mathrm{crys}}$ carrying $\mathrm{Filt}(\mu_0^{-1})$ into $\mathrm{Filt}(\mu_W)$. After possibly replacing β with its composite with an element of (a unipotent subgroup) of $G(B(\mathbb{F}))$ — see, for example, Milne 1990, 1.7, — we may assume that β maps μ_0^{-1} to μ_W. Since β can differ from $B(\mathbb{F}) \otimes_W \alpha$ only by an element of $G(B(\mathbb{F}))$, this shows that μ' is $G(B(\mathbb{F}))$-conjugate to μ_0^{-1}.

Let T be a maximal (split) torus of G_W containing the image of μ'. From its definition (see 3.1), we know μ_0 factors through a specific torus $S \subset G_W$. According to Demazure and Grothendieck 1964, XII 7.1, T and S will be conjugate locally for the étale topology on $\operatorname{Spec} W$, which in our case means that they are conjugate by an element of $G(W)$. We may therefore suppose that μ' and μ_0 both factor through S. But two characters of S are $G(B(\mathbb{F}))$-conjugate if and only if they are conjugate by an element of the Weyl group — see for example Milne 1992, 1.7, — and, because the hyperspecial point fixed by $G(W)$ lies in the apartment corresponding S, $G(W)$ contains a set of representatives for the Weyl group (see 3.1). Thus μ' is conjugate to μ_0^{-1} by an element of $G(W)$.□

The integral comparison conjecture.

Recall that, for an abelian variety over a field k of characteristic zero, $\mathcal{B}(A)$ denotes the $\mathbb{Q}$-algebra of Hodge classes on A (absolute Hodge classes in the sense of Deligne 1982). A Hodge tensor on A is an element of $\bigoplus_{n\geq 0}\mathcal{B}(A^n)$. A Hodge class on A is a family $\gamma = (\gamma_l)_l$ with $\gamma_l \in H^{2*}(A_{k^{\mathrm{al}}}, \mathbb{Q}_l(*))$ for $l \neq \infty$ and $\gamma_\infty \in H^{2*}_{\mathrm{dR}}(A)(*)$.

Let A be an abelian variety over a finite extension K of $B(\mathbb{F})$ contained in $(\mathbb{Q}^{\mathrm{al}})_v$, and suppose that K is sufficiently large that $\mathcal{B}(A) \simeq \mathcal{B}(A_{K^{\mathrm{al}}})$. Then

$$\mathcal{B}_{(p)}(A) \stackrel{\mathrm{def}}{=} \mathcal{B}(A) \cap H^{2*}(A_{K^{\mathrm{al}}}, \mathbb{Z}_p)(*)$$

is a $\mathbb{Z}_{(p)}$-lattice in $\mathcal{B}(A)$.

INTEGRAL COMPARISON CONJECTURE 5.4. *If A has good reduction, and so extends to an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$, then, for $\gamma \in \mathcal{B}(A)$,*

$$\gamma_p \in H^{2*}(A_{k^{\mathrm{al}}}, \mathbb{Z}_p(*)) \implies \gamma_\infty \in H^{2*}_{\mathrm{dR}}(\mathcal{A})(*). \tag{7}$$

The following statement was conjectured in Milne 1995.[18]

[18] More precisely, Milne 1995, Conjecture 0.1, p1, reads:

> Let A be an abelian scheme over $W(\mathbb{F})$. Let $\mathfrak{s} = (s_i)_{i\in I}$ be a family of Hodge tensors on A including a polarization, and, for some fixed inclusion $\tau\colon W(\mathbb{F}) \hookrightarrow \mathbb{C}$, let G be the subgroup of $\mathrm{GL}(H^1((\tau A)(\mathbb{C}), \mathbb{Q}))$ fixing the s_i. Assume that G is reductive, and that the Zariski closure of G in $\mathrm{GL}(H^1(A_{B(\mathbb{F})}, \mathbb{Z}_p))$ is hyperspecial. Then, for some faithfully flat $\mathbb{Z}_p$-algebra R, there exists an isomorphism of W-modules
>
> $$R \otimes_{\mathbb{Z}_p} H^1(A_{B(\mathbb{F})}, \mathbb{Z}_p) \to R \otimes_W H^1_{\mathrm{dR}}(A)$$
>
> mapping the étale component of each s_i to the de Rham component. (In fact, if there exists such an isomorphism for some faithfully flat $\mathbb{Z}_p$-algebra, then there exists an isomorphism with $R = W(\mathbb{F})$.)

THEOREM 5.5. *Let K be a finite extension of $B(\mathbb{F})$, and let A be an abelian variety over $B(\mathbb{F})$ with good reduction (so A extends to an abelian scheme over $\mathcal{O}_K$). Let $(s_i)_{i\in I}$ be a family of Hodge tensors for A that are tensors for $H^1(A,\mathbb{Z}_p)$ and that define a reductive subgroup of $\mathrm{GL}(H^1(A,\mathbb{Z}_p))$. Then there exists an isomorphism of $W(\mathbb{F})$-modules*

$$W(\mathbb{F}) \otimes_{\mathbb{Z}_p} H^1(A,\mathbb{Z}_p) \to H^1_{\mathrm{dR}}(\mathcal{A})$$

mapping the étale component of each s_i to the de Rham component.

PROOF. Apply Kisin 2009, 1.3.6, or Vasiu 2003b. □

Note that Theorem 5.5 implies that the Hodge tensors s_i in the statement satisfy (7).

Construction of the map $\mathcal{M}(W(\mathbb{F})) \to \mathcal{L}$

Define $\mathsf{Mot}(W(\mathbb{F}))$ (respectively $\mathsf{Mot}_{(p)}(W(\mathbb{F}))$) to be the tannakian subcategory of $\mathsf{Mot}(B(\mathbb{F}))$ (respectively $\mathsf{Mot}_{(p)}(B(\mathbb{F}))$) generated by abelian varieties over $B(\mathbb{F})$ with potential good reduction. Note that Conjecture 5.4 implies that there is an exact tensor functor $\omega_{\mathrm{dR}}\colon \mathsf{Mot}_{(p)}(W(\mathbb{F})) \to \mathsf{Mod}_{W(\mathbb{F})}$.

DEFINITION 5.6. An admissible G-motive M over $B(\mathbb{F})$ is *special* if it takes values in $\mathsf{Mot}(W(\mathbb{F}))$ and there exists an exact tensor functor $M'\colon \mathsf{Rep}(G) \to \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ making the following diagram commute:

$$\begin{tikzcd}
 & \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \arrow[r] & \mathsf{Mot}(B(\mathbb{F})^{\mathrm{al}}) \\
\mathsf{Rep}(G) \arrow[ur, "M'"] \arrow[r, "M"] & \mathsf{Mot}(W(\mathbb{F})) \arrow[r, "\subset"] & \mathsf{Mot}(B(\mathbb{F})) \arrow[u]
\end{tikzcd}$$

We say that an equivalence class m of admissible G-motives over $B(\mathbb{F})$ is *special* if it contains a special G-motive M, in which case we let $\mathcal{M}(m) = \mathcal{M}(M')$. We define $\mathcal{M}(G,X)(W(\mathbb{F}))$ to be the subset $\bigsqcup_{m \text{ special}} \mathcal{M}(m)$ of $\mathcal{M}(G,X)(B(\mathbb{F}))$.

PROPOSITION 5.7. *Assume the integral comparison conjecture. Then there is a canonical equivariant map*

$$\mathcal{M}(G,X)(W(\mathbb{F})) \to \mathcal{L}(G,X).$$

The map becomes surjective when we omit the p-component; that is, when we replace $\mathcal{L}(G,X)$ with its quotient

$$\mathcal{L}^p(G,X) \overset{\text{def}}{=} \bigsqcup\nolimits_m \mathcal{L}^p(m), \quad \mathcal{L}^p(m) \overset{\text{def}}{=} \varprojlim_{M\in m} \mathcal{L}^p(M), \quad \mathcal{L}^p(M) = I(M)\backslash(X^p(M)/Z^p).$$

PROOF. Let M be special. The composite

$$\mathsf{Rep}(G) \xrightarrow{M'} \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \xrightarrow{R} \mathsf{Mot}(\mathbb{F})$$

is a special G-motive $\bar{M}$ over $\mathbb{F}$. Let $\eta \in X^p(M)$, and let $\underline{\Lambda}_p \in X_p(M)$. Then $(M, \underline{\Lambda}_p)$ defines an exact tensor functor $\mathsf{Rep}(G) \to \mathsf{Mot}_{(p)}(W(\mathbb{F}))$, whose composite with ω_{dR} is an element of $X_p(\bar{M})$ (because of 5.1 and 5.3). The specialization map in étale cohomology defines an isomorphism $\omega^p_f(M) \to \omega^p_f(\bar{M})$, and so η defines an element of $X^p(\bar{M})$. Therefore, we have a map $\mathcal{M}(M) \to \mathcal{M}(\bar{M})$. On combining these maps for different M, we obtain a map $\mathcal{M}(W(\mathbb{F})) \to \mathcal{L}$. The second statement is obvious from the definitions. □

QUESTION 5.8. Is there an elementary proof that every admissible G-motive over $B(\mathbb{F})$ that takes values in $\mathsf{Mot}(W(\mathbb{F}))$ is special? Perhaps this can be deduced from an analogue of Proposition 3.9. A positive answer would give an elementary proof of the special-points conjecture (see the proof of Theorem 6.8).

REMARK 5.9. When (G, X) is of Hodge type, then Proposition 5.7 follows from Theorem 5.5, i.e., it is not necessary to assume the integral comparison conjecture. To see this, choose a defining representation and tensors for G.

6 Proof of Conjecture LR+ for certain Shimura varieties

Let (G, X) be a Shimura p-datum satisfying (SV1-4,6), and let Sh_p be a canonical integral model. Consider the diagram

$$\begin{tikzcd} \mathcal{M}(W(\mathbb{F})) \subset \mathcal{M}(B(\mathbb{F})) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Sh}_p(B(\mathbb{F})) \simeq \mathrm{Sh}_p(W(\mathbb{F})) \arrow[d, two heads] \\ \mathcal{L} \arrow[r, dashed, "?"] & \mathrm{Sh}_p(\mathbb{F}) \end{tikzcd} \tag{8}$$

in which the isomorphism on the top row is the map $\alpha(B(\mathbb{F}))$ of Theorem 4.5. We say that Conjecture LR holds for (G, X) if there exists an equivariant isomorphism $\mathcal{L} \to \mathrm{Sh}_p(\mathbb{F})$ making the diagram commute.

Siegel modular varieties

Let $(G(\psi), X(\psi))$ be the Shimura datum attached to a symplectic space (V, ψ) over $\mathbb{Q}$. Thus $G(\psi) = \mathrm{GSp}(\psi)$ and $X(\psi)$ consists of the Hodge structures h on $V_{\mathbb{R}}$ for which $(x, y) \mapsto \psi(x, h(i)y)$ is definite (either positive or negative). For any $\mathbb{Z}_{(p)}$-lattice Λ in V such that ψ restricts to a perfect $\mathbb{Z}_{(p)}$-valued pairing on Λ, the subgroup G of $G(\psi)$ stabilizing Λ is a reductive group over $\mathbb{Z}_{(p)}$ with generic fibre $G(\psi)$. Thus $(G, X(\psi))$ is a Shimura p-datum. Any Shimura p-datum arising in this way will be called a *Siegel p-datum*. In this subsection, we prove Conjecture LR for Siegel p-data.

LEMMA 6.1. *Let (A, λ) be a polarized motive over $\mathbb{F}$. For any $\mathbb{Q}$-algebra R, let $H(R)$ be the group of automorphisms of A as an object of $\mathsf{Mot}(\mathbb{F})_{(R)}$ fixing λ. Then H is an algebraic group over $\mathbb{Q}$ satisfying the Hasse principle for H^1.*

PROOF. We may assume that A is isotypic. Let $L = \mathrm{End}(A)$, let E be the centre of L, and let † denote the involution of L defined by λ. For any $\mathbb{Q}$-algebra R,

$$H(R) = \{a \in L \otimes_{\mathbb{Q}} R \mid aa^\dagger = 1\}.$$

There are two cases to consider: (a) $E = \mathbb{Q}$; (b) E is a CM-field (cf. Milne 1994a, 2.16). Choose a fibre functor $\bar{\omega}$ over $\mathbb{Q}^{\mathrm{al}}$, and let $V = \bar{\omega}(A)$. In case (a), $\mathrm{End}(A)_{\mathbb{Q}^{\mathrm{al}}} = \mathrm{End}(V)$ and $H_{\mathbb{Q}^{\mathrm{al}}}$ is the symplectic group attached to the alternating form on V defined by λ. Hence H is a simply connected semisimple group, and so it satisfies the Hasse principle for H^1. In case (b), V decomposes into a direct sum of spaces $W \oplus W^\vee$. Correspondingly, $H_{\mathbb{Q}^{\mathrm{al}}}$ decomposes as a product of groups G_W, where $G_W = \mathrm{End}(W) \oplus \mathrm{End}(W^\vee)$. Moreover, $(a,b)^\dagger = (b^{\mathrm{tr}}, a)$, and so $G_W \simeq \mathrm{GL}_W$. From this, one deduces that $H = \mathrm{Res}_{F/\mathbb{Q}} H'$ where H' is an algebraic group over the largest totally real subfield F of E; the reduce norm defines a homomorphism $H' \to \mathbb{G}_m$ and the kernel is a form of SL_n for some n. From the diagram

$$\begin{tikzcd}
 & F^\times \arrow[r] \arrow[d, "\text{dense}"', "\text{image}"] & H^1(F, H'^{\mathrm{der}}) \arrow[r] \arrow[d, "\simeq"] & H^1(F, H') \arrow[r] \arrow[d] & 0 \\
\prod_{v|\infty} H'(F_v) \arrow[r, "\text{open}", "\text{image}"'] & \prod_{v|\infty} F_v^\times \arrow[r] & \prod_{v|\infty} H^1(F_v, H'^{\mathrm{der}}) \arrow[r] & \prod_v H^1(F_v, H') &
\end{tikzcd}$$

we see that H' satisfies the Hasse principle over F, and so H satisfies the Hasse principle over $\mathbb{Q}$.□

LEMMA 6.2. *Let (A,λ) and (A',λ') be polarized CM abelian varieties over $\mathbb{Q}^{\mathrm{al}}$. If there exists an isogeny $A'_0 \to A_0$ sending λ'_0 to λ_0, then there exists an isomorphism $\hbar^1(A) \to \hbar^1(A')$ sending $\hbar(\lambda)$ to $\hbar(\lambda')$.*

PROOF. For a $\mathbb{Q}$-algebra R, let $\mathcal{F}(R)$ be the set of isomorphisms from $\hbar^1 A$ to $\hbar^1 A'$ in the category $\mathsf{Mot}(\mathbb{F})_{(R)}$ sending $\hbar\lambda$ to $\hbar\lambda'$. We have to show that $\mathcal{F}(\mathbb{Q})$ is nonempty. Clearly $\mathcal{F}$ is a torsor for the algebraic group H in Lemma 6.1, and so it suffices to show that $\mathcal{F}(\mathbb{Q}_l)$ is nonempty for all l.

Let l be a prime $\neq p, \infty$, and let R'_l be the category of finite-dimensional $\mathbb{Q}_l$-vector spaces equipped with a germ of a Frobenius map. Then ζ_l defines a fully faithful functor $\mathsf{Mot}(\mathbb{F})_{(\mathbb{Q}_l)} \to \mathsf{R}'_l$ (cf. Milne 1994a, 3.7). Moreover, $\zeta_l(\hbar^1 A) = \xi_l(h^1 A)$ (by definition), and $\xi_l(h^1 A) = H^1_{\mathrm{et}}(A, \mathbb{Q}_l) \simeq H^1_{\mathrm{et}}(A_0, \mathbb{Q}_l)$ (as objects of R'_l). An isogeny $A'_0 \to A_0$ sending λ'_0 to λ'_0 defines an isomorphism $H^1_{\mathrm{et}}(A_0, \mathbb{Q}_l) \to H^1_{\mathrm{et}}(A'_0, \mathbb{Q}_l)$ sending $H^1_{\mathrm{et}}(\lambda_0)$ to $H^1_{\mathrm{et}}(\lambda'_0)$, and hence an element of $\mathcal{F}(\mathbb{Q}_l)$.

For p, we define R'_p to the category of finite dimensional vector spaces V over $\mathbb{Q}_p^{\mathrm{un}}$ equipped with a σ-linear isomorphism $V \to V$. Then ζ_l defines a fully faithful functor $\mathsf{Mot}(\mathbb{F})_{(\mathbb{Q}_p)} \to \mathsf{R}'_p$. The same argument as in the last paragraph shows that $\mathcal{F}(\mathbb{Q}_p)$ is nonempty.

For ∞, we need to use the categories of Lefschetz motives $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}})$ and $\mathsf{LMot}(\mathbb{F})$ (see Milne 1999). Each of these categories has a canonical polarization for which the geometric polarizations are positive, and the quotient map $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{LMot}(\mathbb{F})$ preserves the polarizations (see Milne 2002, 3.7). These polarizations define (isomorphism classes of) functors from $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}})$ and $\mathsf{LMot}(\mathbb{F})$ to R_∞ (see Deligne and Milne 1982, 5.20). We can choose the functor on $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}})$ to be the composite $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{R}_\infty$; then we can choose the functor on $\mathsf{LMot}(\mathbb{F})$ to be compatible, via the quotient map, with that on $\mathsf{LCM}(\mathbb{Q}^{\mathrm{al}})$. The functor ζ_∞ defines a fully faithful functor from $\mathsf{Mot}(\mathbb{F})_{(\mathbb{R})}$ to the category of objects of R_∞ equipped with a germ of a Frobenius operator (cf. Milne 1994a, 3.6). Now, the same argument as before shows that $\mathcal{F}(\mathbb{R})$ is nonempty.□

THEOREM 6.3. *Let (G,X) be a Siegel p-datum. The map $\mathcal{M}(W(\mathbb{F})) \to \mathrm{Sh}_p(\mathbb{F})$ of (8) induces an isomorphism $\mathcal{L} \to \mathrm{Sh}_p(\mathbb{F})$.*

PROOF. The group G has a natural Tate triple structure (w,t), and we prefer to work with (G,w,t)-objects (see p11). Now fix a polarized CM abelian variety (A,λ) over $\mathbb{Q}^{\mathrm{al}}$, and let M be the corresponding (G,w,t)-motive over $B(\mathbb{F})$ (see 1.8). It follows from Lemma 6.2 and deformation theory (starting from the Serre-Tate theorem; cf Norman 1981) that the fibres of the maps $\mathcal{M}(M) \to \mathcal{L}$ and

$\mathcal{M}(M) \to \mathrm{Sh}_p(\mathbb{F})$ are equal. Therefore, we get a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(M) & \xrightarrow{\text{bijection}} & {}'\mathrm{Sh}_p(B(\mathbb{F})) \\ \downarrow{\scriptstyle \text{onto}} & & \downarrow{\scriptstyle \text{onto}} \\ {}'\mathcal{L} & \xrightarrow{\text{bijection}} & {}'\mathrm{Sh}_p(\mathbb{F}) \end{array}$$

where a prime denotes the image of $\mathcal{M}(M)$. Using the second statement of Proposition 5.7, one sees that the lower arrow extends canonically to an injection $\mathcal{L} \to \mathrm{Sh}_p(\mathbb{F})$, which is a bijection by Zink's theorem (Zink 1983, 2.7). □

Shimura subvarieties

Recall that a map $(G, X) \to (G', X')$ of Shimura data defines a morphism $\mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X')$ of Shimura varieties over $\mathbb{C}$. Deligne (1971, 1.15) shows that the morphism is a closed immersion if $G \to G'$ is injective.

His argument proves a similar statement for Shimura p-data.[19] In particular, if $(G, X) \to (G', X')$ is a map of Shimura p-data such that $G \to G'$ is injective (as a map of group schemes over $\mathbb{Z}_{(p)}$), then $\mathrm{Sh}_p(G, X)(B(\mathbb{F})) \to \mathrm{Sh}_p(G', X')(B(\mathbb{F}))$ is injective. Similarly, $\mathcal{L}(G, X) \to \mathcal{L}(G', X')$ is injective. Write $?'$ for $?(G', X')$. For suitable integral models of the Shimura varieties, there will be a homomorphism of diagrams

$$\begin{array}{ccc} \mathcal{M}(W(\mathbb{F})) & \xrightarrow{\subset} & \mathrm{Sh}_p(W(\mathbb{F})) \\ \downarrow & & \downarrow \\ \mathcal{L} & & \mathrm{Sh}_p(\mathbb{F}) \end{array} \quad \longrightarrow \quad \begin{array}{ccc} \mathcal{M}'(W(\mathbb{F})) & \xrightarrow{\subset} & \mathrm{Sh}'_p(W(\mathbb{F})) \\ \downarrow & & \downarrow \\ \mathcal{L}' & & \mathrm{Sh}'_p(\mathbb{F}). \end{array}$$

PROPOSITION 6.4. *Assume that the map $\mathrm{Sh}_p(\mathbb{F}) \to \mathrm{Sh}'_p(\mathbb{F})$ is injective. If the map $\mathcal{M}'(W(\mathbb{F})) \to \mathrm{Sh}'_p(\mathbb{F})$ factors through ${}'\mathcal{L}'$ and defines a bijection ${}'\mathcal{L}' \to {}'\mathrm{Sh}'_p(\mathbb{F})$, then the same statement is true for $\mathcal{M}(W(\mathbb{F})) \to \mathrm{Sh}_p(\mathbb{F})$.*

PROOF. This is obvious from the above statements. □

Shimura p-data of Hodge type

We say that a Shimura p-datum (G, X) is of *Hodge type* if there exists a Siegel p-datum (G', X') and a map $(G, X) \to (G', X')$ with $G \to G'$ injective. We let Sh_p and Sh'_p denote the canonical integral models.

THEOREM 6.5. *Let (G, X) be a Shimura p-datum of Hodge type, and assume that the map $\mathrm{Sh}_p(\mathbb{F}) \to \mathrm{Sh}'_p(\mathbb{F})$ is injective for some Siegel embedding. Then the map $\mathcal{M}(W(\mathbb{F})) \to \mathrm{Sh}_p(\mathbb{F})$ of (8), p31, defines an injection $\mathcal{L} \to \mathrm{Sh}_p(\mathbb{F})$, which is surjective if the special-points conjecture holds for (G, X).*

PROOF. Theorem 6.3 and Proposition 6.4 show that the map $\mathcal{M}(W(\mathbb{F})) \to \mathrm{Sh}_p(\mathbb{F})$ defines a bijective map ${}'\mathcal{L} \to {}'\mathrm{Sh}_p(\mathbb{F})$, which can be extended to an injection map $\mathcal{L} \to \mathrm{Sh}_p(\mathbb{F})$ using the second part of Proposition 5.7. If the special-points conjecture is true for (G, X), this map is surjective. □

[19] This is written out, for example, in Giguère 1998, 2.1.9.1 and in Kisin 2009, 2.1.2.

6.6. We deduce:

(a) Conjecture LR+ is true for all Shimura varieties of PEL-type (by Zink's theorem; in this case, it is known that the canonical integral model has the property that $\mathrm{Sh}_p(\mathbb{F}) \to \mathrm{Sh}'_p(\mathbb{F})$ is injective).

(b) Conjecture LRo is true for a Shimura variety of p-Hodge type if (and only if) the special-points conjecture holds for the Shimura variety (Conjecture LRo allows us to take the integral model of Sh_p to be the closure of Sh_p in the integral model of Sh'_p).

(c) Conjecture LR+ is true for a Shimura variety of p-Hodge type if the special-points conjecture holds for it and canonical integral model has the property that $\mathrm{Sh}_p(\mathbb{F}) \to \mathrm{Sh}'_p(\mathbb{F})$ is injective (the canonical integral model is not always known to have this last property, because present constructions of it require a normalization; according to Vasiu, this should not be necessary).[20]

Complements

Restatement in terms of groupoids Choose a $\mathbb{Q}^{\mathrm{al}}$-valued fibre functor $\bar{\omega}$ on $\mathsf{Mot}(\mathbb{F})$ and isomorphisms

$$\mathbb{C} \otimes_{\mathbb{Q}^{\mathrm{al}}} \bar{\omega} \to \omega_\infty^{\mathbb{F}}, \quad \mathbb{Q}_\ell^{\mathrm{al}} \otimes_{\mathbb{Q}^{\mathrm{al}}} \bar{\omega} \to \mathbb{Q}_\ell^{\mathrm{al}} \otimes_{\mathbb{Q}_\ell} \omega_\ell^{\mathbb{F}}, \quad B(\mathbb{F})^{\mathrm{al}} \otimes_{\mathbb{Q}^{\mathrm{al}}} \bar{\omega} \to B(\mathbb{F})^{\mathrm{al}} \otimes_{B(\mathbb{F})} \omega_p^{\mathbb{F}}. \tag{9}$$

Then $\mathfrak{P} \stackrel{\mathrm{def}}{=} \underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\bar{\omega})$ is a transitive affine $\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}$-groupoid (Deligne 1990, 1.11, 1.12). Let $\mathfrak{G}_l$ be the groupoid attached to the category R_l and its canonical (forgetful) fibre functor. Then

$$\underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}_l}(\omega_l^{\mathbb{F}}) \simeq \mathfrak{G}_l, \quad l = 2, 3, 5, \ldots, \infty,$$

and so the isomorphisms (9) define homomorphisms $\check{\zeta}_l \colon \mathfrak{G}_l \to \mathfrak{P}(l)$ where $\mathfrak{P}(l)$ is the $\mathbb{Q}_l^{\mathrm{al}}/\mathbb{Q}_l$-groupoid obtained from $\mathfrak{P}$ by extension of scalars. The kernel $\mathfrak{G}_l^{\Delta}$ of $\mathfrak{G}_l$ is $\mathbb{G}_m$ for $l = \infty$, 1 for $l \neq p, \infty$, and $\mathbb{G}$ for $l = p$.

PROPOSITION 6.7. *The system $(\mathfrak{P}, (\check{\zeta}_l)_{l=2,3,5,\ldots,\infty})$ satisfies the following conditions*

(a) $(\mathfrak{P}^{\Delta}, \check{\zeta}_p^{\Delta}, \check{\zeta}_\infty^{\Delta}) = (P, z_p, z_\infty)$;

(b) the morphisms $\check{\zeta}_\ell$ for $\ell \neq p, \infty$ are defined by a section of $\mathfrak{P}$ over $\overline{\mathbb{A}_f^p} \otimes_{\mathbb{Q}} \overline{\mathbb{A}_f^p}$ where $\overline{\mathbb{A}_f^p}$ is the image of $\mathbb{Q}^{\mathrm{al}} \otimes \mathbb{A}_f^p$ in $\prod_{\ell \neq p, \infty} \mathbb{Q}_\ell^{\mathrm{al}}$.

PROOF. Straightforward (cf. Milne 2003, §6). □

Thus $(\mathfrak{P}, (\check{\zeta}_l))$ is a pseudomotivic groupoid in the sense of Milne 1992, 3.27 (corrected in Reimann 1997, p120), which is essentially the same as the object that Langlands and Rapoport (1987) call a "pseudomotivische Galoisgruppe" . On restating Conjecture 3.5 in terms of $(\mathfrak{P}, (\check{\zeta}_l))$, we recover Conjecture 5.e, p169, of Langlands and Rapoport 1987 in the good reduction case; see also Milne 1992, 4.4, and Reimann 1997, Appendix B3.

Langlands and Rapoport also state their conjecture for Shimura varieties whose weight is not rational (that is, which fail SV4) in terms of a "quasimotivische Galoisgruppe", but, as Pfau and

[20]Note that, as the canonical integral model is currently constructed, the map $\mathrm{Sh}_p(W(\mathbb{F})) \to \mathrm{Sh}'_p(W(\mathbb{F}))$ is injective (because it is a submap of the injective map $\mathrm{Sh}_p(B(\mathbb{F})) \to \mathrm{Sh}'_p(B(\mathbb{F}))$).

Reimann have pointed out, their definition of this is incorrect. Following Pfau, we define a *quasi-motivic groupoid* by[21]

$$\mathfrak{Q} = \mathfrak{P} \times_S (\mathbb{G}_m)_{\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}}.$$

Then it is possible Conjecture LR+ for all Shimura varieties in terms of $\mathfrak{Q}$.

Non simply connected derived groups Langlands and Rapoport originally stated their conjecture only for pairs (G,X) with G^{der} simply connected (in fact, their statement becomes false without this condition). In Milne 1992, the conjecture is restated so that it applies to all Shimura varieties, and the condition that the bijection be equivariant for $Z(\mathbb{Q}_p)$ is added. For the improved conjecture, the following statement is proved (ibid. 4.19):

> Let $(G,X) \to (G',X')$ be a morphism of Shimura p-data with $G_{\mathbb{Q}} \to G'_{\mathbb{Q}}$ an isogeny; if the improved Langlands-Rapoport conjecture is true for (G,X), then it is true for (G',X').

Similar arguments prove this for the Conjecture LR+. Thus, once one knows Conjecture LR+ to be true for some Shimura p-data, one obtains it for many more.

Changing the centre of G Pfau (1993, 1996b,a) stated a "refined" Langlands-Rapoport conjecture, and he proved the following statement:

> Let (G,X) and (G',X') be Shimura p-data whose associated *connected* Shimura p-data are isomorphic; if the refined Langlands-Rapoport conjecture is true for one of the Shimura varieties, then it is true for both.

Again, similar arguments prove this statement Conjecture LR+. In fact, the arguments become somewhat simpler. Thus Theorem 6.5 implies Conjecture LR+ for many Shimura varieties whose weight is not rational.

Shimura varieties of abelian type The above statements almost suffice to prove Conjecture LR+ for all Shimura varieties of abelian type (assuming the special-points conjecture). The main obstacle is that, in Theorem 6.5 we required that the Shimura p-datum be of Hodge type, whereas we need to know the conjecture for all (G,X) such that $(G_{\mathbb{Q}},X)$ is of Hodge type. In other words, in Theorem 6.5, we required that there exist an embedding $(G_{\mathbb{Q}},X) \hookrightarrow (G(\psi),X(\psi))$ such that $G(\mathbb{Z}_p)$ maps into a hyperspecial subgroup of $G(\psi)$; we need to prove the theorem without the last condition (or prove that it always holds).

General Shimura varieties It is natural to pose the following problem:

> Let (G,X) be a Shimura p-datum (whose weight is defined over $\mathbb{Q}$, if you wish). Define (in a natural way) an $\mathbb{F}$-scheme L with a continuous action of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p)$ such that $L(\mathbb{F}) = \mathcal{L}(G,X)$.

[21]Those who don't wish to assume that their Shimura varieties satisfy SV6, will need to replace $(\mathbb{G}_m)_{\mathbb{Q}^{\mathrm{cm}}/\mathbb{Q}}$ with $(\mathbb{G}_m)_{\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}}$.

Once this problem has been solved, it becomes possible to state the following conjecture:

> Show that there exists an equivariant isomorphism of $\mathbb{F}$-schemes $L \to \mathrm{Sh}_p$.

Once the problem has been solved for all Shimura varieties and the conjecture has been proved for (certain) Shimura varieties of type A_n, it should be possible to deduce the conjecture for all Shimura varieties by the methods of Milne 1983 and Borovoĭ 1984, 1987.

The rationality conjecture

We refer to Milne 2009, 4.1, for the statement of the rationality conjecture.

If the rationality conjecture is true for all CM abelian varieties, then there is a unique good theory of rational Tate classes $A \mapsto \mathcal{R}(A)$ on abelian varieties over $\mathbb{F}$, and we can define $\mathsf{Mot}(\mathbb{F})$ to be the category whose objects are triples (A, e, m) with A an abelian variety *over* $\mathbb{F}$ and e an idempotent in $\mathcal{R}^{\dim A}(A \times A)$. The reduction functor realizes $\mathsf{Mot}(\mathbb{F})$ as a quotient of the tannakian category $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$, and so it defines a fibre functor ω_0 on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^P$. The quotient category defined by ω_0 is, of course, just $\mathsf{Mot}(\mathbb{F})$. The advantage now is that we know directly that $\mathsf{Mot}(\mathbb{F})$ contains the motives of abelian varieties over $\mathbb{F}$.

THEOREM 6.8. *If the special-points conjecture is true, then the rationality conjecture for CM abelian varieties implies the rationality conjecture for all abelian varieties whose Mumford-Tate group is unramified at p. Conversely, if the rationality conjecture is true for all abelian varieties, then the special-points conjecture is true for Shimura varieties of Hodge type with simply connected derived group (and hence for all Shimura varieties of abelian type A, B, or C).*

PROOF. Let A be an abelian variety over $\mathbb{Q}^{\mathrm{al}}$ with good reduction at v to an abelian variety A_0 over $\mathbb{F}$, and let γ be a Hodge class on A and δ a Lefschetz class on A_0. If the special points conjecture is true, then there exists a CM abelian variety A' over $\mathbb{Q}^{\mathrm{al}}$, a Hodge class γ' on A', and an isogeny $\alpha\colon A'_0 \to A_0$ sending γ'_0 to γ_0. Then

$$\langle \gamma_0 \cdot \delta \rangle \in \langle \gamma'_0 \cdot \alpha^* \delta \rangle \mathbb{Q}.$$

If the rationality conjecture is true for A', then $\langle \gamma'_0 \cdot \alpha^* \delta \rangle \in \mathbb{Q}$, and so $\langle \gamma_0 \cdot \delta \rangle \in \mathbb{Q}$.

Conversely, if the rationality conjecture is true for all abelian varieties, then the reduction functor is defined on the tannakian subcategory $\mathsf{Mot}(W(\mathbb{F}))$ of $\mathsf{Mot}(B(\mathbb{F}))$ generated by abelian varieties over $B(\mathbb{F})$ with good reduction. A point P of $\mathrm{Sh}_p(\mathbb{F})$ arises from an element $[M, \eta, \underline{\Lambda}_p]$ of $\mathcal{M}(B(\mathbb{F}))$ (see (8), p31). If $(G_{\mathbb{Q}}, X)$ is of Hodge type, then M takes values in $\mathsf{Mot}(W(\mathbb{F}))$, and the composite

$$\mathsf{Rep}_{\mathbb{Q}}(G) \to \mathsf{Mot}(W(\mathbb{F})) \to \mathsf{Mot}(\mathbb{F})$$

satisfies the conditions (a,b,c,d) of Proposition 3.9. Therefore, if G^{der} is simply connected, then M is special, which implies that P lifts to a special point. □

If we knew the rationality conjecture was true, this would open up the possibility of extending the motivic moduli description of the points on a Shimura variety of abelian type and rational weight from characteristic zero to characteristic p.

So long as the Tate conjecture remains inaccessible, the rationality conjecture is the most important problem in the theory of abelian varieties over finite fields.

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