Gerbes and Abelian Motives

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Abstract

Assuming the Hodge conjecture for abelian varieties of CM-type, one obtains a good category of abelian motives over $\mathbb{F}_p^{\mathrm{al}}$ and a reduction functor to it from the category of CM-motives over $\mathbb{Q}^{\mathrm{al}}$. Consequently, one obtains a morphism of gerbes of fibre functors with certain properties. We prove unconditionally that there exists a morphism of gerbes with these properties, and we classify them.

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Introduction

Fix a p-adic prime of $\mathbb{Q}^{\mathrm{al}}$, and denote its residue field by $\mathbb{F}$. Let $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ be the category of motives based on the abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$, and let $\mathsf{Mot}(\mathbb{F})$ be the similar category based on the abelian varieties over $\mathbb{F}$; in both cases, the correspondences are to be the algebraic cycles modulo numerical equivalence. Both are Tannakian $\mathbb{Q}$-categories and, because every abelian variety of CM-type has good reduction, there is an exact tensor "reduction" functor

$$R\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}(\mathbb{F}).$$

From R, we obtain a morphism

$$R^{\vee}\colon \mathsf{Mot}(\mathbb{F})^{\vee} \to \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\vee}$$

of gerbes of fibre functors. The bands of $\mathsf{Mot}(\mathbb{F})^{\vee}$ and $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\vee}$ are commutative, and so $R^{\vee}$ is bound by a homomorphism

$$\rho\colon P \to S$$

of commutative affine group schemes over $\mathbb{Q}$.

It is known how to construct an explicit homomorphism $\rho\colon P \to S$ of commutative affine group schemes (Grothendieck, Langlands, Rapoport,...) and to prove that it becomes the homomorphism in the last paragraph when the Hodge conjecture is assumed for abelian varieties of CM-type (this is explained in Milne 1994, §2, §4, under the additional assumption of the Tate conjecture for abelian varieties over finite fields, but that assumption is shown to be superfluous in Milne 1999).

In this paper, we construct (unconditionally) an explicit morphism

$$\mathsf{P} \to \mathsf{S}$$

of gerbes bound by the homomorphism $\rho\colon P \to S$ in the last paragraph, and we prove that it becomes the motivic morphism in the first paragraph when the Hodge conjecture is assumed for abelian varieties of CM-type. Moreover, we classify the morphisms $\mathsf{P} \to \mathsf{S}$ bound by ρ and satisfying certain natural conditions.

Upon choosing an $x \in \mathsf{P}(\mathbb{Q}^{\mathrm{al}})$, we obtain a morphism of groupoids

$$\mathsf{P}(x) \to \mathsf{S}(x)$$

such that the map on kernels

$$\mathsf{P}(x)^{\Delta} \to \mathsf{S}(x)^{\Delta}$$

is the homomorphism ρ.

We now describe the contents of the paper in more detail.

In §1 we compute some inverse limits. In particular, we show that certain inverse limits are infinite (see, for example, 1.11), and hence can not be ignored as they have been in previous works.

After some preliminaries on the cohomology of protori in §2, we construct and classify certain "fundamental" cohomology classes in §§3–4.

In §5, we review part of the theory of gerbes and their classification (Giraud 1971, Debremaeker 1977).

In §6, we prove the existence of gerbes P having the properties expected of $\mathsf{Mot}(\mathbb{F})^{\vee}$, and we classify them.

Finally, in §7 we prove the existence of morphisms of gerbes $\mathsf{P} \to \mathsf{S}$ having the properties expected of $\mathsf{Mot}(\mathbb{F})^{\vee} \to \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\vee}$, and we classify them.

In large part, this article is a critical re-examination of the results and proofs in §§2–4 (pp118–165) of Langlands and Rapoport 1987. In particular, we eliminate the confusion between fpqc cohomology groups and inverse limits of Galois cohomology groups that has persisted in the literature for fifteen years (e.g., Langlands and Rapoport 1987, Milne 1994, Reimann 1997), which amounted to setting certain $\varprojlim^1$s equal to zero. Also we avoid the confusion between gerbes and groupoids to be found in Langlands and Rapoport 1987. Finally, we are concerned, not just with the existence of the various object, but also with their classification.

For the convenience of the reader, I have made this article independent of earlier articles on the topic.

Notations and conventions

The algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted $\mathbb{Q}^{\mathrm{al}}$. The symbol p denotes a fixed finite prime of $\mathbb{Q}$ and ∞ denotes the real prime.

Complex conjugation on $\mathbb{C}$, or a subfield, is denoted by ι. A CM-field is a finite extension E of $\mathbb{Q}$ admitting an involution $\iota_E \neq 1$ such that $\rho \circ \iota_E = \iota \circ \rho$ for all homomorphisms $\rho\colon E \to \mathbb{C}$. The composite of the CM-subfields of $\mathbb{Q}^{\mathrm{al}}$ is denoted $\mathbb{Q}^{\mathrm{cm}}$.

The set $\mathbb{N} \smallsetminus \{0\}$, partially ordered by divisibility, is denoted $\mathbb{N}^{\times}$. For a finite set S, $\mathbb{Z}^S = \mathrm{Hom}(S, \mathbb{Z})$.

For a perfect field k of characteristic p, $W(k)$ is the ring of Witt vectors with coefficients in k and $B(k)$ is the field of fractions of $W(k)$. The automorphism of $W(k)$ inducing $x \mapsto x^p$ on the residue field is denoted σ.

For a finite extension of fields $K \supset k$, $(\mathbb{G}_m)_{K/k}$ is the torus over k obtained from $\mathbb{G}_m$ over K by restriction of scalars.

Groupoid will always mean affine transitive groupoid scheme (Deligne 1990).

For a group scheme G, a right G-object X, and a left G-object Y, $X \wedge^G Y$ denotes the contracted product of X and Y, that is, the quotient of $X \times Y$ by the diagonal action of G, $(x,y)g = (xg, g^{-1}y)$. When $G \to H$ is a homomorphism of group schemes, $X \wedge^G H$ is the H-object obtained from X by extension of the structure group. In this last case, if X is a G-torsor, then $X \wedge^G H$ is also an H-torsor.

The notation $X \approx Y$ means that X and Y are isomorphic, and $X \cong Y$ means that X and Y are canonically isomorphic (or that a particular isomorphism is given).

Direct and inverse limits are always with respect to directed index sets (partially ordered set I such that, for all $i, j \in I$, there exists a $k \in K$ for which $i \le k$ and $j \le k$). An inverse system is *strict* if its transition maps are surjective.

For class field theory we use the sign convention that the local Artin map sends a prime element to the Frobenius automorphism (that inducing $x \mapsto x^q$ on the residue field); for Hodge structures we use the convention that $h(z)$ acts on $V^{p,q}$ as $z^p \bar{z}^q$.

Philosophy

We adopt the following definitions.

0.1. An element of a set is *well-defined* (by a property, construction, condition, etc.) when it is uniquely determined (by the property, construction, condition etc.).

0.2. An object of a category is *well-defined* when it is uniquely determined up to a uniquely given isomorphism. For example, an object defined by a universal property is well-defined in this sense. When X is well-defined in this sense and X' also has the defining property, then each element of X corresponds to a well-defined element of X'.

0.3. A category is *well-defined* when it is uniquely determined up to a category-equivalence[1] that is itself uniquely determined up to a uniquely given isomorphism. When C is well-defined in this sense and C' also has the defining property, then each object of C corresponds to a well-defined object of C'.

Advice to the reader

The article has been written in logical order. The reader is advised to begin with §§5,6,7 and refer back to the earlier sections only as needed. Also, the results in §6 on the gerbe

[1]Recall that a category equivalence is a functor $F\colon \mathsf{C} \to \mathsf{C}'$ for which there exists a functor $G\colon \mathsf{C}' \to \mathsf{C}$ and isomorphisms $\phi\colon \mathrm{id} \to GF$ and $\psi\colon 1 \to FG$ such that $F\phi = \psi F$ (Bucur and Deleanu 1968, 1.6; $F\phi$ and ψF are morphisms $F \to FGF$). Our condition says that if C and C' have the defining property, then there is a distinguished class of equivalences $F\colon \mathsf{C} \to \mathsf{C}'$ and a distinguished class of equivalences $F'\colon \mathsf{C}' \to \mathsf{C}$; if $F_1\colon \mathsf{C} \to \mathsf{C}'$ and $F_2\colon \mathsf{C} \to \mathsf{C}'$ are both in the distinguished class, then there is given a (unique) isomorphism $F_1 \to F_2$.

conjecturally attached to $\mathsf{Mot}(\mathbb{F})$ require very little of the rest of the paper.

1 Some inverse limits.

We compute some $\varprojlim$s and $\varprojlim^1$s that are needed in the rest of the paper.

Review of higher inverse limits

For an inverse system (A_n, u_n) of abelian groups indexed by $(\mathbb{N}, \leq)$, $\varprojlim A_n$ and $\varprojlim^1 A_n$ can be defined to be the kernel and cokernel respectively of

$$(\ldots, a_n, \ldots) \mapsto (\ldots, a_n - u_{n+1}(a_{n+1}), \ldots)\colon \prod_n A_n \xrightarrow{1-u} \prod_n A_n. \tag{1}$$

From the snake lemma, we see that an inverse system of short exact sequences

$$0 \to (A_n) \to (B_n) \to (C_n) \to 0$$

gives rise to an exact sequence

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \varprojlim{}^1 A_n \to \varprojlim{}^1 B_n \to \varprojlim{}^1 C_n \to 0. \tag{2}$$

In particular, $\varprojlim$ is left exact and $\varprojlim^1$ is right exact.

Recall that an inverse system $(A_n, u_n)_{n\in\mathbb{N}}$ is said to satisfy the condition (ML) if, for each n, the decreasing chain in A_n of the images of the A_i for $i \geq n$ is eventually constant.

PROPOSITION 1.1. *The group* $\varprojlim^1_{n\in\mathbb{N}} A_n = 0$ *if*

(a) *the* A_n *are compact and the transition maps are continuous, or*

(b) *the system* $(A_n)_{n\in\mathbb{N}}$ *satisfies (ML).*

PROOF. Standard.[2] □

REMARK 1.2. Consider an inverse system $(A_n)_{n\in\mathbb{N}}$ of finite groups. If $\varprojlim A_n$ is infinite, then it is uncountable. In proving this we may assume[3] that the transition maps are surjective. Because $\varprojlim A_n$ is infinite, the orders of the A_n are unbounded, and the Cantor diagonalization process can be applied.

REMARK 1.3. We shall frequently make use of the following (obvious) criterion:

$$\varprojlim A_n = 0 \text{ if, for all } n,\ \cap_i \operatorname{Im}(A_{n+i} \to A_n) = 0.$$

[2] Consider the map

$$(a_0, \ldots, a_n, \ldots, a_{N+1}) \mapsto (\ldots, a_n - u_{n+1}(a_{n+1}), \ldots)\colon \prod_{0\leq n\leq N+1} A_n \to \prod_{0\leq n\leq N} A_n.$$

Let $x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$, and let P_N be the inverse image of $(x_n)_{0\leq n\leq N}$ in $\prod_{0\leq n\leq N+1} A_n$. We have to show that $\varprojlim P_N$ is nonempty. The projection $(a_n)_{0\leq n\leq N+1} \mapsto a_{N+1}\colon P_N \to A_{N+1}$ is bijective. In case (a), the P_N are compact, and so this case follows from Bourbaki 1989, I §9.6, Proposition 8. In case (b), $(P_N)_{N\in\mathbb{N}}$ satisfies (ML). Let $Q_N = \cap_i \operatorname{Im}(P_{N+i} \to P_N)$. Then each Q_N is nonempty, and $(Q_N)_{N\in\mathbb{N}}$ is a strict inverse system. Hence $\varprojlim Q_N$ is (obviously) nonempty. As $\varprojlim Q_N = \varprojlim P_N$, this proves (b).

[3] As in the preceding footnote.

PROPOSITION 1.4. *If the inverse system $(A_n)_{n\in\mathbb{N}}$ fails (ML) and the A_n are countable, then $\varprojlim^1 A_n$ is uncountable.*

PROOF. See, for example, Milne 2002b, 1.1b. □

EXAMPLE 1.5. Let $(G_n)_{n\in\mathbb{N}}$ be an inverse system of commutative affine algebraic groups over $\mathbb{Q}$ with surjective transition maps. The inverse system $(G_n(\mathbb{Q}))_{n\in\mathbb{N}}$ will usually fail (ML), and so $\varprojlim^1 G_n(\mathbb{Q})$ will usually be uncountable. If G_n contains no $\mathbb{Q}$-split torus, then $G_n(\mathbb{A})/G_n(\mathbb{Q})$ is compact (Platonov and Rapinchuk 1994, Theorem 5.5, p260), and so $\varprojlim{}^1 G_n(\mathbb{A})/G_n(\mathbb{Q}) = 0$; thus

$$\varprojlim{}^1 G_n(\mathbb{Q}) \to \varprojlim{}^1 G_n(\mathbb{A})$$

is surjective.

REMARK 1.6. For an arbitrary directed set I, the category of inverse systems of abelian groups indexed by I has enough injectives, and so there are right derived functors $\varprojlim^i_I$ of $\varprojlim_I$ (Jensen 1972 , §1). For $I = (\mathbb{N}, \leq)$, they agree with those defined above (ibid. p13). If J is a cofinal subset of J, then $\varprojlim^i_J A_\alpha \cong \varprojlim^i_I A_\alpha$. This is obvious for $i = 0$ and it follows for a general i by the usual derived-functor argument (cf. ibid. 1.9 and statement p12). If $(I, \leq)$ is a countable directed set, then I contains a cofinal subset isomorphic to $(\mathbb{N}, \leq)$ or to a finite segment of $(\mathbb{N}, \leq)$. In the first case, the above statements apply to inverse systems indexed by I.

EXERCISE 1.7. For an abelian group A, let (A, m) denote the inverse system indexed by $\mathbb{N}^\times$

$$\cdots \leftarrow \underset{n}{A} \overset{m}{\leftarrow} \underset{mn}{A} \leftarrow \cdots .$$

Show:

(a) $\varprojlim(A, m) = 0 = \varprojlim^1(A, m)$ if $NA = 0$ for some integer N;

(b) $\varprojlim(\mathbb{Z}, m) = 0$, $\varprojlim^1(\mathbb{Z}, m) \cong \hat{\mathbb{Z}}/\mathbb{Z}$;

(c) $\varprojlim(\mathbb{Q}/\mathbb{Z}, m) \cong \mathbb{A}_f$, $\varprojlim^1(\mathbb{Q}/\mathbb{Z}, m) = 0$. [Hint: Regard $(\mathbb{Q}/\mathbb{Z}, m)$ as the projective system $(\mathbb{Q}/m\mathbb{Z}, 1)$, and note that $\mathbb{Q}/\mathbb{Z} \cong \oplus \mathbb{Q}_\ell/\mathbb{Z}_\ell$.]

Notations from class field theory

For a number field L, $U(L)$ is the group of units in the ring of integers of L, $I(L)$ is the group of fractional ideals, and $\mathbb{I}(L)$ is the group of idèles. For a modulus $\mathfrak{m}$ of L, $C_\mathfrak{m}(L)$ is the ray class group, and $H_\mathfrak{m}(L)$ is the corresponding ray class field. When $\mathfrak{m} = 1$, we omit it from the notation. Thus, $C(L)$ is the usual ideal class group and $H(L)$ is the Hilbert class field.

For a finite extension L/F of number fields and moduli $\mathfrak{m}$ of F and $\mathfrak{n}$ of L such that $\mathfrak{m}|\operatorname{Nm}_{L/F}(\mathfrak{n})$, there is a commutative diagram

$$\begin{array}{ccc} C_{\mathfrak{n}}(L) & \xrightarrow{\cong} & \operatorname{Gal}(H_{\mathfrak{n}}(L)/L) \\ \downarrow{\scriptstyle \operatorname{Nm}_{L/F}} & & \downarrow{\scriptstyle \sigma\mapsto\sigma|H_{\mathfrak{m}}(F)} \\ C_{\mathfrak{m}}(F) & \xrightarrow{\cong} & \operatorname{Gal}(H_{\mathfrak{m}}(F)/F) \end{array} \qquad (3)$$

(Tate 1967, 3.2, p166). The horizontal isomorphisms are defined by the Artin symbols $(-,H_{\mathfrak{n}}(L)/L)$ and $(-,H_{\mathfrak{m}}(F)/F)$.

Inverse systems indexed by fields

1.8. Let $\mathcal{F}$ be an infinite set of subfields of $\mathbb{Q}^{\mathrm{al}}$ such that

(a) each field in $\mathcal{F}$ is of finite degree over $\mathbb{Q}$;

(b) the composite of any two fields in $\mathcal{F}$ is again in $\mathcal{F}$.

Then $(\mathcal{F},\subset)$ is a directed set admitting a cofinal subset isomorphic to $(\mathbb{N},\leq)$. For such a set $\mathcal{F}$, the norm maps define inverse systems $(F^{\times})_{F\in\mathcal{F}}$, $(I(F))_{F\in\mathcal{F}}$, etc..

EXAMPLE 1.9. (a) The set of all totally real fields in $\mathbb{Q}^{\mathrm{al}}$ of finite degree over $\mathbb{Q}$ satisfies (1.8a,b). If $f\in\mathbb{R}[X]$ is monic with $\deg(f)$ distinct real roots, then any monic polynomial in $\mathbb{R}[X]$ sufficiently close to f will have the same property (because any quadratic factor of it will have discriminant >0). From this it is easy to construct many totally real extensions of $\mathbb{Q}$. For example, Krasner's lemma shows that, for any finite extension L of $\mathbb{Q}_p$, there exists a finite totally real extension F of $\mathbb{Q}$ contained in L such that $[F\colon\mathbb{Q}]=[L\colon\mathbb{Q}_p]$ and $F\cdot\mathbb{Q}_p=L$. Also, a standard argument in Galois theory can be modified to show that, for all n, there exist totally real fields with Galois group S_n over $\mathbb{Q}$ (Wei 1993, 1.6.7).

(b) The set of all CM fields in $\mathbb{Q}^{\mathrm{al}}$ of finite degree over $\mathbb{Q}$ satisfies (1.8a,b). Let $d\in\mathbb{Z}$, $d>0$. For any totally real field F, $F[\sqrt{-d}]$ is a CM-field, and every CM-field containing $\sqrt{-d}$ is of this form. As F runs over the totally real subfields of $\mathbb{Q}^{\mathrm{al}}$, $F[\sqrt{-d}]$ runs over a cofinal set of CM-subfields of $\mathbb{Q}^{\mathrm{al}}$.

The inverse system $(C(K))$

PROPOSITION 1.10. *Let $\mathcal{F}$ be the set of totally real fields in $\mathbb{Q}^{al}$ of finite degree over $\mathbb{Q}$. Then $\varprojlim_{\mathcal{F}} C(F)=0$.*

PROOF. The Hilbert class field $H(F)$ of a totally real field F is again totally real, and diagram (3) shows that, for any totally real field L containing $H(F)$, the norm map $C(L)\to C(F)$ is zero. Therefore, $\varprojlim C(F)=0$ by (1.3). □

PROPOSITION 1.11. *Let $\mathcal{F}$ be the set of CM-fields in $\mathbb{Q}^{al}$ of finite degree over $\mathbb{Q}$. Then $\varprojlim_{\mathcal{F}} C(K)$ is uncountable.*

Let K be a CM-field with largest real subfield K_+. Let $\mathfrak{m}$ be a modulus for K such that $\iota_K\mathfrak{m} = \mathfrak{m}$ and the exponent of any finite prime ramified in K/K_+ is even. Then $\mathfrak{m}$ is the extension to K of a unique modulus $\mathfrak{m}_+$ for K_+ such that $(\mathfrak{m}_+)_\infty = 1$. This last condition implies that $H_{\mathfrak{m}_+}(K_+) \cap K = K_+$, and so the norm map $C_\mathfrak{m}(K) \to C_{\mathfrak{m}_+}(K_+)$ is surjective (apply (3)). Define $C^-_\mathfrak{m}(K)$ by the exact sequence

$$0 \to C^-_\mathfrak{m}(K) \longrightarrow C_\mathfrak{m}(K) \xrightarrow{\mathrm{Nm}_{K/K_+}} C_{\mathfrak{m}_+}(K_+) \to 0. \tag{4}$$

LEMMA 1.12. *With the above notations, any CM-subfield L of $H_\mathfrak{m}(K)$ containing K is fixed by the subgroup $2 \cdot C^-_\mathfrak{m}(K)$ of $C_\mathfrak{m}(K)$, i.e., $2 \cdot C^-_\mathfrak{m}(K) \hookrightarrow \mathrm{Gal}(H_\mathfrak{m}(K)/L)$.*

PROOF. We shall need the general fact:

> (*). Let L/F be a finite Galois extension of number fields, and let τ be a homomorphism $L \to \mathbb{Q}^{\mathrm{al}}$. For any prime ideal $\mathfrak{P}$ of L unramified in L/F,
>
> $$(\tau\mathfrak{P}, \tau L/\tau F) = \tau \circ (\mathfrak{P}, L/F) \circ \tau^{-1}$$
>
> (equality of Artin symbols). In particular, if L/F is abelian, $\tau L = L$, and $\tau F = F$, then for any prime ideal $\mathfrak{p}$ of F unramified in L, $(\tau\mathfrak{p}, L/F) = \tau \circ (\mathfrak{p}, L/F) \circ \tau^{-1}$.

Because $\iota_K\mathfrak{m} = \mathfrak{m}$, $H_\mathfrak{m}(K)$ is stable under ι, and hence is Galois over K_+. The Galois closure of L over K_+ will again be CM, and so we may assume L to be Galois over K_+. Let H be the subgroup of $C_\mathfrak{m}(K)$ fixing L, and consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_\mathfrak{m}(K) & \longrightarrow & \mathrm{Gal}(H_\mathfrak{m}(K)/K_+) & \longrightarrow & \mathrm{Gal}(K/K_+) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C_\mathfrak{m}(K)/H & \longrightarrow & \mathrm{Gal}(L/K_+) & \longrightarrow & \mathrm{Gal}(K/K_+) & \longrightarrow & 0
\end{array}$$

in which we have used the reciprocity map to replace $\mathrm{Gal}(H_\mathfrak{m}(K)/K)$ with $C_\mathfrak{m}(K)$. According to (*), the action of $\mathrm{Gal}(K/K_+) = \langle \iota_K \rangle$ on $C_\mathfrak{m}(K)$ defined by the upper sequence is the natural action. Because L is a CM-field, the action of ι_K on $C_\mathfrak{m}(K)/H$ defined by the lower sequence is trivial. The composite of the maps

$$C_\mathfrak{m}(K) \xrightarrow{\mathrm{Nm}_{K/K_+}} C_{\mathfrak{m}_+}(K_+) \longrightarrow C_\mathfrak{m}(K)$$

is $c \mapsto c \cdot \iota c$, and so ι_K acts as -1 on $C^-_\mathfrak{m}(K)$. Thus $H \supset 2C^-_\mathfrak{m}(K)$. □

For a finite abelian group A, let $A(\mathrm{odd})$ denote the subgroup of elements of odd order in A. Note that

$$C(K)(\mathrm{odd}) \cong C^-(K)(\mathrm{odd}) \oplus C(K_+)(\mathrm{odd}). \tag{5}$$

LEMMA 1.13. *For any CM-field L containing K, the norm map*

$$C^-(L)(\mathrm{odd}) \to C^-(K)(\mathrm{odd})$$

is surjective.

PROOF. The norm limitation theorem (Tate 1967, p202, paragraph 3) and (5) allow us to assume that L is abelian over K. Therefore, we may suppose $L \subset H_{\mathfrak{m}}(K)$ for some modulus $\mathfrak{m}$ as above. Moreover, we may suppose that L is the largest CM-subfield of $H_{\mathfrak{m}}(K)$. The field $H_{\mathfrak{m}_+}(K_+) \cdot K$ is CM, and so is contained in L. The field $H_{\mathfrak{m}_+}(L_+) \cdot L$ is also CM, and so

$$(H_{\mathfrak{m}_+}(L_+) \cdot L) \cap H_{\mathfrak{m}}(K) = L. \tag{6}$$

Consider the diagram (cf. (3))

$$\begin{array}{ccc}
C^-_{\mathfrak{m}}(L) & \xrightarrow{\cong} & \mathrm{Gal}(H_{\mathfrak{m}}(L)/H_{\mathfrak{m}_+}(L_+) \cdot L) \\
\downarrow{\scriptstyle \mathrm{Nm}_{L/K}} & & \downarrow{\scriptstyle \sigma \mapsto \sigma | H_{\mathfrak{m}}(K)} \\
C^-_{\mathfrak{m}}(K) & \xrightarrow{\cong} & \mathrm{Gal}(H_{\mathfrak{m}}(K)/H_{\mathfrak{m}_+}(K_+) \cdot K)
\end{array}$$

According to (6), the image of $\sigma \mapsto \sigma|H_{\mathfrak{m}}(K)$ is $\mathrm{Gal}(H_{\mathfrak{m}}(K)/L)$, and so by (1.12) the image of $\mathrm{Nm}_{L/K}$ contains $2 \cdot C^-_{\mathfrak{m}}(K)$. Therefore, $\mathrm{Nm}_{L/K} \colon C^-_{\mathfrak{m}}(L)(\mathrm{odd}) \to C^-_{\mathfrak{m}}(K)(\mathrm{odd})$ is surjective, which implies the similar statement without the $\mathfrak{m}$'s. □

PROOF OF PROPOSITION 1.11. As $\varprojlim C(K_+) = 0$ (1.10), the inverse system of exact sequences (4) gives an isomorphism

$$\varprojlim C^-(K) \cong \varprojlim C(K).$$

Because of (1.13), for every CM-field K_0, the projection

$$\varprojlim C^-(K)(\mathrm{odd}) \to C^-(K_0)(\mathrm{odd})$$

is surjective. An irregular prime l divides the order of $C^-(\mathbb{Q}[\zeta_l])$ (Washington 1997, 5.16, p62). Therefore the order of the profinite group $\varprojlim C^-(K)(\mathrm{odd})$ is divisible by every irregular prime. Since there are infinitely many irregular primes (ibid. 5.17), this implies that $\varprojlim C^-(K)(\mathrm{odd})$ is infinite and hence uncountable.[4] □

The inverse system $(K^\times/U(K))$

LEMMA 1.14. *For any set $\mathcal{F}$ satisfying (1.8a,b), there is an exact sequence*

$$0 \to \varprojlim{}_{\mathcal{F}} F^\times/U(F) \to \varprojlim{}_{\mathcal{F}} I(F) \to \varprojlim{}_{\mathcal{F}} C(F) \to \varprojlim{}^1_{\mathcal{F}} F^\times/U(F) \to \varprojlim{}^1_{\mathcal{F}} I(F) \to 0.$$

PROOF. Because the groups $C(F)$ are finite, $\varprojlim{}^1 C(F) = 0$ (1.1), and so this is the exact sequence (2) attached to the inverse system of short exact sequences

$$0 \to F^\times/U(F) \to I(F) \to C(F) \to 0.$$

□

[4] Let h^-_K be the order of $C^-(K)$ (the relative class number). According to the Brauer-Siegel theorem, as K ranges over CM number fields of a given degree, h^-_K is asymptotic to $\log(\sqrt{d_K/d_{K_+}})$ and goes to infinity with d_K (the discriminant of K). The Generalized Riemann Hypothesis implies a similar statement for the exponents of the class groups (Louboutin and Okasaki 2003).

LEMMA 1.15. *Let $\mathcal{F}$ be the set of totally real fields in $\mathbb{Q}^{al}$ of finite degree over $\mathbb{Q}$. Then*

$$\varprojlim{}_{\mathcal{F}} F^{\times} = 0 \tag{7}$$

$$\varprojlim{}_{\mathcal{F}} \mathbb{I}(F) \cong \varprojlim{}_{\mathcal{F}} (F \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \tag{8}$$

$$\varprojlim{}_{\mathcal{F}} F^{\times}/U(F) = 0, \tag{9}$$

$$\varprojlim{}^{1}_{\mathcal{F}} F^{\times}/U(F) \cong \varprojlim{}^{1}_{\mathcal{F}} I(F). \tag{10}$$

Moreover, $\varprojlim{}^{1}_{\mathcal{F}} F^{\times}/U(F)$ is uncountable.

PROOF. The equality (8) follows from the discussion (1.9) and the fact that a nonarchimedean local field has no universal norms. Equality (8) implies (7) because a global norm is a local norm.

An ideal $\mathfrak{a} = \mathfrak{p}^{m} \cdots \in I(F)$, $m \neq 0$, will not be in the image of

$$\mathrm{Nm}_{L/F} \colon I(L) \to I(F)$$

once L is so large that none of the residue class degrees of the primes above $\mathfrak{p}$ divides m. It follows that $\varprojlim I(F) = 0$ and that $(I(F))_{F \in \mathcal{F}}$ fails (ML). Thus, (1.14) proves (9) and that there is an exact sequence

$$0 \to \varprojlim C(F) \to \varprojlim{}^{1} F^{\times}/U(F) \to \varprojlim{}^{1} I(F) \to 0.$$

But $\varprojlim C(F) = 0$ (1.10) and so $\varprojlim^{1} F^{\times}/U(F) \cong \varprojlim^{1} I(F)$, which is uncountable (1.1). $\square$

LEMMA 1.16. *Let $\mathcal{F}$ be the set of CM-subfields of $\mathbb{Q}^{al}$ of finite degree over $\mathbb{Q}$. Then*

$$\varprojlim{}_{\mathcal{F}} K^{\times} = 0 \tag{11}$$

$$\varprojlim{}_{\mathcal{F}} \mathbb{I}(K) \cong \varprojlim{}_{\mathcal{F}} (K \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \tag{12}$$

$$\varprojlim{}_{\mathcal{F}} K^{\times}/U(K) = 0 \tag{13}$$

and there is an exact sequence

$$0 \to \varprojlim{}_{\mathcal{F}} C(K) \to \varprojlim{}^{1}_{\mathcal{F}} K^{\times}/U(K) \to \varprojlim{}^{1}_{\mathcal{F}} I(K) \to 0. \tag{14}$$

PROOF. The proofs of (11), (12), and (13) are similar to those of the corresponding equalities in (1.15). In particular, $\varprojlim I(K) = 0$, and so (14) follows from 1.14. $\square$

The inverse system $(K^{\times}/K_{+}^{\times})$

Let $\mathcal{F}$ be the set of CM fields in $\mathbb{Q}^{\mathrm{al}}$ of finite degree over $\mathbb{Q}$.

LEMMA 1.17. *The quotient map*

$$\varprojlim{}^{1}_{\mathcal{F}} K^{\times}/K_{+}^{\times} \to \varprojlim{}^{1}_{\mathcal{F}} (K^{\times}/U(K) \cdot K_{+}^{\times})$$

is an isomorphism.

PROOF. The homomorphism $U(K_+) \to U(K)$ has finite cokernel, and so

$$\varprojlim{}^1 U(K_+) \to \varprojlim{}^1 U(K)$$

is surjective. Now the statement follows from the diagram

$$\begin{tikzcd}
\varprojlim{}^1 U(K_+) \arrow[r] \arrow[d] & \varprojlim{}^1 K_+^\times \arrow[r] \arrow[d] & \varprojlim{}^1 K_+^\times/U(K_+) \arrow[r] \arrow[d] & 0 \\
\varprojlim{}^1 U(K) \arrow[r] \arrow[d] & \varprojlim{}^1 K^\times \arrow[r] \arrow[d] & \varprojlim{}^1 K^\times/U(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim{}^1 K^\times/K_+^\times \arrow[r] \arrow[d] & \varprojlim{}^1 K^\times/U(K)\cdot K_+^\times \arrow[d] & \\
& 0 & 0 & .
\end{tikzcd}$$

□

LEMMA 1.18. *The quotient map*

$$\varprojlim{}^1_{\mathcal{F}} \mathbb{I}(K)/\mathbb{I}(K_+) \to \varprojlim{}^1_{\mathcal{F}} I(K)/I(K_+)$$

is an isomorphism.

PROOF. For a number field L, let $V(L)$ denote the kernel of $\mathbb{I}(L) \to I(L)$. Let $r = [K_+ \colon \mathbb{Q}]$. Then $V(K_+) \approx (\mathbb{R}^\times)^r \times \{\text{compact}\}$ and $V(K) \approx (\mathbb{C}^\times)^r \times \{\text{compact}\}$, and so $V(K)/V(K_+)$ is compact. Therefore, $\varprojlim^1 V(K)/V(K_+) = 1$ by (1.1a), and the limit of the exact sequences

$$0 \to V(K)/V(K_+) \to \mathbb{I}(K)/\mathbb{I}(K_+) \to I(K)/I(K_+) \to 0$$

gives the required isomorphism. □

PROPOSITION 1.19. *There is an exact sequence*

$$0 \to \varprojlim{}_{\mathcal{F}} C(K) \to \varprojlim{}^1_{\mathcal{F}} K^\times/K_+^\times \to \varprojlim{}^1_{\mathcal{F}} \mathbb{I}(K)/\mathbb{I}(K_+) \to 0.$$

PROOF. Lemmas 1.15 and 1.16 show that the rows in

$$\begin{tikzcd}
& 0 \arrow[r] & \varprojlim{}^1_{\mathcal{F}} K_+^\times/U(K_+) \arrow[r] \arrow[d] & \varprojlim{}^1_{\mathcal{F}} I(K_+) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim{}_{\mathcal{F}} C(K) \arrow[r] & \varprojlim{}^1_{\mathcal{F}} K^\times/U(K) \arrow[r] & \varprojlim{}^1_{\mathcal{F}} I(K) \arrow[r] & 0
\end{tikzcd}$$

are exact, from which we deduce an exact sequence

$$0 \to \varprojlim{}_{\mathcal{F}} C(K) \to \varprojlim{}^1_{\mathcal{F}} (K^\times/U(K)\cdot K_+^\times) \to \varprojlim{}^1_{\mathcal{F}} I(K)/I(K_+) \to 0.$$

Now (1.17) and (1.18) allow us to replace the last two terms with $\varprojlim{}^1_{\mathcal{F}} K^\times/K_+^\times$ and $\varprojlim{}^1_{\mathcal{F}} \mathbb{I}(K)/\mathbb{I}(K_+)$. □

The inverse system $(K(w)^{\times}/K(w)^{\times}_{+})$

Fix a finite prime w of $\mathbb{Q}^{\mathrm{al}}$, and write w_L (or just w) for the prime it defines on a subfield L of $\mathbb{Q}^{\mathrm{al}}$. For an L that is Galois over $\mathbb{Q}$, let $D(w_L)$ denote the decomposition group, and let $L(w) = L^{D(w_L)}$. For a system $\mathcal{F}$ satisfying (1.8) and whose members are Galois over $\mathbb{Q}$, we define $(L(w)^{\times})_{L\in\mathcal{F}}$ to be the inverse system with transition maps

$$\left(\mathrm{Nm}_{L'(w)/L(w)}\right)^{[L'_w\colon L_w]}\colon L'(w)^{\times} \to L(w)^{\times}.$$

We define the inverse systems $(C(L(w)))_{F\in\mathcal{F}}$, $(I(L(w))_{F\in\mathcal{F}}$, etc., similarly.

PROPOSITION 1.20. *Let $\mathcal{F}$ be the set of totally real fields in $\mathbb{Q}^{al}$ of finite degree and Galois over $\mathbb{Q}$. Then*

$$\varprojlim{}_{\mathcal{F}}(C(F(w)) = 0$$

and the map

$$\varprojlim{}^1_{\mathcal{F}} F(w)^{\times}/U(F(w)) \to \varprojlim{}^1_{\mathcal{F}} I(F(w))$$

is an isomorphism.

PROOF. Let m be the exponent of $C(F(w))$. The F' in $\mathcal{F}$ such that $F' \supset F$ and $m|[F'_w\colon F_w]$ form a cofinal subset of $\mathcal{F}$.[5] For such an F', the map

$$\left(\mathrm{Nm}_{F'(w)/F(w)}\right)^{[F'_w\colon F_w]}\colon C(F'(w)) \to C(F(w))$$

is zero, and so $\varprojlim{}_{\mathcal{F}}(C(F(w)) = 0$ by the criterion (1.3). The second statement now follows from (1.14). □

PROPOSITION 1.21. *Let $\mathcal{F}$ be the set of CM fields in $\mathbb{Q}^{al}$ of finite degree and Galois over $\mathbb{Q}$. Then*

$$\varprojlim{}_{\mathcal{F}} C(K(w)) = 0 \tag{15}$$

and

$$\varprojlim{}^1_{\mathcal{F}} K(w)^{\times}/U(K(w)) \cong \varprojlim{}^1_{\mathcal{F}} I(K(w)) \tag{16}$$

$$\varprojlim{}^1_{\mathcal{F}} K(w)^{\times}/K(w)^{\times}_{+} \cong \varprojlim{}^1_{\mathcal{F}} K(w)^{\times}/U(K(w)) \cdot K(w)^{\times}_{+} \tag{17}$$

$$\varprojlim{}^1_{\mathcal{F}} \mathbb{I}(K(w))/\mathbb{I}(K(w)_{+}) \cong \varprojlim{}^1_{\mathcal{F}} I(K(w))/I(K(w)_{+}) \tag{18}$$

$$\varprojlim{}^1_{\mathcal{F}} K(w)^{\times}/K(w)^{\times}_{+} \cong \varprojlim{}^1_{\mathcal{F}} \mathbb{I}(K(w))/\mathbb{I}(K(w)_{+}). \tag{19}$$

PROOF. The proof of (15) is similar to that of (1.20) (cf. 1.9b), and (16) then follows from (1.14). The proofs of (16) and (17) are similar to those of (1.17) and (1.18). For (19), the horizontal arrows in the commutative diagram

$$\begin{array}{ccc}
\varprojlim{}^1_{\mathcal{F}} K(w)^{\times}_{+}/U(K(w)_{+}) & \xrightarrow{\cong} & \varprojlim{}^1_{\mathcal{F}} I(K(w)_{+}) \\
\downarrow & & \downarrow \\
\varprojlim{}^1_{\mathcal{F}} K(w)^{\times}/U(K(w)) & \xrightarrow{\cong} & \varprojlim{}^1_{\mathcal{F}} I(K(w)).
\end{array}$$

[5]Let f_0 be a monic polynomial in $\mathbb{Q}[X]$ whose splitting field is F; let L be a finite extension of F_w such that $m|[L\colon F_w]$, and write $L \approx \mathbb{Q}_p[X]/(g(X))$ with g monic; choose a monic $f \in \mathbb{Q}[X]$ such that f is close w-adically to g and really close to a monic polynomial that splits over $\mathbb{R}$; then, by Krasner's lemma, the splitting field F'_0 of $f_0 \cdot f$ will have the required property, as will any field in $\mathcal{F}$ containg F'_0.

are isomorphisms by Proposition 1.20 and (15). Thus,

$$\varprojlim{}^1_{\mathcal{F}}(K(w)^\times/U(K(w))\cdot K(w)^\times_+)\cong\varprojlim{}^1_{\mathcal{F}}I(K(w))/I(K(w)_+),$$

and (1.21b) and (1.21c) allow us to replace the terms with $\varprojlim{}^1_{\mathcal{F}}K(w)^\times/K(w)^\times_+$ and $\varprojlim{}^1_{\mathcal{F}}\mathbb{I}(K(w))/\mathbb{I}(K(w)_+)$. □

Conclusions

PROPOSITION 1.22. *Let $\mathcal{F}$ be the set of CM fields in $\mathbb{Q}^{al}$ of finite degree and Galois over $\mathbb{Q}$. In the diagram,*

$$\begin{array}{ccc}
\varprojlim^1_{\mathcal{F}} K(w)^\times/K(w)^\times_+ & \longrightarrow & \varprojlim^1_{\mathcal{F}} \mathbb{I}(K(w))/\mathbb{I}(K(w)_+ \\
\downarrow c & & \downarrow \\
\varprojlim^1_{\mathcal{F}} K^\times/K^\times_+ & \xrightarrow{d} & \varprojlim^1_{\mathcal{F}} \mathbb{I}(K)/\mathbb{I}(K^+),
\end{array}$$

$\mathrm{Ker}(d\circ c)=0$ *and* $\mathrm{Ker}(d)\cong\varprojlim C(K)$.

PROOF. The second statement is proved in (1.19). Since the top horizontal arrow is an isomorphism (19), for the first statement, it suffices to show that right vertical map is injective. Because of (1.18) and (18), this is equivalent to showing that

$$\varprojlim{}^1 I(K(w))/I(K(w)_+)\to\varprojlim{}^1 I(K)/I(K_+)$$

is injective. From the exact sequence (2) attached to

$$0\to I(K(w))/I(K(w)_+)\to I(K)/I(K_+)\to I(K)/\left(I(K_+)+I(K(w))\right)\to 0$$

we see that it suffices to show that

$$\varprojlim I(K)/\left(I(K_+)+I(K(w))\right)=0.$$

Let $a\in I(K)$ represent a nonzero element of $I(K)/\left(I(K_+)+I(K(w))\right)$. For some m, $a\notin I(K_+)+I(K(w))+mI(K)$. There exists a $K'\in\mathcal{F}$ containing K and such that, for every prime v' of K' lying over the same prime of K_+ or $K(w)$ as a prime v of K for which the v-component of a is nonzero, m divides the residue class degree $f(v'/v)$. For such a K', $a\notin I(K_+)+I(K(w))+\mathrm{Nm}\,I(K')$.[6] Thus, $\varprojlim I(K)/I(K_+)\cdot I(K(w))=0$ by the criterion (1.3). □

[6] Define an equivalence relation on the finite primes of K as follows: $v\sim v'$ if $v|K_+=v'|K_+$ or $v|K(w)=v'|K(w)$. Then $I(K)/(I(K_+)+I(K(w)))$ decomposes into a direct sum over the equivalences classes. To obtain the statement, consider a direct summand for which the component of a is nonzero.

2 The cohomology of protori

Throughout this section, k is a field of characteristic zero, k^{al} is an algebraic closure of k, and $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$.

Review of affine group schemes

2.1. Every affine group scheme over k is the inverse limit of a strict inverse system of algebraic groups (Demazure and Gabriel 1970, III, §3, 7.5, p355).

2.2. An affine group scheme G over k defines a sheaf $\tilde{G}\colon U \mapsto G(U)$ on $(\mathrm{Spec}\, k)_{\mathrm{fpqc}}$, and the functor $G \mapsto \tilde{G}$ is fully faithful (ibid. III, §1, 3.3, p297). When N is an affine normal subgroup scheme of G, the quotient sheaf $\tilde{G}/\tilde{N}$ is in the essential image of the functor (ibid. III, §3, 7.2, p353).

2.3. Let $(G_\alpha)_{\alpha\in I}$ be an inverse system of affine group schemes over k.[7] The inverse limit $G = \varprojlim G_\alpha$ in the category of k-schemes (so $G(S) = \varprojlim G_\alpha(S)$ for all k-schemes S) is an affine group scheme, and is the inverse limit in the category of affine group schemes (ibid. III, §3, 7.5, p355). Note that $\tilde{G} = \varprojlim \tilde{G}_\alpha$.[8]

2.4. The category of commutative affine group schemes over k is abelian. A sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

is exact if and only if $A \to B$ is a closed immersion, $B \to C$ is fully faithful, and $a\colon A \to B$ is a kernel of b (ibid. III, §3, 7.4, p355). The category is pro-artinian, and so inverse limits of exact sequences are exact (ibid. V, §2, 2, pp563–5).

2.5. The functor $G \mapsto \tilde{G}$ from the category of commutative affine group schemes over k to the category of sheaves of commutative groups on $(\mathrm{Spec}\, k)_{\mathrm{fpqc}}$ is exact (left exactness is obvious, and right exactness follows from 2.2).

2.6. We say that an affine group scheme G is *separable*[9] if the set of affine normal subgroups N of G for which G/N algebraic is countable. Such a G is the inverse limit of a strict inverse system of algebraic groups indexed by $(\mathbb{N}, \leq)$ (apply 2.1).

Continuous cohomology

Let T be a separable protorus over k. Then

$$T(k^{\mathrm{al}}) = \varprojlim Q(k^{\mathrm{al}})$$

[7] Recall (Notations) that this means directed inverse system.

[8] Recall that to form an inverse limit in the category of sheaves, form the inverse limit in the category of presheaves (this is the obvious object), and then the resulting presheaf is a sheaf and is the inverse limit in the category of sheaves.

[9] Compare: a topological space is said to be separable if it has a countable dense subset; a profinite group G is separable if and only if the set of its open subgroups is countable, in which case G is the limit of a strict inverse system of finite groups indexed by $(\mathbb{N}, \leq)$.

where Q runs over the algebraic quotients of T. Endow each group $Q(k^{\mathrm{al}})$ with the discrete topology, and endow $T(k^{\mathrm{al}})$ with the inverse limit topology. Define $H^r_{\mathrm{cts}}(k,T)$ to be the rth cohomology group of the complex

$$\cdots \to C^r(\Gamma,T) \xrightarrow{d^r} C^{r+1}(\Gamma,T) \to \cdots$$

where $C^r(\Gamma,T)$ is the group of continuous maps $\Gamma^r \to T(k^{\mathrm{al}})$ and d^r is the usual boundary map (see, for example, Tate 1976, §2). Note that, if $T = \varprojlim_n T_n$, then $C^r(\Gamma,T) = \varprojlim_n C^r(\Gamma,T_n)$.

When T is a torus, $H^r_{\mathrm{cts}}(k,T)$ is the usual Galois cohomology group

$$H^r_{\mathrm{cts}}(k,T) = H^r(\Gamma, T(k^{\mathrm{al}})) = \varinjlim H^r(\Gamma_{K/k}, T(K))$$

(limit over the finite Galois extensions K of k contained in k^{al}; $\Gamma_{K/k} = \mathrm{Gal}(K/k)$). In this case, we usually omit the subscript "cts".

LEMMA 2.7. *Let*

$$0 \to T' \to T \to T'' \to 0 \tag{20}$$

be the limit of a short exact sequence of countable strict inverse systems of tori. Then the sequence of complexes

$$0 \to C^\bullet(\Gamma,T') \to C^\bullet(\Gamma,T) \to C^\bullet(\Gamma,T'') \to 0 \tag{21}$$

is exact, and so gives rise to a long exact sequence

$$\cdots \to H^r_{cts}(k,T') \to H^r_{cts}(k,T) \to H^r_{cts}(k,T'') \to H^{r+1}_{cts}(k,T') \to \cdots .$$

PROOF. By assumption, (20) is the inverse limit of a system

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & T'_{n+1} & \longrightarrow & T_{n+1} & \longrightarrow & T''_{n+1} & \longrightarrow & 0 \\
 & & \downarrow\text{onto} & & \downarrow\text{onto} & & \downarrow\text{onto} & & \\
0 & \longrightarrow & T'_n & \longrightarrow & T_n & \longrightarrow & T''_n & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Because the transition maps $T'_{n+1}(k^{\mathrm{al}}) \to T'_n(k^{\mathrm{al}})$ are surjective, the limit of the short exact sequences

$$0 \to T'_n(k^{\mathrm{al}}) \to T_n(k^{\mathrm{al}}) \to T''_n(k^{\mathrm{al}}) \to 0$$

is an exact sequence

$$0 \to T'(k^{\mathrm{al}}) \to T(k^{\mathrm{al}}) \to T''(k^{\mathrm{al}}) \to 0$$

(1.1(b)). To show that (21) is exact, it suffices to show that

(a) the topology on $T'(k^{\mathrm{al}})$ is induced from that on $T(k^{\mathrm{al}})$;

(b) the map $T(k^{\mathrm{al}}) \to T''(k^{\mathrm{al}})$ has a continuous section (not necessarily a homomorphism).

The topology on an inverse limit is that inherited by it as a subset of the product. Thus (a) follows from the similar statement for products (Bourbaki 1989, I 4.1, Corollary to Proposition 3, p46). As all the groups in the diagram

$$\begin{array}{ccc} T_n(k^{\mathrm{al}}) & \longrightarrow & T''_n(k^{\mathrm{al}}) \\ \uparrow & & \uparrow \\ T_{n+1}(k^{\mathrm{al}}) & \longrightarrow & T''_{n+1}(k^{\mathrm{al}}) \end{array}$$

are discrete and all the maps surjective, it is possible to successively choose compatible sections to the maps $T_n(k^{\mathrm{al}}) \to T''_n(k^{\mathrm{al}})$. Their limit is the section required for (b). □

PROPOSITION 2.8. *Let T be a separable protorus over k, and write it as the limit $T = \varprojlim_n T_n$ of a strict inverse system of tori. For each $r \geq 0$, there is an exact sequence*

$$0 \to \varprojlim{}^1 H^{r-1}(k, T_n) \to H^r_{cts}(k, T) \to \varprojlim H^r(k, T_n) \to 0.$$

PROOF. The map $T_{n+1}(k^{\mathrm{al}}) \to T_n(k^{\mathrm{al}})$ is surjective, and admits a continuous section because $T_n(k^{\mathrm{al}})$ is discrete. Hence $C^r(\Gamma, T_{n+1}) \to C^r(\Gamma, T_n)$ is surjective. Thus, by (1.1),

$$\varprojlim{}^1 C^r(\Gamma, T_n) = 0,$$

and so we can apply the next lemma to the inverse system of complexes $(C^\bullet(\Gamma, T_n))_{n\in\mathbb{N}}$. □

LEMMA 2.9. *Let $(C_n^\bullet)_{n\in\mathbb{N}}$ be an inverse system of complexes of abelian groups such that $\varprojlim_n^1 C_n^r = 0$ for all r. Then there is a canonical exact sequence*

$$0 \to \varprojlim{}^1_n H^{r-1}(C_n^\bullet) \to H^r(\varprojlim C_n^\bullet) \to \varprojlim H^r(C_n^\bullet) \to 0. \tag{22}$$

PROOF. This is a standard result. [The condition $\varprojlim_n^1 C_n^r = 0$ implies that the sequence of complexes

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_n C_n^r & \longrightarrow & \prod_n C_n^r & \xrightarrow{1-u} & \prod_n C_n^r & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_n C_n^{r+1} & \longrightarrow & \prod_n C_n^{r+1} & \xrightarrow{1-u} & \prod_n C_n^{r+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

is exact (the maps $1-u$ are as in (1)). The associated long exact sequence is

$$\cdots \xrightarrow{1-u} \prod_n H^{r-1}(C_n^\bullet) \to H^r(\varprojlim C_n^\bullet) \to \prod_n H^r(C_n^\bullet) \xrightarrow{1-u} \prod_n H^r(C_n^\bullet) \to \cdots,$$

which gives (22).] □

LEMMA 2.10. *For a separable protorus T, $H^1_{\mathit{fpqc}}(k,T) \cong H^1_{\mathit{cts}}(k,T)$.*

PROOF. The group $H^1_{\text{fpqc}}(k,T)$ is canonically isomorphic to the group of isomorphism classes of torsors for T (Giraud 1971, III 3.5.4, p169). Let P be such a torsor. Because T is separable, it is the limit of a strict inverse system $(T_n)_{n\in\mathbb{N}}$, and correspondingly P is the inverse limit $P = \varprojlim P_n$ of the strict inverse system $(P_n)_{n\in\mathbb{N}} = \left(P \wedge^T T_n\right)_{n\in\mathbb{N}}$. The set $P(k^{\text{al}}) = \varprojlim P_n(k^{\text{al}})$ is nonempty. Choose a $p \in P(k^{\text{al}})$, and for $\sigma \in \Gamma$ write $\sigma p = p \cdot a_\sigma(P)$ with $a_\sigma(P) \in T(k^{\text{al}})$. Then $\sigma \mapsto a_\sigma(P)$ is a cocycle, which is continuous because its projection to each $T_n(k^{\text{al}})$ is continuous, and its class in $H^1_{\text{cts}}(k,T)$ depends only on the isomorphism class of P. The map

$$[P] \mapsto [a_\sigma(P)]\colon H^1_{\text{fpqc}}(k,T) \to H^1_{\text{cts}}(k,T)$$

is easily seen to be injective, and the flat descent theorems show that it is surjective. (See Milne 2002b, 1.20, for a slightly different proof.) $\square$

EXAMPLE 2.11. Define the universal covering $\tilde{T}$ of a torus T to be the inverse limit of the inverse system indexed by $\mathbb{N}^\times$

$$\cdots \leftarrow \underset{n}{T} \overset{m}{\leftarrow} \underset{mn}{T} \leftarrow \cdots.$$

Then, with the notations of (1.7), there is an exact sequence

$$0 \to \varprojlim{}^1 (H^{r-1}(k,T), m) \to H^r(k,\tilde{T}) \to \varprojlim (H^r(k,T), m) \to 0.$$

For any torus T over $\mathbb{Q}$, $\tilde{T}(\mathbb{Q}) = 0$ (e.g., Milne 1994, 3.16).[10] For any torus T over $\mathbb{Q}$ such that $T(\mathbb{R})$ is compact, $\tilde{T}(\mathbb{A}_f) = 0$ (ibid. 3.21).[11]

REMARK 2.12. Let T be a separable torus, $T = \varprojlim_n T_n$, over an algebraically closed field k. Then the maps $T_{n+1}(k) \to T_n(k)$ are surjective, and so $\varprojlim^1 T_n(k) = 0$. Moreover, $H^i(k,T_n) = 0$ for all n and all $i > 0$, and so $H^i(k,T) = 0$ for all $i > 0$.

Adèlic cohomology

We now take $k = \mathbb{Q}$, so that $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$. For a finite set S of primes of $\mathbb{Q}$, $\mathbb{A}^S$ is the restricted product of the $\mathbb{Q}_l$ for $l \notin S$, and for a finite number field L, $\mathbb{A}^S_L = L \otimes_{\mathbb{Q}} \mathbb{A}^S$. When S is empty, we omit it from the notation.

For a torus T over $\mathbb{Q}$, define

$$H^r(\mathbb{A}^S, T) = \varinjlim_L H^r(\Gamma_{L/\mathbb{Q}}, T(\mathbb{A}^S_L))$$

(limit over the finite Galois extensions L of $\mathbb{Q}$ contained in $\mathbb{Q}^{\text{al}}$).

[10] An element of $\widetilde{T}(\mathbb{Q})$ is a family $(a_n)_{n\geq 1}$, $a_n \in T(\mathbb{Q})$, such that $a_n = (a_{mn})^m$. In particular, a_n is infinitely divisible. If $T = (\mathbb{G}_m)_{L/\mathbb{Q}}$, then $T(\mathbb{Q}) = L^\times$, and $\cap L^{\times m} = 1$. Every torus T can be embedded in a product of tori of the form $(\mathbb{G}_m)_{L/\mathbb{Q}}$, and so again $\cap T(\mathbb{Q})^m = 1$.

[11] First, $T(\mathbb{Q})$ is discrete in $T(\mathbb{A})$: let k be a finite splitting field for T; then $T(k)$ is discrete in $T(\mathbb{A}_k)$ by algebraic number theory; now use that $T(\mathbb{Q}) = T(k) \cap T(\mathbb{A}_{\mathbb{Q}})$.

The hypothesis implies that $T(\mathbb{A}_f)/T(\mathbb{Q})$ is compact. Therefore, for any choice of $\mathbb{Z}$-structure on T, the map $T(\hat{\mathbb{Z}}) \to T(\mathbb{A}_f)/T(\mathbb{Q})$ has finite kernel and cokernel. Now use that $\cap T(\hat{\mathbb{Z}})^m = 1$.

PROPOSITION 2.13. *Let T be a torus over $\mathbb{Q}$. For all $r \geq 1$, there is a canonical isomorphism*

$$H^r(\mathbb{A}^S, T) \cong \oplus_{l \notin S} H^r(\mathbb{Q}_l, T).$$

PROOF. Let $L/\mathbb{Q}$ be a finite Galois extension. For each prime l of $\mathbb{Q}$, choose a prime v of L lying over it, and set $L^l = L_v$. Then

$$H^r(\Gamma_{L^l/\mathbb{Q}_l}, T(L^l)) \stackrel{\text{df}}{=} H^r(\Gamma_{L_v/\mathbb{Q}_l}, T(L_v))$$

is independent of the choice of v up to a canonical isomorphism (i.e., it is well-defined). Moreover,

$$H^r(\Gamma_{L/\mathbb{Q}}, T(\mathbb{A}_L^S)) \cong \oplus_{l \notin S} H^r(L^l/\mathbb{Q}_l, T(L^l))$$

(Platonov and Rapinchuk 1994, Proposition 6.7, p298). Now pass to the direct limit over L. □

COROLLARY 2.14. *A short exact sequence*

$$0 \to T' \to T \to T'' \to 0$$

of tori gives rise to a long exact sequence

$$\cdots \to H^r(\mathbb{A}^S, T') \to H^r(\mathbb{A}^S, T) \to H^r(\mathbb{A}^S, T'') \to H^{r+1}(\mathbb{A}^S, T') \to \cdots.$$

PROOF. Take the direct sum of the cohomology sequences over the $\mathbb{Q}_l$ and apply the proposition. □

For a torus T, let

$$T(\bar{\mathbb{A}}^S) = \varinjlim_{L} T(\mathbb{A}_L^S), \quad \text{(limit over } L \subset \mathbb{Q}^{\text{al}} \text{ with } [L\colon \mathbb{Q}] < \infty\text{)},$$

and, for a separable protorus T, let

$$T(\bar{\mathbb{A}}^S) = \varprojlim Q(\bar{\mathbb{A}}^S), \quad \text{(limit over the algebraic quotients of } T\text{)}.$$

Endow each $Q(\bar{\mathbb{A}}^S)$ with the discrete topology and $T(\bar{\mathbb{A}}^S)$ with the inverse limit topology, and define

$$H^r(\mathbb{A}^S, T) = H^r(\Gamma, T(\bar{\mathbb{A}}^S))$$

where $H^r(\Gamma, T(\bar{\mathbb{A}}^S))$ is computed using continuous cochains (profinite topology on Γ). When T is a torus, this coincides with the previous definition.

PROPOSITION 2.15. *Let T be a separable protorus over $\mathbb{Q}$.*

(a) *There is a canonical homomorphism*

$$H^1(\mathbb{Q}, T) \to H^1(\mathbb{A}^S, T).$$

(b) *Write T as the limit of a strict inverse system of tori, $T = \varprojlim T_n$. For each $r \geq 0$, there is a canonical exact sequence*

$$0 \to \varprojlim{}^1 T_n(\mathbb{A}^S) \to H^1(\mathbb{A}^S, T) \to \varprojlim H^1(\mathbb{A}^S, T_n) \to 0.$$

PROOF. (a) For each algebraic quotient Q of T, $Q(\mathbb{Q}^{\text{al}}) \hookrightarrow Q(\bar{\mathbb{A}}^S)$ is continuous (both groups are discrete), and hence the inverse limit $T(\mathbb{Q}^{\text{al}}) \to T(\bar{\mathbb{A}}^S)$ of these homomorphisms is continuous.

(b) The map $T_{n+1}(\bar{\mathbb{A}}^S) \to T_n(\bar{\mathbb{A}}^S)$ is surjective (in fact, $T_{n+1}(\mathbb{A}^S_L) \to T_n(\mathbb{A}^S_L)$ will be surjective once L is large enough to split T), and admits a continuous section because $T_n(\bar{\mathbb{A}}^S)$ is discrete, and so the proof of (2.8) applies. □

REMARK 2.16. For any finite set S of primes of $\mathbb{Q}$,

$$T(\bar{\mathbb{A}}) \cong T(\bar{\mathbb{A}}^S) \times T((\prod_{l \in S} \mathbb{Q}_l) \otimes \mathbb{Q}^{\text{al}})$$

is a topological group, and so

$$H^r(\mathbb{A}, T) \cong H^r(\mathbb{A}^S, T) \times \textstyle\prod_{l \in S} H^r(\mathbb{Q}_l, T).$$

REMARK 2.17. Let $T = \varprojlim (T_n, u_n)$ be a separable pro-torus over $\mathbb{Q}$. It may happen that each T_n satisfies the Hasse principle but T does not. In this case, we get a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim{}^1_n T_n(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, T) & \longrightarrow & \varprojlim_n H^1(\mathbb{Q}, T_n) & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & \varprojlim{}^1_n T_n(\mathbb{A}) & \longrightarrow & H^1(\mathbb{A}, T) & \longrightarrow & \varprojlim_n H^1(\mathbb{A}, T_n) & \longrightarrow & 0
\end{array}$$

in which c is injective and

$$\operatorname{Ker}(a) \cong \operatorname{Ker}(b) \neq 0.$$

Let (t_n) be an element of $\prod T_n(\mathbb{Q})$ that is not in the image of $1 - u$ on $\prod T_n(\mathbb{Q})$ but is in the image of $1 - u$ on $\prod T_n(\mathbb{A})$. Then

$$P = \varprojlim (T_n, u_n \cdot t_{n-1})$$

is a nontrivial T-torsor under over $\mathbb{Q}$ that becomes trivial over $\mathbb{A}$.

3 The cohomology of the Serre and Weil-number protori.

Throughout this section, $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and $\mathcal{F}$ is the set of CM-subfields K of $\mathbb{Q}^{\mathrm{al}}$, finite and Galois over $\mathbb{Q}$.

The Serre protorus S.

For $K \in \mathcal{F}$, the Serre group S^K for K is the quotient of $(\mathbb{G}_m)_{K/\mathbb{Q}}$ such that

$$X^*(S^K) = \left\{ f\colon \Gamma_{K/\mathbb{Q}} \to \mathbb{Z} \mid f(\sigma) + f(\iota\sigma) \text{ is constant} \right\}.$$

The constant value of $f(\sigma) + f(\iota\sigma)$ is called the weight of f. There is an exact sequence

$$0 \to (\mathbb{G}_m)_{K_+/\mathbb{Q}} \xrightarrow{(\text{incl.},\mathrm{Nm}_{F/\mathbb{Q}}^{-1})} (\mathbb{G}_m)_{K/\mathbb{Q}} \times \mathbb{G}_m \to S^K \to 0 \tag{23}$$

of tori over $\mathbb{Q}$ corresponding to the exact sequence

$$0 \longrightarrow X^*(S^K) \xrightarrow[(f,wt(f))]{f\mapsto} \mathbb{Z}^{\Gamma_{K/\mathbb{Q}}} \oplus \mathbb{Z} \xrightarrow[f|\Gamma_{K_+/\mathbb{Q}} - m\sum_{\sigma\in\Gamma_{K_+/\mathbb{Q}}}\sigma]{(f,m)\mapsto} \mathbb{Z}^{\Gamma_{K_+/\mathbb{Q}}} \longrightarrow 0$$

of character groups. The maps $\mathrm{Nm}_{K'/K} \times \mathrm{id}$ induce homomorphisms $S^{K'} \to S^K$, and the Serre group is defined to be

$$S = \varprojlim_{\mathcal{F}} S^K.$$

LEMMA 3.1. *For $K \in \mathcal{F}$, there are exact sequences*

$$0 \to H^1(\mathbb{Q}, S^K) \to \mathrm{Br}(K_+) \to \mathrm{Br}(K) \oplus \mathrm{Br}(\mathbb{Q}) \to H^2(\mathbb{Q}, S^K) \to 0$$
$$0 \to H^1(\mathbb{A}, S^K) \to \oplus_l \mathrm{Br}((K_+)_l) \to \oplus_l \mathrm{Br}(K_l) \oplus \oplus_l \mathrm{Br}(\mathbb{Q}_l) \to H^2(\mathbb{A}, S^K) \to 0.$$

(The subscript l denotes $- \otimes \mathbb{Q}_l$.)

PROOF. Except for the zero at right, the exactness of the first sequence follows from (23) and Hilbert's Theorem 90, but a theorem of Tate (Milne 1986, I 4.21, p80) shows that

$$H^3(K_+, \mathbb{G}_m) \cong \oplus_{v \text{ real}} H^3((K_+)_v, \mathbb{G}_m),$$

and $H^3(\mathbb{R}, \mathbb{G}_m) \cong H^1(\mathbb{R}, \mathbb{G}_m) = 0$ (periodicity of the cohomology of finite cyclic groups). The proof that the second sequence is exact is similar. □

PROPOSITION 3.2. *For $K \in \mathcal{F}$,*

$$H^1(\mathbb{Q}, S^K) \xrightarrow{\cong} H^1(\mathbb{A}, S^K),$$
$$H^2(\mathbb{Q}, S^K) \hookrightarrow H^2(\mathbb{A}, S^K).$$

PROOF. Apply Lemma 3.1 and the snake lemma to the diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(K_+) & \longrightarrow & \oplus_l \mathrm{Br}((K_+)_l) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle x\mapsto(2x,x)} & & \\ 0 & \longrightarrow & \mathrm{Br}(K)\oplus\mathrm{Br}(\mathbb{Q}) & \longrightarrow & \oplus_l \mathrm{Br}(K_l)\oplus\oplus_l\mathrm{Br}(\mathbb{Q}_l) & \longrightarrow & \mathbb{Q}/\mathbb{Z}\oplus\mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

in which the subscript l denotes $-\otimes\mathbb{Q}_l$. For the vertical map at right we have used that

$$\begin{aligned} \mathrm{inv}_E \circ \mathrm{Res} &= [E\colon F]\cdot \mathrm{inv}_F \\ \mathrm{inv}_F \circ \mathrm{Cor} &= \mathrm{inv}_E \end{aligned}$$

for an extension $F\subset E$; see Serre 1962, XI §2, Proposition 1. □

PROPOSITION 3.3. *There is a canonical commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Q}^\times & \longrightarrow & S(\mathbb{Q}) & \longrightarrow & \varprojlim{}^1 K_+^\times \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{A}^\times\times\varprojlim\frac{(K_+\otimes\mathbb{C})^\times}{(K_+\otimes\mathbb{R})^\times} & \longrightarrow & S(\mathbb{A}) & \longrightarrow & \varprojlim{}^1\mathbb{I}(K_+) \end{array}$$

and a canonical exact sequence

$$0\to\varprojlim{}_{\mathcal{F}}C(K)\to\varprojlim{}^1_{\mathcal{F}}S^K(\mathbb{Q})\to\varprojlim{}^1_{\mathcal{F}}S^K(\mathbb{A})\to 0.$$

PROOF. From (23), we obtain an inverse system of exact sequences

$$0\to K_+^\times\to K^\times\times\mathbb{Q}^\times\to S^K(\mathbb{Q})\to 0,$$

and hence an exact sequence (1.15, 1.16),

$$0\to\mathbb{Q}^\times\to S(\mathbb{Q})\to\varprojlim{}^1K_+^\times\to\varprojlim{}^1K^\times\to\varprojlim{}^1S^K(\mathbb{Q})\to 0. \tag{24}$$

Similarly, there is an exact sequence

$$0\to\mathbb{A}^\times\times\varprojlim\frac{(K_+\otimes\mathbb{C})^\times}{(K_+\otimes\mathbb{R})^\times}\to S(\mathbb{A})\to\varprojlim{}^1\mathbb{I}(K_+)\to\varprojlim{}^1\mathbb{I}(K)\to\varprojlim{}^1S^K(\mathbb{A})\to 0. \tag{25}$$

From these sequences, we obtain the commutative diagram and isomorphisms

$$\begin{aligned} \varprojlim{}^1\left(K^\times/K_+^\times\right) &\cong \varprojlim{}^1S^K(\mathbb{Q}) \\ \varprojlim{}^1\left(\mathbb{I}(K)/\mathbb{I}(K_+)\right) &\cong \varprojlim{}^1S^K(\mathbb{A}), \end{aligned}$$

and so the exact sequence follows from Proposition 1.19. □

The Weil-number protorus P

We now fix a p-adic prime w of $\mathbb{Q}^{\mathrm{al}}$, and we write w_K (or w) for the prime it induces on a subfield K of $\mathbb{Q}^{\mathrm{al}}$. The completion $(\mathbb{Q}^{\mathrm{al}})_w$ of $\mathbb{Q}^{\mathrm{al}}$ at w is algebraically closed,[12] and we let $\mathbb{Q}_p^{\mathrm{al}}$ denote the algebraic closure of $\mathbb{Q}_p$ in $(\mathbb{Q}^{\mathrm{al}})_w$. Its residue field, which we denote $\mathbb{F}$, is an algebraic closure of $\mathbb{F}_p$.

A *Weil p^n-number* is an algebraic number π for which there exists an integer m (the *weight* of π) such that $\rho\pi \cdot \overline{\rho\pi} = (p^n)^m$ for all $\rho\colon \mathbb{Q}[\pi] \to \mathbb{C}$. Let $W(p^n)$ be the set of all Weil p^n-numbers in $\mathbb{Q}^{\mathrm{al}}$. It is an abelian group, and for $n|n'$, $\pi \mapsto \pi^{n'/n}$ is a homomorphism $W(p^n) \to W(p^{nn'})$. Define

$$W = \varinjlim W(p^n).$$

There is an action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on W, and P is defined to be the protorus over $\mathbb{Q}$ such that

$$X^*(P) = W.$$

For $\pi \in W(p^n)$ and a p-adic prime v of a finite number field containing π, define

$$s_\pi(v) = \frac{\mathrm{ord}_v(\pi)}{\mathrm{ord}_v(p^n)}.$$

Then $s_\pi(v)$ is well-defined for $\pi \in W$, i.e., it does not depend on the choice of a representative of π.

Let K be a CM field in $\mathbb{Q}^{\mathrm{al}}$, finite and Galois over $\mathbb{Q}$. Define W^K to be the set of $\pi \in W$ having a representative in K and such that

$$f_\pi^K(v) \overset{\mathrm{df}}{=} s_\pi(v) \cdot [K_v \colon \mathbb{Q}_p] \in \mathbb{Z}$$

for all p-adic primes of K, and define P^K to be the torus over $\mathbb{Q}$ such that

$$X^*(P^K) = W^K.$$

Then

$$W = \varinjlim_{\mathcal{F}} W^K, \quad P = \varprojlim_{\mathcal{F}} P^K.$$

Let X and Y respectively be the sets of p-adic primes of K and K_+. Then (e.g., Milne 2001, A.6), there is an exact sequence

$$0 \longrightarrow W^K \xrightarrow[(f_\pi^K, wt(\pi))]{\pi \mapsto} \mathbb{Z}^X \times \mathbb{Z} \xrightarrow[f|Y - n(w_K)\cdot m \cdot \sum_{v\in Y} v]{(f,m)\mapsto} \mathbb{Z}^Y \longrightarrow 0 \tag{26}$$

where $n(w_K)$ is the local degree $[K_w : \mathbb{Q}_p]$. Using the fixed p-adic prime w, we can identify X with $\Gamma_{K/\mathbb{Q}}/D(w_K)$ and Y with $\Gamma_{K_+/\mathbb{Q}}/D(w_{K_+})$, in which case (26) becomes the sequence of character groups of an exact sequence

$$0 \longrightarrow (\mathbb{G}_m)_{K(w)_+/\mathbb{Q}} \xrightarrow{\binom{\text{incl.}}{\mathrm{Nm}^{-n(w_K)}}} (\mathbb{G}_m)_{K(w)/\mathbb{Q}} \times \mathbb{G}_m \longrightarrow P^K \longrightarrow 0. \tag{27}$$

[12] If not, there is a monic irreducible polynomial f of degree > 1 in $(\mathbb{Q}^{\mathrm{al}})_w[X]$. After a substitution $X \mapsto p^m X$, we may suppose that f has w-integral coefficients. Choose a monic g in $\mathbb{Q}^{\mathrm{al}}[X]$ that is close to f. Then Hensel's lemma allows us to refine a root of g in $\mathbb{Q}^{\mathrm{al}}$ to a root of f in $(\mathbb{Q}^{\mathrm{al}})_w$, contradicting the irreducibility of f.

Let $K' \supset K$ with $K' \in \mathcal{F}$. Then $W^K \subset W^{K'}$, and there is map $X' \to X$ from the p-adic primes of K' to those of K. If $v' \mapsto v$, then $s_\pi(v') = s_\pi(v)$, and so $f_\pi(v') = [K'_w\colon K_w] \cdot f_\pi(v)$. Therefore, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^K & \longrightarrow & \mathbb{Z}^X \times \mathbb{Z} & \longrightarrow & \mathbb{Z}^Y & \longrightarrow & 0 \\ & & \cap & & \downarrow{\scriptstyle a\times \mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & W^{K'} & \longrightarrow & \mathbb{Z}^{X'} \times \mathbb{Z} & \longrightarrow & \mathbb{Z}^{Y'} & \longrightarrow & 0 \end{array}$$

commutes with a equal to $[K'_w\colon K_w]\times$ the map induced by $X' \to X$. Therefore,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{G}_m)_{K'(w)_+/\mathbb{Q}} & \longrightarrow & (\mathbb{G}_m)_{K'(w)/\mathbb{Q}} \times \mathbb{G}_m & \longrightarrow & P^{K'} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Nm}^{[K'_w\colon K_w]}_{K'(w)_+/K(w)_+}} & & \downarrow{\scriptstyle \mathrm{Nm}^{[K'_w\colon K_w]}_{K'(w)/K(w)} \times \mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & (\mathbb{G}_m)_{K(w)_+/\mathbb{Q}} & \longrightarrow & (\mathbb{G}_m)_{K(w)/\mathbb{Q}} \times \mathbb{G}_m & \longrightarrow & P^K & \longrightarrow & 0. \end{array} \tag{28}$$

commutes.

LEMMA 3.4. *There is an exact sequence*

$$0 \to H^1(\mathbb{Q}, P^K) \to \mathrm{Br}(K(w)_+) \to \mathrm{Br}(K(w)) \oplus \mathrm{Br}(\mathbb{Q}) \to H^2(\mathbb{Q}, P^K) \to 0.$$

PROOF. Same as that of Lemma 3.1. □

PROPOSITION 3.5. *Let $K \in \mathcal{F}$.*

(a) *If the local degree $n(w_K) = [K_w\colon \mathbb{Q}_p]$ is even, then there is an exact sequence*

$$0 \to H^1(\mathbb{Q}, P^K) \to H^1(\mathbb{A}, P^K) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} \to 0;$$

otherwise,

$$H^1(\mathbb{Q}, P^K) \cong H^1(\mathbb{A}, P^K)$$

(b) *The map*

$$H^2(\mathbb{Q}, P^K) \to H^2(\mathbb{A}, P^K)$$

is injective.

PROOF. If complex conjugation $\iota \in D(w_K)$, then $P^K = \mathbb{G}_m$ and the statement is obvious. Thus, we may assume $\iota \notin D(w_K)$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(K(w)_+) & \longrightarrow & \oplus_l \mathrm{Br}((K(w)_+)_l) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle x\mapsto(2x,n(w_K)x)} & & \\ 0 & \longrightarrow & \mathrm{Br}(K(w)) \oplus \mathrm{Br}(\mathbb{Q}) & \longrightarrow & \oplus_l \mathrm{Br}(K(w)_l) \oplus \oplus_l \mathrm{Br}(\mathbb{Q}_l) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0. \end{array}$$

When $n(w_K)$ is odd, $(2, n(w_K))$ is injective and the snake lemma shows that that $H^1(\mathbb{Q}, P^K) \cong H^1(\mathbb{A}, P^K)$. When $n(w_K)$ is even, the sequence of kernels is

$$0 \to \mathrm{Br}(K(w)/K(w)_+) \to \oplus_l \mathrm{Br}(K(w)_l/(K(w)_+)_l) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z} \to 0,$$

which class field theory shows to be exact. Again, the snake lemma gives the result. □

PROPOSITION 3.6. *The canonical map*

$$\varprojlim{}^1_{\mathcal{F}} P^K(\mathbb{Q}) \to \varprojlim{}^1_{\mathcal{F}} P^K(\mathbb{A})$$

is an isomorphsim.

PROOF. As in the proof of 3.3, there are canonical isomorphism

$$\varprojlim{}^1 \left(K(w)^\times / K(w)_+^\times\right) \cong \varprojlim{}^1 P^K(\mathbb{Q})$$
$$\varprojlim{}^1 \left(\mathbb{I}(K(w)/\mathbb{I}(K(w)_+)\right) \cong \varprojlim{}^1 P^K(\mathbb{A})$$

and so the statement follows from Proposition 1.21 (19). □

LEMMA 3.7. *For each $K \in \mathcal{F}$, there exists an $K' \in \mathcal{F}$ containing K for which the map*

$$H^1(\mathbb{Q}, P^{K'}) \to H^1(\mathbb{Q}, P^K)$$

is zero.

PROOF. After possibly enlarging K, we may assume that $\iota \notin D(w_K)$. Denote the map $\mathrm{Nm}^{[K'_{+w}\colon K_{+w}]}_{K'(w)_+/K(w)_+}\colon (\mathbb{G}_m)_{K'(w)/\mathbb{Q}} \to (\mathbb{G}_m)_{K(w)/\mathbb{Q}}$ of (28) by b. The kernel of $\mathrm{Br}(K'(w)_+) \to \mathrm{Br}(K(w))$ is killed by 2, and so if K' is chosen so that $2|[K'(w)\colon K(w)]$, then $H^2(b)$ is zero on this kernel. Then $H^1(\mathbb{Q}, P^{K'}) \to H^1(\mathbb{Q}, P^K)$ is zero. □

LEMMA 3.8. *The groups $\varprojlim_{\mathcal{F}} H^1(\mathbb{Q}, P^K)$ and $\varprojlim^1_{\mathcal{F}} H^1(\mathbb{Q}, P^K)$ are both zero.*

PROOF. Lemma 3.7 shows that $(H^1(\mathbb{Q}, P^K))_{K\in\mathcal{F}}$ admits a cofinal subsystem in which the transition maps are zero, and so the map u in (1) is zero. □

LEMMA 3.9. *The groups $\varprojlim_{\mathcal{F}} H^1(\mathbb{A}, P^K)$ and $\varprojlim^1_{\mathcal{F}} H^1(\mathbb{A}, P^K)$ are both zero.*

PROOF. For K sufficiently large, (3.5) shows that $H^1(\mathbb{Q}, P^K) \cong H^1(\mathbb{A}, P^K)$ and so this follows from the previous lemma. □

PROPOSITION 3.10. *There are canonical isomorphisms*

$$\varprojlim{}^1 P^K(\mathbb{Q}) \cong H^1(\mathbb{Q}, P) \tag{29}$$
$$H^2(\mathbb{Q}, P) \cong \varprojlim H^2(\mathbb{Q}, P^K) \tag{30}$$
$$\varprojlim{}^1 P^K(\mathbb{A}) \cong H^1(\mathbb{A}, P) \tag{31}$$
$$H^2(\mathbb{A}, P) \cong \varprojlim H^2(\mathbb{A}, P^K). \tag{32}$$

PROOF. Combine Proposition 2.8 with Lemmas 3.8 and 3.9. □

REMARK 3.11. Assume $\iota \notin D(w_K)$. The argument in the proof of Lemma 3.7 shows that, if the local degree $[L_w\colon K_w]$ is even, then all the vertical maps in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\mathbb{Q}, P^L) & \longrightarrow & H^1(\mathbb{A}, P^L) & \longrightarrow & \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(\mathbb{Q}, P^K) & \longrightarrow & H^1(\mathbb{A}, P^K) & \longrightarrow & \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0
\end{array}$$

are zero.

PROPOSITION 3.12. *There are canonical isomorphisms*

$$P(\mathbb{Q}) \cong \mathbb{Q}^{\times}$$
$$P(\mathbb{A}_f) \cong \mathbb{A}_f^{\times}.$$

PROOF. For K sufficiently large, $P^K \cong P_0^K \oplus \mathbb{G}_m$ where $X^*(P_0^K)$ consists of the Weil numbers of weight 0. Thus $P \cong P_0 \oplus \mathbb{G}_m$. For each sufficiently large K, the projection $P_0 \to P_0^K$ factors into $P_0 \twoheadrightarrow \widetilde{P_0^K} \to P_0^K$ where $\widetilde{P_0^K}$ is the universal covering of P_0^K. Therefore, $P_0 = \varprojlim_K \widetilde{P_0^K}$, and $P_0(\mathbb{Q}) = \varprojlim \widetilde{P_0^K}(\mathbb{Q})$, which is 0 by (2.11). Similarly, $P_0(\mathbb{A}_f) = \varprojlim \widetilde{P_0^K}(\mathbb{A}_f) = 0$. □

The cohomology of S/P.

Let $K \in \mathcal{F}$, and assume $\iota \notin D(w_K)$. Let $\mathfrak{p}$ be the prime ideal in $\mathcal{O}_K$ corresponding to w_K. For some h, $\mathfrak{p}^h$ will be principal, say $\mathfrak{p}^h = (a)$. Let $\alpha = a^{2n}$ where $n = (U(K)\colon U(K_+))$. Then, for $f \in X^*(S^K)$, $f(\alpha)$ is independent of the choice of a, and it is a Weil $p^{2nf(\mathfrak{p}/p)}$-number of weight $wt(f)$. The map $f \mapsto f(\alpha)\colon X^*(S^K) \to W^K$ is a surjective homomorphism (e.g., Milne 2001, A.8). Thus, it corresponds to an injective homomorphism $\rho^K\colon P^K \to S^K$ which can also be characterized as the unique homomorphism rendering

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{G}_m)_{K(w)_+/\mathbb{Q}} & \longrightarrow & (\mathbb{G}_m)_{K(w)/\mathbb{Q}} \times \mathbb{G}_m & \longrightarrow & P^K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \text{incl.}} & & \downarrow{\scriptstyle \text{incl.}\times\text{id}} & & \downarrow{\scriptstyle \rho_K} & & \\
0 & \longrightarrow & (\mathbb{G}_m)_{K_+/\mathbb{Q}} & \longrightarrow & (\mathbb{G}_m)_{K/\mathbb{Q}} \times \mathbb{G}_m & \longrightarrow & S^K & \longrightarrow & 0
\end{array}$$

commutative. For varying K, the ρ^K define a morphism of inverse systems. Therefore, on passing to the inverse limit, we obtain an injective homomorphism $\rho\colon P \to S$ of protori.

PROPOSITION 3.13. *The map*

$$H^1(\mathbb{Q}, S^K/P^K) \to H^1(\mathbb{A}, S^K/P^K)$$

is injective on the image of $H^1(\mathbb{Q}, S^L/P^L)$ for any $L \supset K$ such that the local degree $[L_w\colon K_w]$ at w is even. Therefore, the map

$$\varprojlim H^1(\mathbb{Q}, S^K/P^K) \to \varprojlim H^1(\mathbb{A}, S^K/P^K)$$

is injective.

PROOF. Diagram chase in

$$\begin{array}{ccccccc}
H^1(\mathbb{Q}, P^K) & \longrightarrow & H^1(\mathbb{Q}, S^K) & \longrightarrow & H^1(\mathbb{Q}, S^K/P^K) & \longrightarrow & H^2(\mathbb{Q}, P^K) \\
\downarrow{\scriptstyle (3.5)} & & \cong\downarrow{\scriptstyle (3.2)} & & \downarrow & & \text{inj.}\downarrow{\scriptstyle (3.5)} \\
H^1(\mathbb{A}, P^K) & \longrightarrow & H^1(\mathbb{A}, S^K) & \longrightarrow & H^1(\mathbb{A}, S^K/P^K) & \longrightarrow & H^2(\mathbb{A}, P^K),
\end{array}$$

using (3.11). □

PROPOSITION 3.14. *An element $a \in H^1(\mathbb{A}, S^K/P^K)$ arises from an element of $H^1(\mathbb{Q}, S^K/P^K)$ if and only if its image in $H^2(\mathbb{A}, P^K)$ under the connecting homomorphism arises from an element of $H^2(\mathbb{Q}, P^K)$.*

PROOF. Diagram chase in

$$\begin{tikzcd}
H^1(\mathbb{Q}, S^K) \arrow[r] \arrow[d, "\cong"', "(3.2)"] & H^1(\mathbb{Q}, S^K/P^K) \arrow[r] \arrow[d] & H^2(\mathbb{Q}, P^K) \arrow[r] \arrow[d] & H^2(\mathbb{Q}, S^K) \arrow[d, "\text{inj.}"', "(3.2)"] \\
H^1(\mathbb{A}, S^K) \arrow[r] & H^1(\mathbb{A}, S^K/P^K) \arrow[r] & H^2(\mathbb{A}, P^K) \arrow[r] & H^2(\mathbb{A}, S^K).
\end{tikzcd}$$

□

PROPOSITION 3.15. *There is a canonical exact sequence*

$$0 \to \varprojlim_{\mathcal{F}} C(K) \to H^1(\mathbb{Q}, S/P) \xrightarrow{a} H^1(\mathbb{A}, S/P).$$

PROOF. Because $\varprojlim H^1(\mathbb{Q}, S^K/P^K) \to \varprojlim H^1(\mathbb{A}, S^K/P^K)$ is injective,

$$\operatorname{Ker}(a) \cong \operatorname{Ker}(\varprojlim{}^1(S^K/P^K)(\mathbb{Q}) \xrightarrow{b} \varprojlim{}^1(S^K/P^K)(\mathbb{A}));$$

cf. (2.17). On comparing (23) and (27), we obtain an exact sequence

$$0 \to (\mathbb{G}_m)_{K(w)/\mathbb{Q}}/(\mathbb{G}_m)_{K(w)_+/\mathbb{Q}} \to (\mathbb{G}_m)_{K/\mathbb{Q}}/(\mathbb{G}_m)_{K_+/\mathbb{Q}} \to S^K/P^K \to 0,$$

which gives rise to an exact commutative diagram

$$\begin{tikzcd}
\varprojlim^1(K(w)^\times/K(w)^\times_+) \arrow[r, "c"] \arrow[d, "\text{surj.}"', "(1.5)"] & \varprojlim^1(K^\times/K^\times_+) \arrow[r] \arrow[d, "d"] & \varprojlim^1(S^K/P^K)(\mathbb{Q}) \arrow[r] \arrow[d, "b"] & 0 \\
\varprojlim^1(\mathbb{I}(K(w))/\mathbb{I}(K(w)^\times_+) \arrow[r] & \varprojlim^1(\mathbb{I}(K)/\mathbb{I}(K_+)) \arrow[r] & \varprojlim^1(S^K/P^K)(\mathbb{A}) \arrow[r] & 0.
\end{tikzcd}$$

The left-hand vertical map is surjective by 1.5, and the diagram gives an exact sequence

$$\operatorname{Ker}(d \circ c) \xrightarrow{c} \operatorname{Ker}(d) \to \operatorname{Ker}(b) \to 0.$$

We can now apply Proposition 1.22. □

Notes. Most of the calculations concerning the cohomology of S^K and P^K (but not of S or P themselves) can be found already in Langlands and Rapoport 1987.

4 The fundamental classes

The adèlic fundamental class

The Betti fibre functor ω_B to identifies $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ with $\mathsf{Rep}_{\mathbb{Q}}(S)$. Let w_{can} denote the cocharacter of S such that $w_{\mathrm{can}}(a)$ acts on an object of $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ of weight m as a^m, and let μ_{can} denote the cocharacter of $S_{\mathbb{Q}^{\mathrm{al}}}$ such that, for $f \in X^*(S^K)$, $\langle f, \mu_{\mathrm{can}} \rangle = f(i)$ where i is the given inclusion $K \hookrightarrow \mathbb{Q}^{\mathrm{al}}$.

The local fundamental class at ∞

Let R_∞ (realization category at ∞) be the category of pairs (V, F) consisting of a $\mathbb{Z}$-graded finite-dimensional complex vector space $V = \oplus_{m\in\mathbb{Z}} V^m$ and a semilinear endomorphism F such that $F^2 = (-1)^m$. With the obvious tensor structure, R_∞ becomes a Tannakian category with fundamental group $\mathbb{G}_m$.

Let (V, r) be a real representation of the Serre group S. Then $w(r) =_{\mathrm{df}} r \circ w_{\mathrm{can}}$ defines a $\mathbb{Z}$-gradation on $V \otimes \mathbb{C}$. Let F be the map

$$v \mapsto r(\mu_{\mathrm{can}}(i))\bar{v} \colon V \otimes \mathbb{C} \to V \otimes \mathbb{C}.$$

Then $(V \otimes \mathbb{C}, F)$ is an object of R_∞, and $(V, r) \mapsto (V \otimes \mathbb{C}, F)$ defines a tensor functor

$$\xi_\infty \colon \mathsf{Rep}_{\mathbb{R}}(S) \to \mathsf{R}_\infty.$$

The functor ξ_∞ defines a homomorphism $x_\infty \colon \mathbb{G}_m \to S_{\mathbb{R}}$, which is equal to $w_{\mathrm{can}/\mathbb{R}}$.

Let $\mathsf{R}_\infty^{\mathbb{G}_m}$ be the full subcategory of R_∞ of objects of weight zero. For any (V, F) of weight zero,

$$V^F = \{v \in V \mid Fv = v\}$$

is a real form of V, and $(V, F) \mapsto V^F$ is a fibre functor on $\mathsf{R}_\infty^{\mathbb{G}_m}$. Its composite with $\mathsf{Rep}_{\mathbb{R}}(S)^{\mathbb{G}_m} \to \mathsf{R}_\infty^{\mathbb{G}_m}$ is a fibre functor ω_∞, and $\wp(\omega_\infty) =_{\mathrm{df}} \underline{\mathrm{Hom}}^{\otimes}(\omega_B, \omega_\infty)$ is an $S/\mathbb{G}_m$-torsor. We define c_∞ to be the cohomology class of $\wp(\omega_\infty) \wedge^{S/\mathbb{G}_m} S/P$.

The local fundamental class at p.

Let R_p (realization category at p) be the category of F-isocrystals over $\mathbb{F}$. Thus, an object of R_p is a pair (V, F) consisting of a finite-dimensional vector space V over $B(\mathbb{F})$ and a semilinear isomorphism $F \colon V \to V$. With the obvious tensor structure, R_p becomes a Tannakian category with fundamental group $\mathbb{G} =_{\mathrm{df}} \widetilde{\mathbb{G}_m}$.

There is a tensor functor

$$\xi_p \colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{R}_p$$

such that, for a CM-abelian variety A, idempotent e, and integer m,

$$\xi_p(h(A, e, m)) = e \cdot H^*_{\mathrm{crys}}(A_{\mathbb{F}})(m)$$

where $A_{\mathbb{F}}$ is the reduction of A at w. The functor ξ_p defines a homomorphism $x_p \colon \mathbb{G} \to S_{/\mathbb{Q}_p}$ whose action on $X^*(S^K)$ is $f \mapsto \sum_{\sigma\in D(w_K)} f(\sigma)/(D(w_K) \colon 1)$.

Let $\mathsf{R}_p^{\mathbb{G}}$ be the full subcategory of R_p of objects of slope 0. For any (V, F) of slope zero,

$$V^F = \{v \in V \mid Fv = v\}$$

is a $\mathbb{Q}_p$-form of V, and $(V, F) \mapsto V^F$ is a fibre functor on $\mathsf{R}_p^{\mathbb{G}}$. Its composite with $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\mathbb{G}} \xrightarrow{\xi_\infty} \mathsf{R}_p^{\mathbb{G}}$ is a fibre functor on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\mathbb{G}}$, and $\wp(\omega_p) =_{\mathrm{df}} \underline{\mathrm{Hom}}^{\otimes}(\omega_B, \omega_p)$ is a $S/\mathbb{G}$-torsor. We define c_p to be the cohomology class of $\wp(\omega_p) \wedge^{S/\mathbb{G}} S/P$.

The adèlic fundamental class

We define $c_{\mathbb{A}} \in H^1(\mathbb{A}, S/P)$ to be the class corresponding to $(0, c_p, c_\infty)$ under the isomorphism (2.16)

$$H^1(\mathbb{A}, S/P) \cong H^1(\mathbb{A}^{\{p,\infty\}}, S/P) \times H^1(\mathbb{Q}_p, S/P) \times H^1(\mathbb{R}, S/P).$$

The global fundamental class

THEOREM 4.1. *There exists a $c \in H^1(\mathbb{Q}, S/P)$ mapping to $c_{\mathbb{A}} \in H^1(\mathbb{A}, S/P)$; any two such c's have the same image in $H^1(\mathbb{Q}, S^K/P^K)$ for all K; the set of such c's is a principal homogeneous space for $\varprojlim_{\mathcal{F}} C(K)$ where $\mathcal{F}$ is the set of CM-subfields of $\mathbb{Q}^{al}$ finite over $\mathbb{Q}$.*

LEMMA 4.2. *Let d_p^K and d_∞^K be the images of the classes of R_p^K and R_∞^K in $H^2(\mathbb{Q}_p, P^K)$ and $H^2(\mathbb{R}, P^K)$. There exists a unique element of $H^2(\mathbb{Q}, P^K)$ with image*

$$(0, d_p^K, d_\infty^K) \in H^2(\mathbb{A}^{\{p,\infty\}}, P^K) \times H^2(\mathbb{Q}_p, P^K) \times H^2(\mathbb{R}, P^K).$$

PROOF. The uniqueness follows from (3.5b). The existence is proved in Langlands and Rapoport 1987 (also, Milne 1994, proof of 3.31). □

PROPOSITION 4.3. *Let d_p and d_∞ be the images of the classes of R_p and R_∞ in $H^2(\mathbb{Q}_p, P)$ and $H^2(\mathbb{R}, P)$. There exists a unique element of $H^2(\mathbb{Q}, P)$ with image*

$$(0, d_p, d_\infty) \in H^2(\mathbb{A}^{\{p,\infty\}}, P) \times H^2(\mathbb{Q}_p, P) \times H^2(\mathbb{R}, P).$$

PROOF. Follows from the lemma, using (30) and (32). □

We first prove three lemmas.

LEMMA 4.4. *Let C and Q be Tannakian categories with commutative fundamental groups G and H. Assume C is neutral. Let $\xi\colon \mathsf{C} \to \mathsf{Q}$ be a tensor functor inducing an injective homomorphism $H \to G$. Let ω_ξ denote the fibre functor*

$$\mathsf{C}^H \xrightarrow{\xi} \mathsf{Q}^H \xrightarrow{\mathrm{Hom}(\mathbb{1},-)} \mathsf{Vec}_k.$$

For any fibre functor ω on C, the class of the torsor $\underline{\mathrm{Hom}}^{\otimes}(\omega, \omega_\xi)$ in $H^1(k, G/H)$ maps to the class of Q in $H^2(k, H)$ under the connecting homomorphism.

PROOF. For a Tannakian category T, write $\mathsf{T}^\vee$ for the gerbe of fibre functors on T. We are given a morphism $\xi^\vee\colon \mathsf{Q}^\vee \to \mathsf{C}^\vee$ bound by the injective homomorphism $H \to G$ of commutative affine group schemes. Clearly $\mathsf{Q}^\vee$ is the gerbe of local liftings of $\underline{\mathrm{Hom}}^\otimes(\omega, \omega_\xi)$ (Giraud 1971, IV 2.5.4.1, p238), and so its class is the image of $\underline{\mathrm{Hom}}^\otimes(\omega, \omega_\xi)$ under Giraud's definition of the connecting homomorphism (ibid. IV 4.2), which coincides with the usual connecting homomorphism in the commutative case (ibid. IV 3.4). Finally, the class of Q in $H^1(k, H)$ is defined to be that represented by the gerbe $\mathsf{Q}^\vee$. $\square$

LEMMA 4.5. *The image of $c_{\mathbb{A}}^K$ in $H^2(\mathbb{A}, P^K)$ under the connecting homomorphism arises from an element of $H^2(\mathbb{Q}, P^K)$.*

PROOF. Let d_p^K and d_∞^K be the images of the classes of R_p^K and R_∞^K in $H^2(\mathbb{Q}_p, P^K)$ and $H^2(\mathbb{R}, P^K)$. According to Lemma 4.4, we have to prove that the element

$$(0, d_p^K, d_\infty^K) \in H^2(\mathbb{A}^{\{p,\infty\}}, P^K) \times H^2(\mathbb{Q}_p, P^K) \times H^2(\mathbb{R}, P^K)$$

arises from an element of $H^2(\mathbb{Q}, P^K)$, but this was shown in Lemma 4.2. $\square$

LEMMA 4.6. *For every CM-subfield K of $\mathbb{C}$, finite and Galois over $\mathbb{Q}$, there exists a unique $c^K \in H^1(\mathbb{Q}, S^K/P^K)$ mapping to $c_{\mathbb{A}}^K \in H^1(\mathbb{A}, S^K/P^K)$ and lifting to some $L \supset K$ with $[L_w\colon K_w]$ even.*

PROOF. Apply 4.1, 4.5. $\square$

We now prove the theorem. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim{}^1 (S^K/P^K)(\mathbb{Q}) \arrow[r] \arrow[d, "b"] & H^1(\mathbb{Q}, S/P) \arrow[r] \arrow[d, "a"] & \varprojlim H^1(\mathbb{Q}, S^K/P^K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varprojlim{}^1 (S^K/P^K)(\mathbb{A}) \arrow[r] \arrow[d] & H^1(\mathbb{A}, S/P) \arrow[r] & \varprojlim H^1(\mathbb{A}, S^K/P^K) \arrow[r] & 0 \\
& 0. & & &
\end{tikzcd}$$

The groups S^K/P^K are anisotropic, which explains the 0 at lower left (1.5). We have to show that there is a $c \in H^1(\mathbb{Q}, S/P)$ mapping to $c_{\mathbb{A}} \in H^1(\mathbb{A}, S/P)$. We know (4.6) that, for each K, there is a unique element $c^K \in H^1(\mathbb{Q}, S^K/P^K)$ mapping to the image $c_{\mathbb{A}}^K$ of $c_{\mathbb{A}}$ in $H^1(\mathbb{A}, S^K/P^K)$ and lifting to some $L \supset K$ with $[L_w\colon K_w]$ even. Because of the uniqueness, the c^K define an element $\underleftarrow{c} \in \varprojlim H^1(\mathbb{Q}, S^K/P^K)$. Choose $c \in H^1(\mathbb{Q}, S/P)$ to map to (c^K). A diagram chase using the surjectivity of b shows that c can be chosen to map to $c_{\mathbb{A}} \in H^1(\mathbb{A}, S/P)$.

Any two c's have the same image in $H^1(\mathbb{Q}, S^K/P^K)$ because they have the same image in $H^1(\mathbb{A}, S^K/P^K)$ and we can apply (3.13).

That $a^{-1}(c_{\mathbb{A}})$ is a principal homogeneous space for $\varprojlim C(K)$ follows from Proposition 3.15.

Towards an elementary definition of the fundamental classes

Are there elementary descriptions of the fundamental classes? By elementary, we mean involving only class field theory and the cohomology of affine group schemes. In particular, an elementary description should not mention abelian varieties, much less motives.

The definition of c_∞ given above is elementary in this sense, but the definition of c_p is not. Wintenberger (1991) shows that ξ_p^K has the following description: choose a prime element a in K_w, and let $b = \mathrm{Nm}_{K_w/B}(a)$ where B is the maximal unramified extension of $\mathbb{Q}_p$ contained in K_w; define

$$\xi'(V) = (V \otimes B(\mathbb{F}), x \mapsto (1 \otimes \sigma)(bx)).$$

Then $\xi_p^K \approx \xi'$. This gives an elementary description of c_p^K, but the family $(c_p^K)_{K \in \mathcal{F}}$ does not determine c_p: there is an exact sequence

$$0 \to \varprojlim{}^1 S^K(\mathbb{Q}_p) \to H^1(\mathbb{Q}_p, S) \to \varprojlim H^1(\mathbb{Q}_p, S^K) \to 0,$$

but $\varprojlim^1 S^K(\mathbb{Q}_p) \neq 0$ because it fails the ML condition.

We shall see later that the Hodge conjecture for CM abelian varieties implies that there is exactly one distinguished global fundamental class. It seems doubtful that this can be described elementarily.

5 Review of gerbes

In this section, we review some definitions and results from Giraud 1971 and Debremaeker 1977 in the case we need them. Throughout E is a category with finite fibred products (in particular, a final object S) endowed with a Grothendieck topology. For example, S could be an affine scheme and E the category of affine S-schemes Aff/S endowed with the fpqc topology.

Gerbes bound by a sheaf of commutative groups

Gerbes. Recall that a *gerbe on* S is a stack of groupoids $\varphi\colon \mathsf{G} \to \mathsf{E}$ such that

(a) there exists a covering map $U \to S$ for which G_U is nonempty;

(b) every two objects of a fibre G_U are locally isomorphic (their inverse images under some covering map $V \to U$ are isomorphic).

The gerbe is said to be *neutral* if G_S is nonempty.

Let $\mathbf{x}$ be a cartesian section of $\mathsf{G}/U \to \mathsf{E}/U$. Then $\mathcal{A}ut(\mathbf{x})$ is a sheaf of groups on E/U, which, up to a unique isomorphism, depends only on $\mathbf{x}(U)$. For $x \in \mathrm{ob}(\mathsf{G}_U)$, this allows us to define $\mathcal{A}ut(x) = \mathcal{A}ut(\mathbf{x})$ with $\mathbf{x}$ any cartesian section such that $\mathbf{x}(U) = x$.

A**-gerbes.** Let A be a sheaf of commutative groups on E. An A*-gerbe on* S is a pair (G, j) where G is a gerbe on S and $j(x)$ for $x \in \mathrm{ob}(\mathsf{G})$ is a natural isomorphism $j(x)\colon \mathcal{A}ut(x) \to A|\varphi(x)$. For example, the gerbe $\mathrm{T{\small ORS}}(A) \to E$ with $\mathrm{T{\small ORS}}(A)_U$ equal to the category of $A|U$-torsors on E/U is a (neutral) A-gerbe on S.

f**-morphisms.** Let $f\colon A \to A'$ be a homomorphism of sheaves of commutative groups, and let (G, j) be an A-gerbe on S and (G', j') an A'-gerbe. An f*-morphism* from (G, j) to (G', j') is a cartesian E-functor $\lambda\colon \mathsf{G} \to \mathsf{G}'$ such that

$$\begin{array}{ccc} \mathcal{A}ut(x) & \xrightarrow{\lambda} & \mathcal{A}ut(\lambda x) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ A|U & \xrightarrow{f} & A'|U \end{array}$$

commutes for all U and all $x \in \mathsf{G}_U$. An f-morphism is an f*-equivalence* if it is an equivalence of categories in the usual sense. If f is an isomorphism, then every f-morphism is an f-equivalence. In particular, every id_A-morphism is an id_A-equivalence (in this context, we shall write A-morphism and A-equivalence).

Let G be a trivial A-gerbe on S. The choice of a cartesian section $\mathbf{x}$ to $\mathsf{G} \to \mathsf{E}$ determines an equivalence of A-gerbes $\mathrm{Hom}(\mathbf{x}, -)\colon \mathsf{G} \to \mathrm{T{\small ORS}}(A)$. Thus, condition (a) in the definition of a gerbe implies that every A-gerbe is locally isomorphic to $\mathrm{T{\small ORS}}(A)$.

Morphisms of f-morphisms. A *morphism* $m\colon \lambda_1 \to \lambda_2$ of two f-morphisms is simply a morphism of E-functors. With these definitions, the A-gerbes on S form a 2-category.

Having defined these objects, our next task is to classify them.

Classification of A-gerbes. One checks easily that A-equivalence is an equivalence relation. Giraud (1971, IV 3.1.1, p247) defines $H^2(S,A)$ to be the set of A-equivalence classes of A-gerbes, and he then shows that (in the case that A is a sheaf of *commutative* groups), $H^2(S,A)$ is canonically isomorphic to the usual (derived functor) group (ibid. IV 3.4.2, p261).

Classification of f-morphisms and their morphisms. Let $f\colon A \to A'$ be a homomorphism of sheaves of commutative groups, and let (G,j) be an A-gerbe and (G',j') an A'-gerbe on S. There is an A'-gerbe $\mathsf{HOM}_f(\mathsf{G},\mathsf{G}')$ on S such that $\mathsf{HOM}_f(\mathsf{G},\mathsf{G}')_U$ is the category whose objects are the f-morphisms $\mathsf{G}|U \to \mathsf{G}'|U$ and whose morphisms are the morphisms of f-morphisms (Giraud 1971, IV 2.3.2, p218). Its class in $H^2(S,A')$ is the difference of the class of G' and the image by λ of the class of G.

5.1. We can read off from this the following statements.

(a) There exists an f-morphism $\mathsf{G} \to \mathsf{G}'$ if and only if λ maps the class of G in $H^2(S,A)$ to the class of G' in $H^2(S,A')$ (as A' is commutative, $\mathsf{HOM}_f(\mathsf{G},\mathsf{G}')$ is neutral if and only if its class is zero).

(b) Let $\lambda_0\colon \mathsf{G} \to \mathsf{G}'$ be an f-morphism (assumed to exist). For any other f-morphism $\lambda\colon \mathsf{G} \to \mathsf{G}'$, $\mathcal{H}om(\lambda_0,\lambda)$ is an A'-torsor, and the functor $\lambda \mapsto \mathcal{H}om(\lambda_0,\lambda)$ is an equivalence from the category whose objects are the f-morphisms $\mathsf{G} \to \mathsf{G}'$ to the category $\textsc{Tors}(A')$. In particular, the set of isomorphism classes of f-morphisms $\mathsf{G} \to \mathsf{G}'$ is a principal homogeneous space for $H^1(S,A')$.

(c) Let $\lambda_1,\lambda_2\colon \mathsf{G} \to \mathsf{G}'$ be two f-morphisms. If they are isomorphic, then the set of isomorphisms $\lambda_1 \to \lambda_2$ is a principal homogeneous space for $H^0(S,A') =_{\mathrm{df}} A'(S)$.

EXERCISE 5.2. Let $S = \operatorname{Spec} k$ with k a field, and let $\mathsf{G} \to \mathsf{Aff}/S$ be a gerbe bound by a separable torus. Let $\bar{S} = \operatorname{Spec} k^{\mathrm{al}}$, and let a,b be the projection maps $\bar{S} \times_S \bar{S} \to \bar{S}$. Show that $\mathsf{G}_{\bar{S}}$ is nonempty, and that for any $x \in \operatorname{ob} \mathsf{G}_{\bar{S}}$, a^*x and b^*x are isomorphic. [Hint: use 2.12.]

Gerbes bound by a sheaf of commutative crossed module

Commutative crossed modules. Recall that a *crossed module* is a pair of groups (A,B) together with an action of B on A and a homomorphism $\rho\colon A \to B$ respecting this action. It is said to be *commutative* if A and B are both commutative and the action of B on A is trivial. Thus, a commutative crossed module is nothing more than a homomorphism of commutative groups. Similarly, a sheaf of commutative crossed modules is simply a homomorphism $\rho\colon A \to B$ of sheaves of commutative groups. A homomorphism

$(f, \phi)\colon (A, B, \rho) \to (A', B', \rho')$ of sheaves of commutative crossed modules is a pair of homomorphisms giving a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \rho'} \\ B & \xrightarrow{\phi} & B', \end{array}$$

that is, it is a morphism of complexes.

(A, B)**-gerbes.** Let $\rho\colon A \to B$ be a sheaf of commutative crossed modules. Following Debremaeker 1977, we define an (A, B)-gerbe to be a triple (G, μ, j) with (G, j) an A-gerb and μ a ρ-morphism $A \to \mathrm{T{\small ORS}}(B)$. For example, $\mathrm{T{\small ORS}}(A) \xrightarrow{\lambda_*} \mathrm{T{\small ORS}}(B)$ is an (A, B)-gerbe.

(f, ϕ)**-morphisms.** Let $(f, \phi)\colon (A, B) \to (A', B')$ be a homomorphism of sheaves of commutative crossed modules. Let (G, μ, j) be an (A, B)-gerbe on S and (G', μ', j') an (A', B')-gerbe. An (f, ϕ)-*morphism* from (G, μ, j) to (G', μ', j') is a pair (λ, i) where λ is an f-morphism $(\mathsf{G}, j) \to (\mathsf{G}', j')$ and i is an isomorphism of functors

$$i\colon \phi_* \circ \mu \Rightarrow \mu' \circ \lambda\colon \mathsf{G} \to \mathrm{T{\small ORS}}(B').$$

When $(f, \phi) = (\mathrm{id}_A, \mathrm{id}_B)$, we speak of an (A, B)-morphism $(\lambda, i)\colon (\mathsf{G}, \mu, j) \to (\mathsf{G}', \mu', j')$.

Morphisms of (f, ϕ)**-morphisms.** Let (λ_1, i_1) and (λ_2, i_2) be (f, ϕ)-morphims. A *morphism* $m\colon (\lambda_1, i_1) \to (\lambda_2, i_2)$ is a morphism of E-functors $m\colon \lambda_1 \to \lambda_2$ satisfying the following condition: on applying μ' to m, we obtain morphism of functors $\mu' \cdot m\colon \mu' \circ \lambda_1 \to \mu' \circ \lambda_2$; the composite of this with i_1 is required to equal i_2,

$$(\mu' \cdot m) \circ i_1 = i_2.$$

With these definitions, the (A, B)-gerbes over S form a 2-category.

Again, we wish to classify these objects.

Classification of (A, B)**-gerbes.** An (A, B)-morphism is an (A, B)-equivalence if λ is an equivalence of categories. Again, (A, B)-equivalence is an equivalence relation, and Debremaeker defines $H^2(S, A \to B)$ to be the set of equivalence classes. The forgetful functor $(\mathsf{G}, \mu, j) \mapsto (\mathsf{G}, j)$ defines a map $H^2(S, A \to B) \to H^2(S, A)$.

Because $A \to B$ is sheaf of *commutative* crossed modules, $H^2(S, A \to B)$ is in fact canonically isomorphic to the usual hypercohomology group of the complex $A \to B$.

Classification of the (f, ϕ)**-morphisms.** Let $\rho\colon A \to B$ be a sheaf of commutative crossed modules. Define an (A, B)-*torsor* to be a pair (P, p) with P an A-torsor and p

an S-point of $\rho_* P$. A morphism $(P,p) \to (P',p')$ is a morphism $P \to P'$ of A-torsors carrying p to p'. Define $H^1(S, A \to B)$ to be the set of isomorphism classes of (A,B)-torsors (cf. Deligne 1979, 2.4.3; Milne 2002b, §1).

Now consider a homomorphism $(f,\phi)\colon (A,B) \to (A',B')$. Let (G,μ,j) be an (A,B)-gerbe over S and (G',μ',j') an (A',B')-gerbe. Assume that there exists an (f,ϕ)-morphism $(\lambda_0,i_0)\colon (\mathsf{G},\mu,j) \to (\mathsf{G}',\mu',j')$. As we noted above, the map $\lambda \mapsto P(\lambda) =_{\text{df}} \mathcal{H}om(\lambda_0,\lambda)$ is an equivalence from the category of f-morphisms $(\mathsf{G},j) \to (\mathsf{G}',j')$ to the category of A'-torsors. From i we get a point $p(i) \in (\rho'_* P)(S)$, and $(\lambda,i) \mapsto (P(\lambda),p(i))$ defines an equivalence from the category of (f,ϕ)-morphism $(\mathsf{G},\mu,j) \to (\mathsf{G}',\mu',j')$ to the category of (A,B)-torsors. In particular, we see that the set of (f,ϕ)-morphisms $(\mathsf{G},\mu,j) \to (\mathsf{G}',\mu',j')$ is a principal homogeneous space for $H^1(S, A \to B)$.

Classification of the morphisms of (f,ϕ)-morphisms. Let $(\lambda_1,i_1),(\lambda_2,i_2)\colon (\mathsf{G},\mu,j) \to (\mathsf{G}',\mu',j')$ be two (f,ϕ)-morphisms. If they are isomorphic, then the set of isomorphisms $(\lambda_1,i_1) \to (\lambda_2,i_2)$ is a principal homogeneous space for

$$H^0(S, A \to B) =_{\text{df}} \operatorname{Ker}(A(S) \to B(S)).$$

Gerbes bound by an injective commutative crossed module

Consider an exact sequence

$$0 \to A \xrightarrow{\rho} B \xrightarrow{\sigma} C \to 0$$

of sheaves of commutative groups.

The group $H^2(k, A \to B)$. For any C-torsor P, there is a gerbe $\mathsf{K}(P)$ whose fibre $\mathsf{K}(P)_U$ over U is the category whose objects are pairs (Q,λ) with Q a B-torsor and λ a ρ-morphism $Q \to P$. This has a natural structure of an A-gerbe, and the forgetful functor $(Q,\lambda) \mapsto Q$ endows it with the structure of an (A,B)-gerbe (Debremaeker 1977).

Conversely, let (G,μ,j) be an (A,B)-gerbe. For any object $x \in \operatorname{ob}(\mathsf{G}_U)$, $\sigma_*\mu(x)$ is a C-torsor over U endowed with a canonical descent datum, which gives a C-torsor over S.

These correspondences define inverse isomorphisms

$$H^2(k, A \to B) \cong H^1(k, C).$$

The group $H^1(k, A \to B)$. For any point $c \in C(k)$, $\sigma^{-1}(c)$ is an A-torsor with $\rho_*(\sigma^{-1}(c)) = B$. In particular, $\rho_*(\sigma^{-1}(c))$ has a canonical point (the identity of B), and so $\sigma^{-1}(c)$ has the structure of an (A,B)-torsor.

Conversely, let (P,p) be an (A,B)-torsor. For any point q of P in some covering of k, $q-p$ is an element of B whose image in C lies in $C(k)$.

These correspondences define inverse isomorphisms

$$H^1(k, A \to B) \cong C(k).$$

The group $H^0(k, A \to B)$. By definition,

$$H^0(k, A \to B) = \operatorname{Ker}(A(k) \to B(k)) = 0.$$

6 The motivic gerbe

Grothendieck's construction gives a rigid pseudo-abelian tensor category $\mathsf{Mot}(\mathbb{F})$ of abelian motives over $\mathbb{F}$ based on the abelian varieties over $\mathbb{F}$ and using the numerical equivalence classes of algebraic cycles as correspondences (see, for example, Saavedra Rivano 1972, VI 4.1). In fact, $\mathsf{Mot}(\mathbb{F})$ is Tannakian (Jannsen 1992). When one assumes the Tate conjecture for abelian varieties, the fundamental group of $\mathsf{Mot}(\mathbb{F})$ becomes identified with P (e.g., Milne 1994), and when one assumes Grothendieck's Hodge standard conjecture, $\mathsf{Mot}(\mathbb{F})$ acquires a canonical polarization (e.g., Saavedra Rivano 1972, VI 4.4).

Thus, when we assume these two conjectures, $\mathsf{Mot}(\mathbb{F})$ is a Tannakian $\mathbb{Q}$-category with fundamental group the Weil-number protorus P, an $\mathbb{A}_f^p$-valued fibre functor ω_f^p (étale cohomology), an exact tensor functor $\omega_p\colon \mathsf{Mot}(\mathbb{F}) \to \mathsf{R}_p$ (crystalline cohomology), and an exact tensor functor $\omega_\infty\colon \mathsf{Mot}(\mathbb{F}) \to \mathsf{R}_\infty$ (from the polarization — see Deligne and Milne 1982, 5.20). Note that ω_∞ is uniquely defined up to isomorphism, whereas ω^p and ω_p are uniquely defined up to a unique isomorphism.

Let $\mathsf{P} = \mathsf{Mot}(\mathbb{F})^\vee$ be the gerbe of fibre functors on $\mathsf{Mot}(\mathbb{F})$. It is a P-gerbe endowed with an object $w^p =_{\text{df}} \omega_f^p$ in $\mathsf{P}_{\mathbb{A}_f^p}$ and morphisms $w_p\colon \mathsf{R}_p^\vee \to \mathsf{P}(p)$ and $w_\infty\colon \mathsf{R}_\infty^\vee \to \mathsf{P}(\infty)$ (defined by ω_p and ω_∞), where $\mathsf{P}(p) =_{\text{df}} \mathsf{P}/\operatorname{Spec}(\mathbb{Q}_p)$ and $\mathsf{P}(\infty) = \mathsf{P}/\operatorname{Spec}(\mathbb{R})$.

REMARK 6.1. Assume the Tate conjecture for all smooth projective varieties over $\mathbb{F}$ and the Hodge standard conjecture for all abelian varieties over $\mathbb{F}$. Then the category of motives based on all smooth projective varieties is equivalent with the subcategory based on the abelian varieties (Milne 1994). Therefore, under these assumptions the category of all motives over $\mathbb{F}$ has a system $(\mathsf{P}, w^p, w_p, w_\infty)$ attached to it with P again a P-gerbe.

The (pseudo)motivic gerbe

6.2. We now drop all assumptions, and consider the existence and uniqueness of systems $(\mathsf{P}, w^p, w_p, w_\infty)$ with

- P a P-gerbe over $\operatorname{Spec}(\mathbb{Q})$,
- w^p an object of $\mathsf{P}_{\mathbb{A}_f^p}$,
- w_p an x_p-morphism $\mathsf{R}_p^\vee \to \mathsf{P}(p)$,
- w_∞ an x_∞-morphism $\mathsf{R}_\infty^\vee \to \mathsf{P}(\infty)$.

Here x_p and x_∞ are the homomorphisms $\mathbb{G} \to S_{/\mathbb{Q}_p}$ and $\mathbb{G}_m \to S_{/\mathbb{R}}$ described in §4.

THEOREM 6.3. *There exists a system* $(\mathsf{P}, w^p, w_p, w_\infty)$ *as in 6.2. If* $(\mathsf{P}', w^{p\prime}, w'_p, w'_\infty)$ *is a second such system, then there exists a* P*-equivalence* $\lambda\colon \mathsf{P} \to \mathsf{P}'$ *such that* $\lambda(w^p) \approx w^{p\prime}$, $\lambda \circ w_p \approx w'_p$, $\lambda \circ w_\infty \approx w'_\infty$. *Any two such* λ*'s are isomorphic by an isomorphism that is unique up to a nonzero rational number.*

PROOF. We shall apply the results reviewed in §5 without further reference.

The P-equivalence classes of P-gerbes P are classified by $H^2(\mathbb{Q}, P)$. The condition that there exists a w^p is that $\mathsf{P}/\mathbb{A}_f^p$ is neutral or, equivalently, that the class of P maps to zero in $H^2(\mathbb{A}_f^p, P)$; the condition that there exist a w_p is that the class of $\mathsf{P}(p)$ in $H^2(\mathbb{Q}_p, P)$ is the image of the class of $\mathsf{R}_p^\vee$ under $x_p\colon H^2(\mathbb{Q}_p, \mathbb{G}) \to H^2(\mathbb{Q}_p, P)$; the condition that there exist a w_∞ is that the class of $\mathsf{P}(\infty)$ in $H^2(\mathbb{R}, P)$ is the image of the class of $\mathsf{R}_\infty^\vee$ under $x_\infty\colon H^2(\mathbb{R}, \mathbb{G}_m) \to H^2(\mathbb{R}, P)$. Proposition 4.3 shows that there is a unique class in $H^2(\mathbb{Q}, P)$ with these properties. Thus, there exists a P-gerbe P for which there exist w^p, w_p, w_∞ as in 6.2 and if P' is a second such gerbe, then there exists a P-morphism $P \to P'$.

Now consider two systems $(\mathsf{P}, w^p, w_p, w_\infty)$ and $(\mathsf{P}', w^{p\prime}, w_p', w_\infty')$. The isomorphism classes of P-morphisms $\mathsf{P} \to \mathsf{P}'$ form a principal homogeneous space for $H^1(\mathbb{Q}, P)$. On the other hand, the isomorphism classes of triples (w^p, w_p, w_∞) for P (or P') form a principle homogeneous space for $H^1(\mathbb{A}, P)$. Since the map $H^1(\mathbb{Q}, P) \to H^1(\mathbb{A}, P)$ is an isomorphism (3.6, 3.10), there exists a P-morphism $\lambda\colon \mathsf{P} \to \mathsf{P}'$ carrying the isomorphism class of (w^p, w_p, w_∞) into that of $(w^{p\prime}, w_p', w_\infty')$, and it is unique up to isomorphism. If λ_1 and λ_2 are isomorphic P-morphisms $P \to P'$, then the set of isomorphisms $\lambda_1 \to \lambda_2$ is a principal homogeneous space for $P(\mathbb{Q})$, which equals $\mathbb{Q}^\times$ (3.12). □

REMARK 6.4. When we replace $\mathsf{Mot}(\mathbb{F})$ with its subcategory $\mathsf{Mot}_0(\mathbb{F})$ of motives of weight 0, we obtain similar results, except that its gerbe P_0 is a P_0-gerbe. Theorem 6.3 holds with the only change being that now any two λ's are uniquely isomorphic (because $P_0(\mathbb{Q}) = 0$).

The pseudomotivic groupoid

Let ν_p be the forgetful $B(\mathbb{F})$-valued fibre functor on the Tannakian category R_p, and let $\mathfrak{G}_p = \underline{\mathrm{Aut}}^\otimes(\nu_p)$. It is a $\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p$-groupoid with kernel $\mathfrak{G}_p^\Delta = \mathbb{G}$. The fibre functor ν_p is an object of the gerbe $(\mathsf{R}_p^\vee)_{B(\mathbb{F})}$, and $\mathfrak{G}_p$ can also be described as the groupoid of $\mathbb{Q}_p$-automorphisms of this object (in the sense of Deligne 1990, 3.4).

Let ν_∞ be the forgetful $\mathbb{C}$-valued fibre functor on the Tannakian category R_∞, and let $\mathfrak{G}_\infty = \underline{\mathrm{Aut}}^\otimes(\nu_\infty)$. It is a $\mathbb{C}/\mathbb{R}$-groupoid with kernel $\mathfrak{G}_\infty^\Delta = \mathbb{G}_{m/\mathbb{C}}$. The fibre functor ν_∞ is an object of the gerbe $(\mathsf{R}_\infty^\vee)_\mathbb{C}$, and $\mathfrak{G}_\infty$ can also be described as the groupoid of $\mathbb{R}$-automorphisms of this object.

For $l \neq p, \infty$, let $\mathfrak{G}_l$ be the trivial $\mathbb{Q}_l^{\mathrm{al}}/\mathbb{Q}_l$-groupoid.

Let $(\mathsf{P}, w^p, w_p, w_\infty)$ be as in (6.2). Then, because $H^i(\mathbb{Q}^{\mathrm{al}}, P) = 0$ for $i > 0$ (2.12), there exists an $x \in \mathrm{ob}(\mathsf{P}_{\mathbb{Q}^{\mathrm{al}}})$, and any two such objects are isomorphic. Let $S = \operatorname{Spec}\mathbb{Q}$ and $\bar{S} = \operatorname{Spec}\mathbb{Q}^{\mathrm{al}}$. Let $\mathfrak{P}$ be the $\bar{S}/S$-groupoid of automorphisms of x: for any $\bar{S} \times_S \bar{S}$-scheme $(b, a)\colon T \to \bar{S} \times_S \bar{S}$, $\mathfrak{P}(T)$ is the set of isomorphisms $a^*x \to b^*x$. It admits a section over $\operatorname{Spec}(\mathbb{Q}^{\mathrm{al}} \otimes_\mathbb{Q} \mathbb{Q}^{\mathrm{al}})$, and its kernel $\mathfrak{P}^\Delta = P$. For $l \neq p, \infty$, w^p defines a homomorphism $\zeta_l\colon \mathfrak{G}_l \to \mathfrak{P}(l)$ where $\mathfrak{P}(l)$ is the $\mathbb{Q}_l^{\mathrm{al}}/\mathbb{Q}_l$-groupoid obtained from $\mathfrak{P}$ by base change. Moreover, w_p defines a homomorphism $\zeta_p\colon \mathfrak{G}_p \to \mathfrak{P}(p)$ and w_∞ defines a homomorphism $\zeta_\infty\colon \mathfrak{G}_\infty \to \mathfrak{P}(\infty)$.

PROPOSITION 6.5. (a) *The system* $(\mathfrak{P}, (\zeta_l)_l)$ *satisfies the following conditions:*

i) $(\mathfrak{P}^\Delta, \zeta_p^\Delta, \zeta_\infty^\Delta) = (P, x_p, x_\infty)$;

ii) *the morphisms* ζ_l *for* $l \neq p, \infty$ *are induced by a section of* $\mathfrak{P}$ *over* $\mathrm{Spec}(\overline{\mathbb{A}_f^p} \otimes_{\mathbb{A}_f^p} \overline{\mathbb{A}_f^p})$ *where* $\overline{\mathbb{A}_f^p}$ *is the image of the map* $\bar{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \prod_{l \neq p,\infty} \mathbb{Q}_\ell^{al}$.

(b) *Let* $(\mathfrak{P}', (\zeta_l'))$ *be the system attached to* $x' \in \mathrm{ob}(\mathsf{P}'_{\mathbb{Q}^{al}})$ *for a second quadruple* $(\mathsf{P}', w^{p\prime}, w_p', w_\infty')$. *The choice of a* P*-equivalence* $\lambda\colon \mathsf{P} \to \mathsf{P}'$ *as in Theorem 6.3 and of an isomorphism* $\lambda(x) \to x'$ *determine an isomorphism* $\alpha\colon \mathfrak{P} \to \mathfrak{P}'$ *such that, for all* l, ζ_l' *is isomorphic to* $\alpha \circ \zeta_l$, *and any two* α*'s arising in this way are isomorphic.*

PROOF. Straightforward consequence of Theorem 6.3. □

DEFINITION 6.6. Any system $(\mathfrak{P}, (\zeta_l)_l)$ arising from a system $(\mathsf{P}, w^p, w_p, w_\infty)$ as in (6.2) and an object $x \in P_{\mathbb{Q}^{\mathrm{al}}}$ will be called a *pseudomotivic groupoid*.

Notes. In Milne 1992, 3.27, a pseudomotivic groupoid is defined to be any system $(\mathfrak{P}, (\zeta_l))$ satisfying condition (6.5a(i)). Theorem 3.28 (ibid.) then states: there exists a pseudomotivic groupoid $(\mathfrak{P}, (\zeta_l))$; if $(\mathfrak{P}', (\zeta_l'))$ is a second pseudomotivic groupoid, then there is an isomorphism $\alpha\colon \mathfrak{P} \to \mathfrak{P}'$ such that $\zeta_l' \approx \alpha \circ \zeta_l$, and α is uniquely determined up to isomorphism. Only a brief indication of proof is given, and the theorem is credited to Langlands and Rapoport (1987). A sketch of a proof is given in Milne 1994, 3.31.

As Reimann points out (1997, p120), I should have included the condition (6.5a(ii)) in the definition of the pseudomotivic groupoid (because it is needed for the statement of the conjecture of Langlands and Rapoport concerning Shimura varieties). Moreover, the argument sketched in Milne 1994 (based on that in Langlands and Rapoport 1987) shows only that there exists an isomorphism α for which ζ_l' is algebraically isomorphic to $\alpha \circ \zeta_l$ (i.e., becomes isomorphic when projected to any algebraic quotient).

However, there is a more serious criticism of the four articles just cited (and others), namely, in each article $H^i(k, P)$ is taken to be $\varprojlim H^i(k, P^K)$ instead of the fpqc group $H^i(k, P)$ which is, in fact, the group that classifies the various objects.[13] This amounts to ignoring the terms $\varprojlim^1 H^{i-1}(k, P^K)$, which are not all zero. Thus, some of the proofs in these papers are inadequate.

The papers cited above all work with groupoids.[14] Here, I have preferred to work with gerbes because their attachment to Tannakian categories is canonical (the groupoid of a Tannakian category depends on the choice of a fibre functor), are more directly related to nonabelian cohomology, and are, in some respects, easier to work with.

[13] In Milne 1994, footnote p441, I correctly note that the gerbes are classified by the fpqc group $H^2(k, P)$, but then claim that this group equals $\varprojlim H^2(k, P^K)$, giving as a reference Saavedra Rivano 1972, III 3.1. In fact, Saavedra proves results only for algebraic groups. I don't know why I thought the results held for affine group schemes, except perhaps I confused the (true) statement that cohomology commutes with products with the (false) statement that it commutes with inverse limits.

[14] This is obscured in Langlands and Rapoport 1987 by the authors' calling their groupoids gerbes.

7 The motivic morphism of gerbes

Assume the Hodge conjecture for complex abelian varieties of CM-type. This implies the Tate conjecture and the Hodge standard conjecture for abelian varieties over finite fields (Milne 1999; Milne 2001). Hence we get a category of abelian motives $\mathsf{Mot}(\mathbb{F})$ with fibre functors as in the last section. Moreover, the category $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ of CM-motives over $\mathbb{Q}^{\mathrm{al}}$, defined using Deligne's Hodge classes, coincides with that defined using algebraic cycles modulo numerical equivalence. Thus, we get a reduction functor $R\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathsf{Mot}(\mathbb{F})$ (exact tensor functor of $\mathbb{Q}$-categories), and hence a morphism of gerbes of fibre functors

$$R^{\vee}\colon \mathsf{Mot}(\mathbb{F})^{\vee} \to \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\vee}.$$

It follows from Shimura-Taniyama theory that this is a ρ-morphism, where $\rho\colon P \to S$ is the homomorphism defined in §3, p26.

The Betti fibre functor ω_B on $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})$ defines an S-equivalence of gerbes

$$\omega \mapsto \mathcal{H}om(\omega_B, \omega)\colon \mathsf{CM}(\mathbb{Q}^{\mathrm{al}})^{\vee} \to \mathrm{T{\scriptsize ORS}}(S).$$

On composing $R^{\vee}$ with this, we obtain a ρ-morphism $\mu\colon \mathsf{Mot}(\mathbb{F})^{\vee} \to \mathrm{T{\scriptsize ORS}}(S)$, i.e., a (P,S)-gerbe. For all $l \neq p, \infty$, $\mu(\omega_l)$ is a trivial $S_{\mathbb{Q}_l}$-torsor. Moreover, the composites of w_p and w_∞ with μ are isomorphic to the functors $\xi_p^{\vee}$ and $\xi_\infty^{\vee}$, where ξ_p and ξ_∞ are as in §4.

The (pseudo)motivic morphism of gerbes

7.1. We now drop all assumptions, and consider the existence and uniqueness of systems $(\mathsf{P}, \mu, w^p, w_p, w_\infty)$ with

- (P, μ) a (P,S)-gerbe,
- w^p an object of $\mathsf{P}_{\mathbb{A}_f^p}$ such that $\mu(w^p)$ is a trivial $S_{\mathbb{A}_f^p}$-torsor;
- w_p an x_p-morphism $\mathsf{R}_p^{\vee} \to \mathsf{P}(p)$ such that $\mu \circ w_p \approx \xi_p^{\vee}$;
- w_∞ an x_∞-morphism $\mathsf{R}_\infty^{\vee} \to \mathsf{P}(\infty)$ such that $\mu \circ w_\infty \approx \xi_\infty^{\vee}$.

THEOREM 7.2. *The set of (P,S)-equivalence classes of (P,S)-gerbes (P, μ) that can be completed to a system $(\mathsf{P}, \mu, w^p, w_p, w_\infty)$ as in (7.1) is a $\varprojlim_{\mathcal{F}} C(K)$-principal homogeneous space, where $\mathcal{F}$ is the set of CM-subfields of $\mathbb{Q}^{al}$ of finite degree over $\mathbb{Q}$; in particular, it is uncountable (1.11).*

PROOF. We shall apply the results reviewed in §5 without further reference. The (P,S)-gerbes (P, μ) are classified by $H^2(\mathbb{Q}, P \xrightarrow{\rho} S) \cong H^1(\mathbb{Q}, S/P)$. The condition that there exists a w^p with $\mu(w^p)$ neutral is that (P, μ) becomes equivalent with the trivial (P,S)-gerbe $\mathrm{T{\scriptsize ORS}}(P) \xrightarrow{\rho_*} \mathrm{T{\scriptsize ORS}}(S)$ over $\mathbb{A}_f^p$, i.e., that the class of (P, μ) maps to zero in $H^1(\mathbb{A}_f^p, S/P)$. The condition that there exists a w_p with $\mu \circ w_p \approx \xi_p^{\vee}$ is that the class of (P, μ) over $\mathbb{Q}_p$ is the local fundamental class c_p. The condition that there exists a w_∞ with $\mu \circ w_\infty \approx \xi_\infty^{\vee}$ is that the class of (P, μ) over $\mathbb{R}$ is the local fundamental class c_∞. Theorem 4.1 shows that there exists such a class in $H^1(\mathbb{Q}, S/P)$, and that the set of them is a principal homogeneous space under $\varprojlim C(K)$. This completes the proof. □

REMARK 7.3. If (P,μ) and (P',μ') are (P,S)-gerbes that can be completed to systems as in (7.1), then, for all $K\in\mathcal{F}$, (P^K,μ^K) and (P'^K,μ'^K) are (P^K,S^K)-equivalent (P^K,S^K)-gerbes. The point is that any two fundamental classes have the same image in $H^1(\mathbb{Q},S^K/P^K)$ (4.1).

REMARK 7.4. Let $(\mathsf{P}',\mu',w^{p\prime},w_p',w_\infty')$ be a second system as in 7.1. Then, even if (P,μ) and (P',μ') are (P,S)-equivalent, there may not be a (P,S)-equivalence $(\lambda,i)\colon(\mathsf{P},\mu)\to(\mathsf{P}',\mu')$ such that $\lambda(w^p)\approx w^{p\prime}$, $\lambda\circ w_p\approx w_p'$, $\lambda\circ w_\infty\approx w_\infty'$. The point is that the set of isomorphism classes of (P,S)-equivalences $(\mathsf{P},\mu)\to(\mathsf{P}',\mu')$ (if nonempty) is a principle homogeneous space under $(S/P)(\mathbb{Q})$, while the set of isomorphism classes of triples (w^p,w_p,w_∞) satisfying (7.1) for (P,μ) is a principal homogeneous space under $(S/P)(\mathbb{A})$, and $(S/P)(\mathbb{Q})\to(S/P)(\mathbb{A})$ is not surjective[15].

REMARK 7.5. Let $(\mathsf{P},\mu,w^p,w_p,w_\infty)$ be as in 7.1. On choosing an object $x\in\mathsf{P}_{\mathbb{Q}^{\mathrm{al}}}$ and an isomorphism of $\mu(x)$ with the trivial torsor, we obtain a morphism of groupoids $\mathfrak{P}\to\mathfrak{G}_S$ with $(\mathfrak{P}\to\mathfrak{G}_S)^\Delta\cong(P\xrightarrow{\rho}S)$. Following Pfau 1993, we define the quasimotivic groupoid to be $\mathfrak{P}\times_{\mathfrak{G}_S}\mathfrak{G}_T$ where $T=\varprojlim(\mathbb{G}_m)_{L/\mathbb{Q}}$ (limit over all subfields L of $\mathbb{Q}^{\mathrm{al}}$ of finite degree over $\mathbb{Q}$).

REMARK 7.6. To state the conjecture of Langlands and Rapoport (Langlands and Rapoport 1987, p169) for Shimura varieties $\mathrm{Sh}(G,X)$ with rational weight and G^{der} simply connected, only the pseudomotivic groupoid is needed. For Shimura varieties with rational weight and G^{der} not necessarily simply connected, one needs[16] the morphism $\mathfrak{P}\to\mathfrak{G}_S$ (Milne 1992). For arbitrary Shimura varieties, one needs a quasimotivic groupoid.

REMARK 7.7. André (2003) has shown that motivated classes (in the sense of André 1996) reduce modulo w to motivated classes. Assume the following conjecture (André 1996): the intersection number of any two motivated classes of complementary dimension on a smooth projective variety over $\mathbb{F}$ is a rational number. Then the arguments of Milne 1999 and Milne 2002a show that the Tate conjecture and the Hodge standard conjecture hold for abelian varieties over $\mathbb{F}$ with "algebraic class" replaced with "motivated class".[17] Thus, under the assumption of André's conjecture, there exists a Tannakian category of abelian motives $\mathsf{Mot}(\mathbb{F})$ over $\mathbb{F}$ (defined using motivated correspondences) and a reduction functor $\mathsf{CM}(\mathbb{Q}^{\mathrm{al}})\to\mathsf{Mot}(\mathbb{F})$. From this, we obtain a *well-defined* system (P,μ,w^p,w_p,w_∞) as in 7.1.

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[15]Consider the commutative diagram (3.6, 3.10, 3.12)

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbb{Q}^\times & \longrightarrow & S(\mathbb{Q}) & \longrightarrow & (S/P)(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q},P)\\
& \downarrow & & \downarrow & & \downarrow & & \downarrow\cong\\
0 \longrightarrow & \mathbb{A}^\times & \longrightarrow & S(\mathbb{A}) & \longrightarrow & (S/P)(\mathbb{A}) & \longrightarrow & H^1(\mathbb{A},P)
\end{array}$$

and use that the map $S(\mathbb{Q})\to S(\mathbb{A})$ is far from surjective.

[16]Reimann (1997) overlooks this: in his Conjecture B3.7 it is necessary to require that G^{der} be simply connnected.

[17]The Hodge standard conjecture then holds for algebraic classes on abelian varieties in prime characteristic!

References

André, Yves. 1996. *Pour une théorie inconditionnelle des motifs*, Inst. Hautes Études Sci. Publ. Math., 5–49. MR98m:14022

———. 2003. *Cycles de Tate sur les variétés abéliennes sur les corps finis*, Preprint, to appear in Journal de l'Institut de Mathématiques de Jussieu.

Bourbaki, Nicolas. 1989. *General topology. Chapters 1–4*, Elements of Mathematics, Springer-Verlag, Berlin, ISBN 3-540-19374-X. MR90a:54001a

Bucur, Ion and Aristide Deleanu. 1968. *Introduction to the theory of categories and functors*, With the collaboration of Peter J. Hilton and Nicolae Popescu. Pure and Applied Mathematics, Vol. XIX, Interscience Publication John Wiley & Sons, Ltd., London-New York-Sydney. MR38 #4534

Debremaeker, R. 1977. *Non abelian cohomology*, Bull. Soc. Math. Belg. **29**, 57–72. MR58 #675

Deligne, Pierre. 1979. *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic Forms, Representations and l-Functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., pp. 247–289. MR81i:10032

———. 1990. *Catégories tannakiennes*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, pp. 111–195. MR92d:14002

Deligne, Pierre and James S. Milne. 1982. *Tannakian categories*, Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, pp. 101–228.

Demazure, Michel and Pierre Gabriel. 1970. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris. MR46 #1800

Giraud, Jean. 1971. *Cohomologie non abélienne*, Springer-Verlag, Berlin. MR49 #8992

Jannsen, Uwe. 1992. *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107**, 447–452. MR93g:14009

Jensen, C. U. 1972. *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*, Springer-Verlag, Berlin. MR53 #10874

Langlands, R. P. and M. Rapoport. 1987. *Shimuravarietäten und Gerben*, J. Reine Angew. Math. **378**, 113–220. MR88i:11036

Louboutin, S. and R. Okasaki. 2003. *Exponents of the ideal class groups of cm number fields*, Math. Z. **243**, 155–159.

Milne, J. S. 1986. *Arithmetic duality theorems*, Perspectives in Mathematics, vol. 1, Academic Press Inc., Boston, MA, ISBN 0-12-498040-6. MR88e:14028

———. 1992. *The points on a Shimura variety modulo a prime of good reduction*, The Zeta Functions of Picard Modular Surfaces, Univ. Montréal, Montreal, QC, pp. 151–253. MR94g:11041

———. 1994. *Motives over finite fields*, Motives (Seattle, Wa, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, pp. 401–459. MR95g:11053

———. 1999. *Lefschetz motives and the Tate conjecture*, Compositio Math. **117**, 45–76. MR2000i:11090

———. 2001. *The Tate conjecture for certain abelian varieties over finite fields*, Acta Arith. **100**, 135–166. MR2002j:14050

———. 2002a. *Polarizations and Grothendieck's standard conjectures*, Ann. of Math. (2) **155**, 599–610. MR1 906 596

———. 2002b. *Periods of abelian varieties*, Preprint, 35 pages, available at www.jmilne.org/math/.

Pfau, Matthias. 1993. *The reduction of connected shimura varieties at primes of good reduction*, Ph.D. Thesis, University of Michigan.

Platonov, Vladimir and Andrei Rapinchuk. 1994. *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, MA, ISBN 0-12-558180-7. MR95b:11039

Reimann, Harry. 1997. *The semi-simple zeta function of quaternionic Shimura varieties*, Lecture Notes in Mathematics, vol. 1657, Springer-Verlag, Berlin. MR98k:11076

Saavedra Rivano, Neantro. 1972. *Catégories Tannakiennes*, Springer-Verlag, Berlin. MR49 #2769

Serre, Jean-Pierre. 1962. *Corps locaux*, Publications de l'Institut de Mathématique de l'Université de Nancago, VIII, Actualités Sci. Indust., No. 1296. Hermann, Paris. MR27 #133

Tate, J. T. 1967. *Global class field theory*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., pp. 162–203. MR36 #3749

______. 1976. *Relations between K_2 and Galois cohomology*, Invent. Math. **36**, 257–274. MR55 #2847

Washington, Lawrence C. 1997. *Introduction to cyclotomic fields*, Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, ISBN 0-387-94762-0. MR97h:11130

Wei, Wafa. 1993. *Weil numbers and generating large field extensions*, Ph.D. Thesis, University of Mich.

Wintenberger, Jean-Pierre. 1991. *Torseur entre cohomologie étale p-adique et cohomologie cristalline; le cas abélien*, Duke Math. J. **62**, 511–526. MR92j:14024

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