Semisimple Algebraic Groups in Characteristic Zero *

J.S. Milne

May 9, 2007

Abstract

It is shown that the classification theorems for semisimple algebraic groups in characteristic zero can be derived quite simply and naturally from the corresponding theorems for Lie algebras by using a little of the theory of tensor categories. This article is extracted from Milne 2007.

Introduction

The classical approach to classifying the semisimple algebraic groups over $\mathbb{C}$ (see Borel 1975, §1) is to:

◇ classify the complex semisimple Lie algebras in terms of reduced root systems (Killing, E. Cartan, et al.);

◇ classify the complex semisimple Lie groups with a fixed Lie algebra in terms of certain lattices attached to the root system of the Lie algebra (Weyl, E. Cartan, et al.);

◇ show that a complex semisimple Lie group has a unique structure of an algebraic group compatible with its complex structure.

Chevalley (1956-58, 1960-61) proved that the classification one obtains is valid in all characteristics, but his proof is long and complicated.[1]

Here I show that the classification theorems for semisimple algebraic groups in characteristic zero can be derived quite simply and naturally from the corresponding theorems for Lie algebras by using a little of the theory of tensor categories. In passing, one also obtains a classification of their finite-dimensional representations. Beyond its simplicity, the advantage of this approach is that it makes clear the relation between semisimple Lie algebras, semisimple algebraic groups, and tensor categories in characteristic zero.

The idea of obtaining an algebraic proof of the classification theorems for semisimple algebraic groups in characteristic zero by exploiting their representations is not new — in a somewhat primitive form it can be found already in Cartier's announcement (1956) — but I have not seen an exposition of it in the literature.

Throughout, k is a field of characteristic zero and "representation" of a Lie algebra or affine group means "finite-dimensional linear representation".

I assume that the reader is familiar with the elementary parts of the theories of algebraic groups and tensor categories and with the classification of semisimple Lie algebras; see Milne 2007 for a more detailed account.

[1] See Humphreys 1975, Chapter XI, and Springer 1998, Chapters 10 & 11. Despite its fundamental importance, many books on algebraic groups, e.g., Borel 1991, don't prove the classification, and some, e.g., Tauvel and Yu 2005, don't even state it.

1 Elementary Tannaka duality

1.1. *Let G be an algebraic group, and let R be a k-algebra. Suppose that for each representation (V, r_V) of G on a finite-dimensional k-vector space V, we have an R-linear endomorphism λ_V of $V(R)$. If the family (λ_V) satisfies the conditions,*

- ◇ $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$ *for all representations V, W,*
- ◇ $\lambda_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$ *(here $\mathbb{1} = k$ with the trivial action),*
- ◇ $\lambda_W \circ \alpha_R = \alpha_R \circ \lambda_V$, *for all G-equivariant maps $\alpha\colon V \to W$,*

then there exists a $g \in G(R)$ such that $\lambda_V = r_V(g)$ for all X (Deligne and Milne 1982, 2.8.

Because G admits a faithful finite-dimensional representation, g is uniquely determined by the family (λ_V), and so the map sending $g \in G(R)$ to the family $(r_V(g))$ is a bijection from $G(R)$ onto the set of families satisfying the conditions in the theorem. Therefore we can recover G from the category $\mathsf{Rep}(G)$ of representations of G on finite-dimensional k-vector spaces.

1.2. Let G be an algebraic group over k. For each k-algebra R, let $G'(R)$ be the set of families (λ_V) satisfying the conditions in (1.1). Then G' is a functor from k-algebras to groups, and there is a natural map $G \to G'$. That this map is an isomorphism is often paraphrased by saying that ***Tannaka duality holds for*** G.

2 Gradations on tensor categories

2.1. Let M be a finitely generated abelian group, and let $D(M)$ be the associated diagonalizable algebraic group. An M-***gradation*** on an object X of an abelian category is a family of subobjects $(X^m)_{m \in M}$ such that $X = \bigoplus_{m \in M} X^m$. An M-***gradation*** on a tensor category C is an M-gradation on each object X of C compatible with all arrows in C and with tensor products in the sense that $(X \otimes Y)^m = \bigoplus_{r+s=m} X^r \otimes X^s$. Let (C, ω) be a neutral tannakian category, and let G be its Tannaka dual. To give an M-gradation on C is the same as to give a central homomorphism $D(M) \to G$: a homomorphism corresponds to the M-gradation such that X^m is the subobject of X on which $D(M)$ acts through the character m (Saavedra Rivano 1972; Deligne and Milne 1982, §5).

2.2. Let C be a semsimple k-linear tensor category such that $\mathrm{End}(X) = k$ for every simple object X in C, and let $I(\mathsf{C})$ be the set of isomorphism classes of simple objects in C. For elements $x, x_1, \ldots, x_m$ of $I(\mathsf{C})$ represented by simple objects $X, X_1, \ldots, X_m$, write $x \prec x_1 \otimes \cdots \otimes x_m$ if X is a direct factor of $X_1 \otimes \cdots \otimes X_m$. The following statements are obvious.

(a) Let M be a commutative group. To give an M-gradation on C is the same as to give a map $f\colon I(\mathsf{C}) \to M$ such that

$$x \prec x_1 \otimes x_2 \implies f(x) = f(x_1) + f(x_2).$$

A map from $I(\mathsf{C})$ to a commutative group satisfying this condition will be called a ***tensor map***. For such a map, $f(\mathbb{1}) = 0$, and if X has dual $X^\vee$, then $f([X^\vee]) = -f([X])$.

(b) Let $M(\mathsf{C})$ be the free abelian group with generators the elements of $I(\mathsf{C})$ modulo the relations: $x = x_1 + x_2$ if $x \prec x_1 \otimes x_2$. The obvious map $I(\mathsf{C}) \to M(\mathsf{C})$ is a universal tensor map, i.e., it is a tensor map, and every other tensor map $I(\mathsf{C}) \to M$ factors uniquely through it. Note that $I(\mathsf{C}) \to M(\mathsf{C})$ is surjective.

2.3. Let (C,ω) be a neutral tannakian category such that C is semisimple and $\mathrm{End}(V)=k$ for every simple object in C. Let Z be the centre of $G\overset{\text{def}}{=}\underline{\mathrm{Aut}}^{\otimes}(\omega)$. Because C is semisimple, G is reductive, and so Z is of multiplicative type. Assume (for simplicity) that Z is split, so that $Z=D(N)$ with N the group of characters of Z. According to (2.1), to give an M-gradation on C is the same as to give a homomorphism $D(M)\to Z$, or, equivalently, a homomorphism $N\to M$. On the other hand, (2.2) shows that to give an M-gradation on C is the same as to give a homomorphism $M(\mathsf{C})\to M$. Therefore $M(\mathsf{C})\simeq N$. In more detail: let X be an object of C; if X is simple, then Z acts on X through a character n of Z, and the tensor map $[X]\mapsto n\colon\ I(\mathsf{C})\to N$ is universal.

2.4. Let (C,ω) be as in (2.3), and define an equivalence relation on $I(\mathsf{C})$ by

$$a\sim a' \iff \text{there exist } x_1,\ldots,x_m\in I(\mathsf{C}) \text{ such that } a,a'\prec x_1\otimes\cdots\otimes x_m.$$

A function f from $I(\mathsf{C})$ to a commutative group defines a gradation on C if and only if $f(a)=f(a')$ whenever $a\sim a'$. Therefore, $M(\mathsf{C})\simeq I(\mathsf{C})/\!\sim$.

3 Representations of split semisimple Lie algebras

Throughout this subsection, $(\mathfrak{g},\mathfrak{h})$ is a split semisimple Lie algebra with root system $R\subset\mathfrak{h}^\vee$, and $\mathfrak{b}$ is the Borel subalgebra of $(\mathfrak{g},\mathfrak{h})$ attached to a base S for R. According to a theorem of Weyl, the representations of $\mathfrak{g}$ are semisimple, and so to classify them it suffices to classify the simple representations.

3.1. *Let $r\colon\ \mathfrak{g}\to\mathfrak{gl}_V$ be a simple representation of $\mathfrak{g}$.*

(a) *There exists a unique one-dimensional subspace L of V stabilized by $\mathfrak{b}$.*

(b) *The L in (a) is a weight space for $\mathfrak{h}$, i.e., $L=V_{\varpi_V}$ for some $\varpi_V\in\mathfrak{h}^\vee$.*

(c) *The ϖ_V in (b) is dominant, i.e., $\varpi_V\in P_{++}$;*

(d) *If ϖ is also a weight for $\mathfrak{h}$ in V, then $\varpi=\varpi_V-\sum_{\alpha\in S}m_\alpha\alpha$ with $m_\alpha\in\mathbb{N}$.*

The Lie-Kolchin theorem shows that there does exist a one-dimensional eigenspace for $\mathfrak{b}$ — the content of (a) is that when V is simple (as a representation of $\mathfrak{g}$), the space is unique. Since L is mapped into itself by $\mathfrak{b}$, it is also mapped into itself by $\mathfrak{h}$, and so lies in a weight space. The content of (b) is that it is the whole weight space. For the proof, see Bourbaki Lie, VIII, §7.

Because of (d), ϖ_V is called the ***heighest weight*** of the simple representation (V,r).

3.2. *Every dominant weight occurs as the highest weight of a simple representation of $\mathfrak{g}$ (ibid.).*

3.3. *Two simple representations of $\mathfrak{g}$ are isomorphic if and only if their highest weights are equal.*

Thus $(V,r)\mapsto\varpi_V$ defines a bijection from the set of isomorphism classes of simple representations of $\mathfrak{g}$ onto the set of dominant weights P_{++}.

3.4. *If (V,r) is a simple representation of $\mathfrak{g}$, then $\mathrm{End}(V,r)\simeq k$.*

To see this, let $V=V_\varpi$ with ϖ dominant. Every isomorphism $V_\varpi\to V_\varpi$ maps the highest weight line L into itself, and is determined by its restriction to L because L generates V_ϖ as a $\mathfrak{g}$-module.

3.5. The category $\mathsf{Rep}(\mathfrak{g})$ of representations of $\mathfrak{g}$ is a semisimple k-linear category to which we can apply (2.2). Statements (3.2, 3.3) allow us to identify the set of isomorphism classes of $\mathsf{Rep}(\mathfrak{g})$ with P_{++}. Let $M(P_{++})$ be the free abelian group with generators the elements of P_{++} and relations

$$\varpi = \varpi_1 + \varpi_1 \text{ if } V_\varpi \subset V_{\varpi_1} \otimes V_{\varpi_2}.$$

Then $P_{++} \to M(P_{++})$ is surjective, and two elements ϖ and ϖ' of P_{++} have the same image in $M(P_{++})$ if and only there exist $\varpi_1, \ldots, \varpi_m \in P_{++}$ such that W_ϖ and $W_{\varpi'}$ are subrepresentations of $W_{\varpi_1} \otimes \cdots \otimes W_{\varpi_m}$ (see 2.4). Later we shall prove that this condition is equivalent to $\varpi - \varpi' \in Q$, and so $M(P_{++}) \simeq P/Q$. In other words, $\mathsf{Rep}(\mathfrak{g})$ has a gradation by $P_{++}/Q \cap P_{++} \simeq P/Q$ but not by any larger quotient.

For example, let $\mathfrak{g} = \mathfrak{sl}_2$, so that $Q = \mathbb{Z}\alpha$ and $P = \mathbb{Z}\frac{\alpha}{2}$. For $n \in \mathbb{N}$, let $V(n)$ be a simple representation of $\mathfrak{g}$ with heighest weight $\frac{n}{2}\alpha$. From the Clebsch-Gordon formula (Bourbaki Lie, VIII, §9), namely,

$$V(m) \otimes V(n) \approx V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n), \quad n \leq m,$$

we see that $\mathsf{Rep}(\mathfrak{g})$ has a natural P/Q-gradation (but not a gradation by any larger quotient of P).

EXERCISE 3.6. [2]Prove that the kernel of $P_{++} \to M(P_{++})$ is $Q \cap P_{++}$ by using the formulas for the characters and multiplicities of the tensor products of simple representations (cf. Humphreys 1972, §24, especially Exercise 12).

4 Basic theory of semisimple algebraic groups

PROPOSITION 4.1. *A connected algebraic group G is semisimple (resp. reductive) if and only if its Lie algebra is semisimple (resp. reductive).*

PROOF. Suppose that $\mathrm{Lie}(G)$ is semisimple, and let N be a normal commutative subgroup of G. Then $\mathrm{Lie}(N)$ is a commutative ideal in $\mathrm{Lie}(G)$, and so is zero. This implies that N is finite.

Conversely, suppose that G is semisimple, and let $\mathfrak{n}$ be a commutative ideal in $\mathfrak{g}$. When G acts on $\mathfrak{g}$ through the adjoint representation, the Lie algebra of $H \overset{\text{def}}{=} C_G(\mathfrak{n})$ is

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid [x, \mathfrak{n}] = 0\},$$

which contains $\mathfrak{n}$. Because $\mathfrak{n}$ is an ideal, so is $\mathfrak{h}$:

$$[x, n] = 0, \quad y \in \mathfrak{g} \implies [[y, x], n] = [y, [x, n]] - [x, [y, n]] = 0.$$

Therefore H° is normal in G, which implies that its centre $Z(H^\circ)$ is normal in G. Because G is semisimple, $Z(H^\circ)$ is finite, and so $z(\mathfrak{h}) = 0$. But $z(\mathfrak{h}) \supset \mathfrak{n}$, and so $\mathfrak{n} = 0$.

The reductive case is similar. □

COROLLARY 4.2. *The Lie algebra of the radical of a connected algebraic group G is the radical of the Lie algebra of $\mathfrak{g}$; in other words,* $\mathrm{Lie}(R(G)) = r(\mathrm{Lie}(G))$.

[2]Not done by the author.

PROOF. Because Lie is an exact functor, the exact sequence

$$1 \to RG \to G \to G/RG \to 1$$

gives rise to an exact sequence

$$0 \to \mathrm{Lie}(RG) \to \mathfrak{g} \to \mathrm{Lie}(G/RG) \to 0$$

in which $\mathrm{Lie}(RG)$ is solvable (obviously) and $\mathrm{Lie}(G/RG)$ is semisimple. The image in $\mathrm{Lie}(G/RG)$ of any solvable ideal in $\mathfrak{g}$ is zero, and so $\mathrm{Lie}(RG)$ is the largest solvable ideal in $\mathfrak{g}$. □

A connected algebraic group G is ***simple*** if it is noncommutative and has no proper normal algebraic subgroups $\neq 1$, and it is ***almost simple*** if it is noncommutative and has no proper normal algebraic subgroups except for finite subgroups. An algebraic group G is said to be the ***almost-direct product*** of its algebraic subgroups $G_1, \ldots, G_n$ if the map

$$(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n \colon G_1 \times \cdots \times G_n \to G$$

is a surjective homomorphism with finite kernel; in particular, this means that the G_i commute with each other and each G_i is normal in G.

THEOREM 4.3. *Every connected semisimple algebraic group G is an almost-direct product*

$$G_1 \times \cdots \times G_r \to G$$

of its minimal connected normal algebraic subgroups. In particular, there are only finitely many such subgroups. Every connected normal algebraic subgroup of G is a product of those G_i that it contains, and is centralized by the remaining ones.

PROOF. Because $\mathrm{Lie}(G)$ is semisimple, it is a direct sum of its simple ideals:

$$\mathrm{Lie}(G) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

Let G_1 be the identity component of $C_G(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r)$. Then

$$\mathrm{Lie}(G_1) = c_{\mathfrak{g}}(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r) = \mathfrak{g}_1,$$

which is an ideal in $\mathrm{Lie}(G)$, and so G_1 is normal in G. If G_1 had a proper normal nonfinite algebraic subgroup, then $\mathfrak{g}_1$would have an ideal other than $\mathfrak{g}_1$ and 0, contradicting its simplicity. Therefore G_1 is almost-simple. Construct $G_2, \ldots, G_r$ similarly. Because $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$, the groups G_i and G_j commute. The subgroup $G_1 \cdots G_r$ of G has Lie algebra $\mathfrak{g}$, and so equals G. Finally,

$$\mathrm{Lie}(G_1 \cap \ldots \cap G_r) = \mathfrak{g}_1 \cap \ldots \cap \mathfrak{g}_r = 0$$

and so $G_1 \cap \ldots \cap G_r$ is finite.

Let H be a connected algebraic subgroup of G. If H is normal, then $\mathrm{Lie}\, H$ is an ideal, and so it is a direct sum of those $\mathfrak{g}_i$ it contains and centralizes the remainder. This implies that H is a product of those G_i it contains, and centralizes the remainder. □

COROLLARY 4.4. *An algebraic group is semisimple if and only if it is an almost direct product of almost-simple algebraic groups.*

COROLLARY 4.5. *All nontrivial connected normal subgroups and quotients of a semisimple algebraic group are semisimple.*

PROOF. They are almost-direct products of almost-simple algebraic groups. □

COROLLARY 4.6. *A semisimple group has no commutative quotients* $\neq 1$.

PROOF. This is obvious for simple groups, and the theorem then implies it for semisimple groups. □

DEFINITION 4.7. A semisimple algebraic group G is said to be ***splittable*** if it has a split maximal subtorus. A ***split semisimple algebraic group*** is a pair (G,T) consisting of a semisimple algebraic group G and a split maximal torus T.

LEMMA 4.8. *If T is a split torus in G, then* $\mathrm{Lie}(T)$ *is a commutative subalgebra of* $\mathrm{Lie}(G)$ *consisting of semisimple elements.*

PROOF. Certainly $\mathrm{Lie}(T)$ is commutative. Let (V,r_V) be a faithful representation of G. Then (V,r_V) decomposes into a direct sum $\bigoplus_{\chi\in X^*(T)} V_\chi$, and $\mathrm{Lie}(T)$ acts (semisimply) on each factor V_χ through the character $d\chi$. As (V,dr_V) is faithful, this shows that $\mathrm{Lie}(T)$ consists of semisimple elements. □

5 Rings of representations of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over k. A ***ring of representations*** of $\mathfrak{g}$ is a collection of representations of $\mathfrak{g}$ that is closed under the formation of direct sums, subquotients, tensor products, and duals. An ***endomorphism*** of such a ring $\mathcal{R}$ is a family

$$\alpha = (\alpha_V)_{V\in\mathcal{R}}, \quad \alpha_V \in \mathrm{End}_{k\text{-linear}}(V),$$

such that

- $\alpha_{V\otimes W} = \alpha_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \alpha_W$ for all $V,W\in\mathcal{R}$,
- $\alpha_V = 0$ if $\mathfrak{g}$ acts trivially on V, and
- for any homomorphism $\beta\colon V\to W$ of representations in $\mathcal{R}$,

$$\alpha_W \circ \beta = \alpha_V \circ \beta.$$

The set $\mathfrak{g}_R$ of all endomorphisms of $\mathcal{R}$ becomes a Lie algebra over k (possibly infinite dimensional) with the bracket

$$[\alpha,\beta]_V = [\alpha_V,\beta_V].$$

EXAMPLE 5.1 (IWAHORI 1954). Let $\mathfrak{g}=k$ with k algebraically closed. To give a represention of $\mathfrak{g}$ on a vector space V is the same as to give an endomorphism α of V, and so the category of representations of $\mathfrak{g}$ is equivalent to the category of pairs (k^n,A), $n\in\mathbb{N}$, with A an $n\times n$ matrix. It follows that to give an endomorphism of the ring $\mathcal{R}$ of all representations of $\mathfrak{g}$ is the same as to give a map $A\mapsto\lambda(A)$ sending a square matrix A to a matrix of the same size and satisfying certain conditions. A pair (g,c) consisting of an additive homomorphism $g\colon k\to k$ and an element c of k defines a λ as follows:

- $\lambda(S) = U\mathrm{diag}(ga_1,\ldots,ga_n)U^{-1}$ if λ is the semisimple matrix $U\mathrm{diag}(a_1,\ldots,a_n)U^{-1}$;
- $\lambda(N) = cN$ if N is nilpotent;

◇ $\lambda(A) = \lambda(S)+\lambda(N)$ if $A = S+N$ is the decomposition of A into its commuting semisimple and nilpotent parts.

Moreover, every λ arises from a unique pair (g, c). Note that $\mathfrak{g}_{\mathcal{R}}$ has infinite dimension.

Let $\mathcal{R}$ be a ring of representations of a Lie algebra $\mathfrak{g}$. For any $x \in \mathfrak{g}$, $(r_V(x))_{V\in\mathcal{R}}$ is an endomorphism of $\mathcal{R}$, and $x \mapsto (r_V(x))$ is a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$.

LEMMA 5.2. *If $\mathcal{R}$ contains a faithful representation of $\mathfrak{g}$, then $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is injective.*

PROOF. For any representation (V, r_V) of $\mathfrak{g}$, the composite

$$\mathfrak{g} \xrightarrow{x\mapsto(r_V(x))} \mathfrak{g}_{\mathcal{R}} \xrightarrow{\lambda\mapsto\lambda_V} \mathfrak{gl}(V).$$

is r_V. Therefore, $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is injective if r_V. □

PROPOSITION 5.3. *Let G be an affine group over k, and let $\mathcal{R}$ be the ring of representations of $\mathfrak{g}$ arising from a representation of G. Then $\mathfrak{g}_{\mathcal{R}} \simeq \mathrm{Lie}(G)$; in particular, $\mathfrak{g}_{\mathcal{R}}$ depends only of G°.*

PROOF. By definition, $\mathrm{Lie}(G)$ is the kernel of $G(k[\varepsilon]) \to G(k)$. Therefore, to give an element of $\mathrm{Lie}(G)$ is the same as to give a family of $k[\varepsilon]$-linear maps

$$\mathrm{id}_V + \alpha_V\varepsilon \colon V[\varepsilon] \to V[\varepsilon]$$

indexed by $V \in \mathcal{R}$ satisfying the three conditions of (1.1). The first of these conditions says that

$$\mathrm{id}_{V\otimes W} + \alpha_{V\otimes W}\varepsilon = (\mathrm{id}_V + \alpha_V\varepsilon) \otimes (\mathrm{id}_W + \alpha_W\varepsilon),$$

i.e., that

$$\alpha_{V\otimes W} = \mathrm{id}_V \otimes \alpha_W + \alpha_V \otimes \mathrm{id}_W .$$

The second condition says that

$$\alpha_{\mathbb{1}} = 0,$$

and the third says that the α_V commute with all G-morphisms (= $\mathfrak{g}$-morphisms). Therefore, to give such a family is the same as to give an element $(\alpha_V)_{V\in\mathcal{R}}$ of $\mathfrak{g}_{\mathcal{R}}$. □

PROPOSITION 5.4. *For a ring $\mathcal{R}$ of representations of a Lie algebra $\mathfrak{g}$, the following statements are equivalent:*

(a) *the map $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is an isomorphism;*

(b) *$\mathfrak{g}$ is the Lie algebra of an affine group G such that G° is algebraic and $\mathcal{R}$ is the ring of all representations of $\mathfrak{g}$ arising from a representation of G.*

PROOF. This is an immediate consequence of (5.3) and the fact that an affine group is algebraic if its Lie algebra is finite-dimensional. □

COROLLARY 5.5. *Let $\mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of $\mathfrak{g}$, and let $\mathcal{R}(V)$ be the ring of representations of $\mathfrak{g}$ generated by V. Then $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is an isomorphism if and only if $\mathfrak{g}$ is algebraic, i.e., the Lie algebra of an algebraic subgroup of GL_V.*

PROOF. Immediate consequence of the proposition. □

REMARK 5.6. Let $\mathfrak{g} \to \mathfrak{gl}(V)$ be a faithful representation of $\mathfrak{g}$, and let $\mathcal{R}(V)$ be the ring of representations of $\mathfrak{g}$ generated by V. When is $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ an isomorphism? It is easy to show, for example, when $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. In particular, $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is an isomorphism when $\mathfrak{g}$ is semisimple. For an abelian Lie group $\mathfrak{g}$, $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is an isomorphism if and only if $\mathfrak{g} \to \mathfrak{gl}(V)$ is a semisimple representation and there exists a lattice in $\mathfrak{g}$ on which the characters of $\mathfrak{g}$ in V take integer values. For the Lie algebra in (Bourbaki Lie, I, §5, Exercise 6), $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}(V)}$ is *never* an isomorphism.

Let $\mathcal{R}$ be the ring of all representations of $\mathfrak{g}$. When $\mathfrak{g} \to \mathfrak{g}_{\mathcal{R}}$ is an isomorphism one says that ***Tannaka duality holds for*** $\mathfrak{g}$. The aside shows that Tannaka duality holds for $\mathfrak{g}$ if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. On the other hand, Example 5.1 shows that Tannaka duality fails when $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, and even that $\mathfrak{g}_{\mathcal{R}}$ has infinite dimension in this case.

6 An adjoint to the functor Lie

Let $\mathfrak{g}$ be a Lie group, and let $\mathcal{R}$ be the ring of all representations of $\mathfrak{g}$. We define $G(\mathfrak{g})$ to be the Tannaka dual of the neutral tannakian category $\mathsf{Rep}(\mathfrak{g})$. Recall that this means that $G(\mathfrak{g})$ is the affine group whose R-points for any k-algebra R are the families

$$\lambda = (\lambda_V)_{V \in \mathcal{R}}, \quad \lambda_V \in \mathrm{End}_{R\text{-linear}}(V(R)),$$

such that

- $\lambda_{V \otimes W} = \lambda_V \otimes \lambda_W$ for all $V \in \mathcal{R}$;
- if $xv = 0$ for all $x \in \mathfrak{g}$ and $v \in V$, then $\lambda_V v = v$ for all $\lambda \in G(\mathfrak{g})(R)$ and $v \in V(R)$;
- for every $\mathfrak{g}$-homomorphism $\beta \colon V \to W$,

$$\lambda_W \circ \beta = \beta \circ \lambda_V.$$

For each $V \in \mathcal{R}$, there is a representation r_V of $G(\mathfrak{g})$ on V defined by

$$r_V(\lambda)v = \lambda_V v, \quad \lambda \in G(\mathfrak{g})(R), \quad v \in V(R), \quad R \text{ a } k\text{-algebra},$$

and $V \rightsquigarrow (V, r_V)$ is an equivalence of categories

$$\mathsf{Rep}(\mathfrak{g}) \xrightarrow{\sim} \mathsf{Rep}(G(\mathfrak{g})). \tag{1}$$

LEMMA 6.1. *The homomorphism $\eta \colon \mathfrak{g} \to \mathrm{Lie}(G(\mathfrak{g}))$ is injective, and the composite of the functors*

$$\mathsf{Rep}(G(\mathfrak{g})) \xrightarrow{(V,r) \rightsquigarrow (V,dr)} \mathsf{Rep}(\mathrm{Lie}(G(\mathfrak{g}))) \xrightarrow{\eta^\vee} \mathsf{Rep}(\mathfrak{g}) \tag{2}$$

is an equivalence of categories.

PROOF. According to (5.3), $\mathrm{Lie}(G(\mathfrak{g})) \simeq \mathfrak{g}_{\mathcal{R}}$, and so the first assertion follows from (5.2) and Ado's theorem. The composite of the functors in (2) is a quasi-inverse to the functor in (1). □

LEMMA 6.2. *The affine group $G(\mathfrak{g})$ is connected.*

PROOF. We have to show that if a representation V of $\mathfrak{g}$ has the property that the category of subquotients of direct sums of copies of V is stable under tensor products, then V is a trivial representation. When $\mathfrak{g} = k$, this is obvious (cf. 5.1), and when $\mathfrak{g}$ is semisimple it follows from (3.1).

Let V be a representation of $\mathfrak{g}$ with the property. It follows from the commutative case that the radical of $\mathfrak{g}$ acts trivially on V, and then it follows from the semisimple case that $\mathfrak{g}$ itself acts trivially. □

PROPOSITION 6.3. *The pair $(G(\mathfrak{g}),\eta)$ is universal: for any algebraic group H and k-algebra homomorphism $a\colon \mathfrak{g}\to \mathrm{Lie}(H)$, there is a unique homomorphism $b\colon G(\mathfrak{g})\to H$ such that $a=\mathrm{Lie}(b)\circ\eta$. In other words, the map sending a homomorphism $b\colon G(\mathfrak{g})\to H$ to the homomorphism $\mathrm{Lie}(b)\circ\eta\colon \mathfrak{g}\to\mathrm{Lie}(H)$ is a bijection*

$$\mathrm{Hom}_{\textit{affine groups}}(G(\mathfrak{g}),H)\to\mathrm{Hom}_{\textit{Lie algebras}}(\mathfrak{g},\mathrm{Lie}(H)). \tag{3}$$

If a is surjective and $\mathsf{Rep}(G(\mathfrak{g}))$ is semisimple, then b is surjective.

PROOF. From a homomorphism $b\colon G(\mathfrak{g})\to H$, we get a commutative diagram

$$\begin{tikzcd} \mathsf{Rep}(H) \arrow[r, "b^\vee"] \arrow[d, "\text{fully faithful}"'] & \mathsf{Rep}(G(\mathfrak{g})) \arrow[d, "\simeq"', "(6.1)"] \\ \mathsf{Rep}(\mathrm{Lie}(H)) \arrow[r, "a^\vee"] & \mathsf{Rep}(\mathfrak{g}) \end{tikzcd} \qquad a\overset{\text{def}}{=}\mathrm{Lie}(b)\circ\eta.$$

If $a=0$, then $a^\vee$ sends all objects to trivial objects, and so the functor $b^\vee$ does the same, which implies that the image of b is 1. Hence (3) is injective.

From a homomorphism $a\colon\mathfrak{g}\to\mathrm{Lie}(H)$, we get a tensor functor

$$\mathsf{Rep}(H)\to\mathsf{Rep}(\mathrm{Lie}(H))\xrightarrow{a^\vee}\mathsf{Rep}(\mathfrak{g})\simeq\mathsf{Rep}(G(\mathfrak{g}))$$

and hence a homomorphism $G(\mathfrak{g})\to H$, which acts as a on the Lie algebras. Hence (3) is surjective.

If a is surjective, then $a^\vee$ is fully faithful, and so $\mathsf{Rep}(H)\to\mathsf{Rep}(G(\mathfrak{g}))$ is fully faithful, which implies that $G(\mathfrak{g})\to G$ is surjective. □

PROPOSITION 6.4. *For any finite extension $k'\supset k$ of fields, $G(\mathfrak{g}_{k'})\simeq G(\mathfrak{g})_{k'}$.*

PROOF. More precisely, we prove that the pair $(G(\mathfrak{g})_{k'},\eta_{k'})$ obtained from $(G(\mathfrak{g}),\eta)$ by extension of the base field has the universal property characterizing $(G(\mathfrak{g}_{k'}),\eta)$. Let H be an algebraic group over k', and let H_* be the group over k obtained from H by restriction of the base field. Then

$$\begin{aligned}\mathrm{Hom}_{k'}(\mathcal{G}(\mathfrak{g})_{k'},H)&\simeq\mathrm{Hom}_k(\mathcal{G}(\mathfrak{g}),H_*)\quad\text{(universal property of } H_*)\\&\simeq\mathrm{Hom}_k(\mathfrak{g},\mathrm{Lie}(H_*))\quad(6.3)\\&\simeq\mathrm{Hom}_{k'}(\mathfrak{g}_{k'},\mathrm{Lie}(H)).\end{aligned}$$

For the last isomorphism, note that

$$\mathrm{Lie}(H_*)\overset{\text{def}}{=}\mathrm{Ker}(H_*(k[\varepsilon])\to H_*(k))\simeq\mathrm{Ker}(H(k'[\varepsilon])\to H(k'))\overset{\text{def}}{=}\mathrm{Lie}(H).$$

In other words, $\mathrm{Lie}(H_*)$ is $\mathrm{Lie}(H)$ regarded as a Lie algebra over k (instead of k'), and the isomorphism is simply the canonical isomorphism in linear algebra,

$$\mathrm{Hom}_{k\text{-linear}}(V,W)\simeq\mathrm{Hom}_{k'\text{-linear}}(V\otimes_k k',W)$$

(V,W vector spaces over k and k' respectively). □

The next theorem shows that, when $\mathfrak{g}$ is semisimple, $G(\mathfrak{g})$ is a semisimple algebraic group with Lie algebra $\mathfrak{g}$, and any other semisimple group with Lie algebra $\mathfrak{g}$ is a quotient of $G(\mathfrak{g})$; moreover, the centre of $G(\mathfrak{g})$ has character group P/Q.

THEOREM 6.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra.*

(a) *The homomorphism $\eta\colon \mathfrak{g} \to \mathrm{Lie}(G(\mathfrak{g}))$ is an isomorphism.*
(b) *The group $G(\mathfrak{g})$ is a connected semisimple group.*
(c) *For any algebraic group H and isomorphism $a\colon \mathfrak{g} \to \mathrm{Lie}(H)$, there exists a unique isogeny $b\colon G(\mathfrak{g}) \to H^{\circ}$ such that $a = \mathrm{Lie}(b) \circ \eta$.*
(d) *Let Z be the centre of $G(\mathfrak{g})$; then $X^{*}(Z) \simeq P/Q$.*

PROOF. (a) Because $\mathsf{Rep}(G(\mathfrak{g}))$ is semisimple, $G(\mathfrak{g})$ is reductive. Therefore $\mathrm{Lie}(G(\mathfrak{g}))$ is reductive (4.1), and so $\mathrm{Lie}(G(\mathfrak{g})) = \eta(\mathfrak{g}) \oplus \mathfrak{a} \oplus \mathfrak{c}$ with $\mathfrak{a}$ is semisimple and $\mathfrak{c}$ commutative. If $\mathfrak{a}$ or $\mathfrak{c}$ is nonzero, then there exists a nontrivial representation r of $G(\mathfrak{g})$ such that $\mathrm{Lie}(r)$ is trivial on $\mathfrak{g}$. But this is impossible because η defines an equivalence $\mathsf{Rep}(G(\mathfrak{g})) \to \mathsf{Rep}(\mathfrak{g})$.

(b) Now (4.1) shows that G is semisimple.

(c) Proposition 6.3 shows that there exists a unique homomorphism b such that $a = \mathrm{Lie}(b) \circ \eta$, which is an isogeny because $\mathrm{Lie}(b)$ is an isomorphism.

(d) In the next subsection, we show that if $\mathfrak{g}$ is splittable, then $X^{*}(Z) \simeq P/Q$ (as abelian groups). As $\mathfrak{g}$ becomes splittable over a finite Galois extension, this implies (d). □

REMARK 6.6. The isomorphism $X^{*}(Z) \simeq P/Q$ in (d) commutes with the natural actions of $\mathrm{Gal}(k^{\mathrm{al}}/k)$.

7 Split semisimple algebraic groups

Let $(\mathfrak{g}, \mathfrak{h})$ be a split semisimple Lie algebra, and let P and Q be the corresponding weight and root lattices. The action of $\mathfrak{h}$ on a $\mathfrak{g}$-module V decomposes it into a direct sum $V = \bigoplus_{\varpi \in P} V_{\varpi}$. Let $D(P)$ be the diagonalizable group attached to P. Then $\mathsf{Rep}(D(P))$ has a natural identification with the category of P-graded vector spaces. The functor $(V, r_V) \mapsto (V, (V_{\varpi})_{\omega \in P})$ is an exact tensor functor $\mathsf{Rep}(\mathfrak{g}) \to \mathsf{Rep}(D(P))$, and hence defines a homomorphism $D(P) \to G(\mathfrak{g})$. Let $T(\mathfrak{h})$ be the image of this homomorphism.

THEOREM 7.1. *With the above notations:*

(a) *The group $T(\mathfrak{h})$ is a split maximal torus in $G(\mathfrak{g})$, and η restricts to an isomorphism $\mathfrak{h} \to \mathrm{Lie}(T(\mathfrak{h}))$.*
(b) *The map $D(P) \to T(\mathfrak{h})$ is an isomorphism; therefore, $X^{*}(T(\mathfrak{h})) \simeq P$.*
(c) *The centre of $G(\mathfrak{g})$ is contained in $T(\mathfrak{h})$ and equals*

$$\bigcap\nolimits_{\alpha \in R} \mathrm{Ker}(\alpha\colon T(\mathfrak{h}) \to \mathbb{G}_m)$$

(and so has character group P/Q).

PROOF. (a) The torus $T(\mathfrak{h})$ is split because it is the quotient of a split torus. Certainly, η restricts to an injective homomorphism $\mathfrak{h} \to \mathrm{Lie}(T(\mathfrak{h}))$. It must be surjective because otherwise $\mathfrak{h}$ wouldn't be a Cartan subalgebra of $\mathfrak{g}$. The torus $T(\mathfrak{h})$ must be maximal because otherwise $\mathfrak{h}$ wouldn't be equal to its normalizer.

(b) Let V be the representation $\bigoplus V_{\varpi}$ of $\mathfrak{g}$ where ϖ runs through a set of fundamental weights. Then $G(\mathfrak{g})$ acts on V, and the map $D(P) \to \mathrm{GL}(V)$ is injective. Therefore, $D(P) \to T(\mathfrak{h})$ is injective.

(c) A gradation on $\mathsf{Rep}(\mathfrak{g})$ is defined by a homomorphism $P \to M(P_{++})$ (see 3.5), and hence by a homomorphism $D(M(P_{++})) \to T(\mathfrak{h})$. This shows that the centre of G is contained in $T(\mathfrak{h})$.

Because the centre of $\mathfrak{g}$ is trivial, the kernel of the adjoint map $\mathrm{Ad}\colon G \to \mathrm{GL}_{\mathfrak{g}}$ is the centre $Z(G)$ of G, and so the kernel of $\mathrm{Ad}\,|T(\mathfrak{h})$ is $Z(G) \cap T(\mathfrak{h}) = Z(G)$. But

$$\mathrm{Ker}(\mathrm{Ad}\,|T(\mathfrak{h})) = \bigcap_{\alpha \in R} \mathrm{Ker}(\alpha),$$

so $Z(G)$ is as described. □

THEOREM 7.2. *Let T and T' be split maximal tori in $G(\mathfrak{g})$. Then $T' = gTg^{-1}$ for some $g \in G(\mathfrak{g})(k)$.*

PROOF. Let x be a nilpotent element of $\mathfrak{g}$. For any representation (V, r_V) of $\mathfrak{g}$, $e^{r_V(x)} \in G(\mathfrak{g})(k)$. There exist nilpotent elements $x_1, \ldots, x_m$ in $\mathfrak{g}$ such that

$$e^{\mathrm{ad}(x_1)} \cdots e^{\mathrm{ad}(x_m)}\,\mathrm{Lie}(T) = \mathrm{Lie}(T').$$

Let $g = e^{\mathrm{ad}(x_1)} \cdots e^{\mathrm{ad}(x_m)}$; then $gTg^{-1} = T'$ because they have the same Lie algebra. □

8 Classification

We can now read off the classification theorems for split semisimple algebraic groups from the similar theorems for split semisimple Lie algebras.

Let (G, T) be a split semisimple algebraic group. Because T is diagonalizable, the k-vector space $\mathfrak{g}$ decomposes into eigenspaces under its action:

$$\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}^{\alpha}.$$

The roots of (G, T) are the nonzero α such that $\mathfrak{g}^{\alpha} \neq 0$. Let R be the set of roots of (G, T).

PROPOSITION 8.1. *The set of roots of (G, T) is a reduced root system R in $V \overset{\text{def}}{=} X^*(T) \otimes \mathbb{Q}$; moreover,*

$$Q(R) \subset X^*(T) \subset P(R). \tag{4}$$

PROOF. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{h} = \mathrm{Lie}\, T$. Then $(\mathfrak{g}, \mathfrak{h})$ is a split semisimple Lie algebra, and, when we identify V with a subspace of $\mathfrak{h}^{\vee} \simeq X^*(T) \otimes k$, the roots of (G, T) coincide with the roots of $(\mathfrak{g}, \mathfrak{h})$ and (4) holds. □

By a ***diagram*** (V, R, X), we mean a reduced root system (V, R) over $\mathbb{Q}$ and a lattice X in V that is contained between $Q(R)$ and $P(R)$.

THEOREM 8.2 (EXISTENCE). *Every diagram arises from a split semisimple algebraic group over k.*

More precisely, we have the following result.

THEOREM 8.3. *Let (V, R, X) be a diagram, and let $(\mathfrak{g}, \mathfrak{h})$ be a split semisimple Lie algebra over k with root system $(V \otimes k, X)$. Let $\mathsf{Rep}(\mathfrak{g})^X$ be the full subcategory of $\mathsf{Rep}(\mathfrak{g})$ whose objects are those whose simple components have heighest weight in X. Then $\mathsf{Rep}(\mathfrak{g})^X$ is a tannakian subcategory of $\mathsf{Rep}(\mathfrak{g})$, and there is a natural functor $\mathsf{Rep}(\mathfrak{g})^X \to \mathsf{Rep}(D(X))$. The Tannaka dual (G, T) of this functor is a split semisimple algebraic group with diagram (V, R, X).*

PROOF. When $X = Q$, $(G,T) = (G(\mathfrak{g}), T(\mathfrak{h}))$, and the statement follows from Theorem 7.1. For an arbitrary X, let

$$N = \bigcap\nolimits_{\chi \in X/Q} \mathrm{Ker}(\chi\colon Z(G(\mathfrak{g})) \to \mathbb{G}_m).$$

Then $\mathsf{Rep}(\mathfrak{g})^X$ is the subcategory of $\mathsf{Rep}(\mathfrak{g})$ on which N acts trivially, and so it is a tannakian category with Tannaka dual $G(\mathfrak{g})/N$. Now it is clear that $(G(\mathfrak{g})/N, T(\mathfrak{h})/N)$ is the Tannaka dual of $\mathsf{Rep}(\mathfrak{g})^X \to \mathsf{Rep}(D(X))$, and that it has diagram (V, R, X). □

THEOREM 8.4 (ISOGENY). *Let (G,T) and (G',T') be split semisimple algebraic groups over k, and let (V,R,X) and (V,R',X') be their associated diagrams. Any isomorphism $V \to V'$ sending R onto R' and X into X' arises from an isogeny $G \to G'$ mapping T onto T'.*

PROOF. Let $(\mathfrak{g},\mathfrak{h})$ and $(\mathfrak{g}',\mathfrak{h}')$ be the split semisimple Lie algebras of (G,T) and (G',T'). An isomorphism $V \to V'$ sending R onto R' and X into X' arises from an isomorphism $(\mathfrak{g},\mathfrak{h}) \xrightarrow{\beta} (\mathfrak{g}',\mathfrak{h}')$. Now β defines an exact tensor functor $\mathsf{Rep}(\mathfrak{g}')^{X'} \to \mathsf{Rep}(\mathfrak{g})^X$, and hence a homomorphism $\alpha\colon G \to G'$, which has the required properties. □

References

BOREL, A. 1975. Linear representations of semi-simple algebraic groups, pp. 421–440. *In* Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974). Amer. Math. Soc., Providence, R.I.

BOREL, A. 1991. Linear algebraic groups, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

BOURBAKI, N. Lie. Groupes et Algèbres de Lie. Elements of Mathematics. Hermann; Masson, Paris. Chap. I, Hermann 1960; Chap. II,III, Hermann 1972; Chap. IV,V,VI, Masson 1981;Chap. VII,VIII, Masson 1975; Chap. IX, Masson 1982 (English translation available from Springer).

CARTIER, P. 1956. Dualité de Tannaka des groupes et des algèbres de Lie. *C. R. Acad. Sci. Paris* 242:322–325.

CHEVALLEY, C. 1960-61. Certains schémas de groupes semi-simples, pp. Exp. No. 219, 219–234. *In* Séminaire Bourbaki, 13 année. Soc. Math. France, Paris. Reprinted 1995 by the SMF.

CHEVALLEY, C. C. 1956-58. Classification des groupes de Lie algébriques, Seminaire ENS, Paris. mimeographed. Reprinted 2005 by Springer Verlag with a postface by P. Cartier.

DELIGNE, P. AND MILNE, J. S. 1982. Tannakian categories, pp. 101–228. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics. Springer-Verlag, Berlin.

HUMPHREYS, J. E. 1972. Introduction to Lie algebras and representation theory. Springer-Verlag, New York.

HUMPHREYS, J. E. 1975. Linear algebraic groups. Springer-Verlag, New York.

IWAHORI, N. 1954. On some matrix operators. *J. Math. Soc. Japan* 6:76–105.

MILNE, J. S. 2007. Semisimple Lie algebras, algebraic groups, and tensor categories. May 9, 2007, available at www.jmilne.org/math/.

SAAVEDRA RIVANO, N. 1972. Catégories Tannakiennes. Springer-Verlag, Berlin.

SPRINGER, T. A. 1998. Linear algebraic groups, volume 9 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA.

TAUVEL, P. AND YU, R. W. T. 2005. Lie algebras and algebraic groups. Springer Monographs in Mathematics. Springer-Verlag, Berlin.