# DESCENT FOR SHIMURA VARIETIES

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ABSTRACT. This note proves that the descent maps provided by Langlands's Conjugacy Conjecture *do* satisfy the continuity condition necessary for them to be effective. Hence the conjecture does imply the existence of canonical models.

In his Corvallis article (1979, §6), Langlands stated a conjecture that identifies the conjugate of a Shimura variety by an automorphism of $\mathbb{C}$ with the Shimura variety defined by different data, and he sketched a proof that his conjecture implies the existence of canonical models. However, as Jörg Wildeshaus and others have pointed out to me, it is not obvious that the descent maps defined by Langlands satisfy the continuity condition necessary for the descent to be effective. In this note, I prove that they do satisfy this condition, and hence that Langlands's conjecture *does* imply the existence of canonical models — this is our only proof of the existence of these models for a general Shimura variety. The proof is quite short and elementary. I give it in Section 2 after reviewing some generalities on the descent of varieties in Section 1.

*Notations and Conventions.* A variety over a field $k$ is a geometrically reduced scheme of finite type over $\operatorname{Spec} k$ (*not* necessarily irreducible). For a variety $V$ over a field $k$ and a homomorphism $\sigma \colon k \to k'$, $\sigma V$ is the variety over $k'$ obtained by base change. The ring of finite adèles for $\mathbb{Q}$ is denoted by $\mathbb{A}_f$.

## 1. DESCENT OF VARIETIES.

In this section, $\Omega$ is an algebraically closed field of characteristic zero. For a field $L \subset \Omega$, $A(\Omega/L)$ denotes the group of automorphisms of $\Omega$ fixing the elements of $L$.

Let $V$ be a variety over $\Omega$, and let $k$ be a subfield of $\Omega$. A family $(f_\sigma)_{\sigma \in A(\Omega/k)}$ of isomorphisms $f_\sigma \colon \sigma V \to V$ will be called a *descent system* if $f_{\sigma\tau} = f_\sigma \circ \sigma f_\tau$ for all $\sigma, \tau \in A(\Omega/k)$. We say that a model $(V_0, f \colon V_{0,\Omega} \to V)$ of $V$ over $k$ *splits* $(f_\sigma)$ if $f_\sigma = f \circ (\sigma f)^{-1}$ for all $\sigma \in A(\Omega/k)$, and that a descent system is *effective* if it is split by some model over $k$. The next theorem restates results of Weil 1956.

THEOREM 1.1. *Assume that $\Omega$ has infinite transcendence degree over $k$. A descent system $(f_\sigma)_{\sigma \in A(\Omega/k)}$ on a quasiprojective variety $V$ over $\Omega$ is effective if, for some subfield $L$ of $\Omega$ finitely generated over $k$, the descent system $(f_\sigma)_{\sigma \in A(\Omega/L)}$ is effective.*

PROOF. Let $k'$ be the algebraic closure of $k$ in $L$ — then $k'$ is a finite extension of $k$ and $L$ is a regular extension of $k'$. Let $(V_{t'}, f' \colon V_{t',\Omega} \to V)$ be the model of $V$ over $L$ splitting $(f_\sigma)_{\sigma \in A(\Omega/L)}$. Let $t \colon L \to k_t$ be a $k'$-isomorphism from $L$ onto a subfield

$k_t$ of $\Omega$ linearly disjoint from $L$ over $k'$, and let $V_t = V_{t'} \otimes_{L,t} k_t$. Zorn's Lemma allows us to extend $t$ to an automorphism $\tau$ of $\Omega$ over $k'$. The isomorphism

$$f_{t,t'} \colon V_{t',\Omega} \xrightarrow{f'} V \xrightarrow{f_\tau^{-1}} \tau V \xrightarrow{(\tau f')^{-1}} V_{t,\Omega}$$

is independent of the choice of $\tau$, is defined over $L \cdot k_t$, and satisfies the hypothesis of Weil 1956, Theorem 6, which gives a model $(W, f)$ of $V$ over $k'$ splitting $(f_\sigma)_{\sigma \in A(\Omega/k')}$.

For $\sigma \in A(\Omega/k)$, $g_\sigma \overset{\mathrm{df}}{=} f_\sigma \circ \sigma f \colon \sigma W_\Omega \to V$ depends only on $\sigma|k'$. For $k$-homomorphisms $\sigma, \tau \colon k' \to \Omega$, define $f_{\tau,\sigma} = g_\tau^{-1} \circ g_\sigma \colon \sigma W \to \tau W$. Then $f_{\tau,\sigma}$ is defined over the Galois closure of $k'$ in $\Omega$ and the family $(f_{\tau,\sigma})$ satisfies the hypotheses of Weil 1956, Theorem 3, which gives a model of $V$ over $k$ splitting $(f_\sigma)_{\sigma \in A(\Omega/k)}$.  $\square$

COROLLARY 1.2. *Let $\Omega$, $k$, and $V$ be as in the theorem, and let $(f_\sigma)_{\sigma \in A(\Omega/k)}$ be a descent system on $V$. If there is a finite set $\Sigma$ of points in $V(\Omega)$ such that*

(a) *any automorphism of $V$ fixing all $P \in \Sigma$ is the identity map, and*
(b) *there exists a subfield $L$ of $\Omega$ finitely generated over $k$ such that $f_\sigma(\sigma P) = P$ for all $P \in \Sigma$ and all $\sigma \in A(\Omega/L)$,*

*then $(f_\sigma)_{\sigma \in A(\Omega/k)}$ is effective.*

PROOF. After possibly replacing the $L$ in (b) with a larger finitely generated extension of $k$, we may suppose that $V$ has a model $(W, f)$ over $L$ for which the points of $\Sigma$ are rational, i.e., such that for each $P \in \Sigma$, $P = f(P')$ for some $P' \in W(L)$. Now, for each $\sigma \in A(\Omega/L)$, $f_\sigma$ and $f \circ \sigma f^{-1}$ are both isomorphisms $\sigma V \to V$ sending $\sigma P$ to $P$, and so hypothesis (a) implies they are equal. Hence $(f_\sigma)_{\sigma \in A(\Omega/L)}$ is effective, and the theorem applies.  $\square$

REMARK 1.3. (a) It is easy to construct noneffective descent systems. For example, take $\Omega$ to be the algebraic closure of $k$, and let $V$ be a variety $k$. A one-cocycle $h \colon A(\Omega/k) \to \mathrm{Aut}(V_\Omega)$ can be regarded as a descent system — identify $h_\sigma$ with a map $\sigma V_\Omega = V_\Omega \to V_\Omega$. If $h$ is not continuous, for example, if it is a homomorphism into $\mathrm{Aut}(V)$ whose kernel is not open, then the descent system will not be effective.
(b) An example (Dieudonné 1964, p 131) shows that the hypothesis that $V$ be quasiprojective in (1.1) is necessary unless the model $V_0$ is allowed to be an algebraic space in the sense of M. Artin.
(c) Theorem 1.1 and its corollary replace Lemma 3.23 of Milne 1994, which omits the continuity conditions.

**Application to moduli problems.** Suppose we have a contravariant functor $\mathcal{M}$ from the category of algebraic varieties over $\Omega$ to the category of sets, and equivalence relations $\sim$ on each of the sets $\mathcal{M}(T)$ compatible with morphisms. The pair $(\mathcal{M}, \sim)$ is then called a *moduli problem* over $\Omega$. A $t \in T(\Omega)$ defines a map

$$m \mapsto m_t \overset{\mathrm{df}}{=} t^* m \colon \mathcal{M}(T) \to \mathcal{M}(\Omega).$$

A *solution to the moduli problem* is a variety $V$ over $\Omega$ together with an isomorphism $\alpha \colon \mathcal{M}(\Omega)/\sim \to V(\Omega)$ such that:

(a) for all varieties $T$ over $\Omega$ and all $m \in \mathcal{M}(T)$, the map $t \mapsto \alpha(m_t) \colon T(\Omega) \to V(\Omega)$ is regular (i.e., defined by a morphism $T \to V$ of $\Omega$-varieties);

(b) for any variety $W$ over $\Omega$ and map $\beta\colon \mathcal{M}(\Omega)/\!\sim\,\to W(\Omega)$ satisfying the condition (a), the map $P \mapsto \beta(\alpha^{-1}(P))\colon V(\Omega) \to W(\Omega)$ is regular.

Clearly, a solution to a moduli problem is unique up to a unique isomorphism when it exists.

Let $(\mathcal{M}, \sim)$ be a moduli problem over $\Omega$, and let $k$ be a subfield $\Omega$. For $\sigma \in A(\Omega/k)$, define ${}^{\sigma}\mathcal{M}$ to be the functor sending an $\Omega$-variety $T$ to $\mathcal{M}(\sigma^{-1}T)$. We say that $(\mathcal{M}, \sim)$ is *rational over* $k$ if there is given a family $(g_{\sigma})_{\sigma \in A(\Omega/k)}$ of isomorphisms $g_{\sigma}\colon {}^{\sigma}\mathcal{M} \to \mathcal{M}$, compatible with $\sim$, such that $g_{\sigma\tau} = g_{\sigma} \circ \sigma g_{\tau}$ for all $\sigma, \tau \in A(\Omega/k)$ — the last equation means that $g_{\sigma\tau}(T) = g_{\sigma}(T) \circ g_{\tau}(\sigma^{-1}T)$ for all varieties $T$. Note that ${}^{\sigma}\mathcal{M}(\Omega) = \mathcal{M}(\Omega)$, and that the rule $\sigma m = g_{\sigma}(m)$ defines an action of $A(\Omega/k)$ on $\mathcal{M}(\Omega)$. A *solution to a moduli problem* $(\mathcal{M}, \sim, (g_{\sigma}))$ *rational over* $k$ is a variety $V_0$ over $k$ together with an isomorphism $\alpha\colon \mathcal{M}(\Omega)/\!\sim\,\to V_0(\Omega)$ such that

(a) $(V_{0,\Omega}, \alpha)$ is a solution to the moduli problem $(\mathcal{M}, \sim)$ over $\Omega$, and
(b) $\alpha$ commutes with the actions of $A(\Omega/k)$ on $\mathcal{M}(\Omega)$ and $V_0(\Omega)$.

Again, $(V_0, \alpha)$ is uniquely determined up to a unique isomorphism (over $k$) when it exists.

THEOREM 1.4. *Assume that* $\Omega$ *has infinite transcendence degree over* $k$. *Let* $(\mathcal{M}, \sim, (g_{\sigma}))$ *be a moduli problem rational over* $k$ *for which* $(\mathcal{M}, \sim)$ *has a solution* $(V, \alpha)$ *over* $\Omega$. *Then* $(\mathcal{M}, \sim, (g_{\sigma}))$ *has a solution over* $k$ *if there exists a finite subset* $\Sigma \subset \mathcal{M}(\Omega)$ *such that*

(a) *any automorphism of* $V$ *fixing* $\alpha(P)$ *for all* $P \in \Sigma$ *is the identity map, and*
(b) *there exists a subfield* $L$ *of* $\Omega$ *finitely generated over* $k$ *such that* $g_{\sigma}(P) \sim P$ *for all* $P \in \Sigma$ *and all* $\sigma \in A(\Omega/L)$.

PROOF. The family $(g_{\sigma})$ defines a descent system on $V$, which Corollary 1.2 shows to be effective. $\qquad\square$

## 2. DESCENT OF SHIMURA VARIETIES.

In this section, all fields will be subfields of $\mathbb{C}$. For a subfield $E$ of $\mathbb{C}$, $E^{\mathrm{ab}}$ denotes the composite of all the finite abelian extensions of $E$ in $\mathbb{C}$.

Let $(G, X)$ be a pair satisfying the axioms (2.1.1.1–2.1.1.3) of Deligne 1979 to define a Shimura variety, and let $\mathrm{Sh}(G, X)$ be the corresponding Shimura variety over $\mathbb{C}$. We regard $\mathrm{Sh}(G, X)$ as a pro-variety endowed with a continuous action of $G(\mathbb{A}_f)$ — in particular (ibid. 2.7.1) this means that $\mathrm{Sh}(G, X)$ is a projective system of varieties $(\mathrm{Sh}_K(G, X))$ indexed by the compact open subgroups $K$ of $G(\mathbb{A}_f)$. Let $[x, a] = ([x, a]_K)_K$ denote the point in $\mathrm{Sh}(G, X)(\mathbb{C})$ defined by a pair $(x, a) \in X \times G(\mathbb{A}_f)$, and let $E(G, X)$ be the reflex field of $(G, X)$. For a special point $x \in X$, let $E(x) \supset E(G, X)$ be the reflex field for $x$ and let

$$r_x\colon \mathrm{Gal}(E(x)^{\mathrm{ab}}/E(x)) \to T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

be the reciprocity map defined in Milne 1992, p164 (inverse to that in Deligne 1979, 2.2.3). Here $T$ is a subtorus of $G$ such that $\mathrm{Im}(h_x) \subset T_{\mathbb{R}}$ and $T(\mathbb{Q})^-$ is the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$. A *model* of $\mathrm{Sh}(G, X)$ over a field $k$ is a pro-variety $S$ over $k$ endowed with an action of $G(\mathbb{A}_f)$ and a $G(\mathbb{A}_f)$-equivariant isomorphism $f\colon S_{\mathbb{C}} \to \mathrm{Sh}(G, X)$. A model of $\mathrm{Sh}(G, X)$ over $E(G, X)$ is *canonical* if, for each special point $x \in X$ and

$a \in G(\mathbb{A}_f)$, $[x, a]$ is rational over $E(x)^{\mathrm{ab}}$ and $\sigma \in \mathrm{Gal}(E(x)^{\mathrm{ab}}/E(x))$ acts on $[x, a]$ according[1] to the rule:

$$\sigma[x, a] = [x, r_x(\sigma) \cdot a].$$

Let $k$ be a field containing $E(G, X)$. A *descent system* for $\mathrm{Sh}(G, X)$ over $k$ is a family of isomorphisms

$$(f_\sigma \colon \sigma \mathrm{Sh}(G, X) \to \mathrm{Sh}(G, X))_{\sigma \in A(\mathbb{C}/k)}$$

such that,

(a) for all $\sigma, \tau \in A(\mathbb{C}/k)$, $f_{\sigma\tau} = f_\sigma \circ \sigma f_\tau$, and
(b) for all $\sigma \in A(\mathbb{C}/k)$, $f_\sigma$ is equivariant for the actions of $G(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X)$ and $\sigma \mathrm{Sh}(G, X)$.

We say that a model $(S, f)$ of $\mathrm{Sh}(G, X)$ over $k$ *splits* $(f_\sigma)$ if $f_\sigma = f \circ \sigma f^{-1}$ for all $\sigma \in A(\Omega/k)$, and that a descent system if *effective* if it is split by some model over $k$. A descent system $(f_\sigma)$ for $\mathrm{Sh}(G, X)$ over $E(G, X)$ is *canonical* if

$$f_\sigma(\sigma[x, a]) = [x, r_x(\sigma|E(x)^{\mathrm{ab}}) \cdot a]$$

whenever $x$ is a special point of $X$, $\sigma \in A(\mathbb{C}/E(x))$, and $a \in G(\mathbb{A}_f)$.

REMARK 2.1. (a) For a Shimura variety $\mathrm{Sh}(G, X)$, there exists at most one canonical descent system for $\mathrm{Sh}(G, X)$ over $E(G, X)$. (Apply Deligne 1971, 5.1, 5.2.)
(b) Let $(S, f)$ be a model of $\mathrm{Sh}(G, X)$ over $E(G, X)$, and let $f_\sigma = f \circ (\sigma f)^{-1}$. Then $(f_\sigma)_{\sigma \in A(\mathbb{C}/k)}$ is a descent system for $\mathrm{Sh}(G, X)$, and $(f_\sigma)$ is canonical if and only if $(S, f)$ is canonical.
(c) Suppose $\mathrm{Sh}(G, X)$ has a canonical descent system $(f_\sigma)_{\sigma \in A(\mathbb{C}/E(G,X))}$; then $\mathrm{Sh}(G, X)$ has a canonical model if and only if $(f_\sigma)$ is effective. (Follows from (a) and (b).)
(d) A descent system $(f_\sigma)_{\sigma \in A(\mathbb{C}/k)}$ on $\mathrm{Sh}(G, X)$ defines for each compact open subgroup $K$ of $G(\mathbb{A}_f)$ a descent system $(f_{\sigma,K})_{\sigma \in A(\mathbb{C}/k)}$ on the variety $\mathrm{Sh}_K(G, X)$ (in the sense of §1). If $(f_\sigma)$ is effective, then so also is $(f_{\sigma,K})$ for all $K$; conversely, if $(f_{\sigma,K})_{\sigma \in A(\mathbb{C}/k)}$ is effective (in the sense of §1) for all sufficiently small $K$, then $(f_\sigma)_{\sigma \in A(\mathbb{C}/k)}$ is effective (in the sense of this section).

LEMMA 2.2. *The automorphism group of the quotient of a bounded symmetric domain by a neat arithmetic group is finite.*

PROOF. According to Mumford 1977, Proposition 4.2, such a quotient is an algebraic variety of logarithmic general type, which implies that its automorphism group is finite (Iitaka 1982, 11.12).

Alternatively, one sees easily that the automorphism group of the quotient of a bounded symmetric domain $D$ by a neat arithmetic subgroup $\Gamma$ is $N/\Gamma$ where $N$ is the normalizer of $\Gamma$ in $\mathrm{Aut}(D)$. Now $N$ is countable and closed (because $\Gamma$ is closed), and hence is discrete (Baire category theorem). Because the quotient of $\mathrm{Aut}(D)$ by $\Gamma$ has finite measure, this implies that $\Gamma$ has finite index in $N$. Cf. Margulis 1991, II 6.3. □

THEOREM 2.3. *Every canonical descent system on a Shimura variety is effective.*

[1]More precisely, the condition for $(S, f)$ to be canonical is the following: if $P \in S(\mathbb{C})$ corresponds under $f$ to $[x, a]$, then $\sigma P$ corresponds under $f$ to $[x, r_x(\sigma) \cdot a]$.

PROOF. Let $(f_\sigma)_{\sigma\in A(\mathbb{C}/E(G,X))}$ be a canonical descent system for the Shimura variety $\mathrm{Sh}(G,X)$. Let $K$ be a compact open subgroup of $G(\mathbb{A}_f)$, chosen so small that the connected components of $\mathrm{Sh}_K(G,X)$ are quotients of bounded symmetric domains by *neat* arithmetic groups. Let $x$ be a special point of $X$. According to Deligne 1971, 5.2, the set $\Sigma = \{[x,a]_K \mid a \in G(\mathbb{A}_f)\}$ is Zariski dense in $\mathrm{Sh}_K(G,X)$. Because the automorphism group of $\mathrm{Sh}_K(G,X)$ is finite, there is a finite subset $\Sigma_f$ of $\Sigma$ such that any automorphism $\alpha$ of $\mathrm{Sh}_K(G,X)$ fixing each $P \in \Sigma_f$ is the identity map.

The rule

$$\sigma * [x,a]_K = [x, r_x(\sigma) \cdot a]_K$$

defines an action of $\mathrm{Gal}(E(x)^{\mathrm{ab}}/E(x))$ on $\Sigma$ for which the stabilizer of each point of $\Sigma$ is open. Therefore, there exists a finite abelian extension $L$ of $E(x)$ such that $\sigma * P = P$ for all $P \in \Sigma_f$ and all $\sigma \in \mathrm{Gal}(E(x)^{\mathrm{ab}}/L)$.

Now, because $(f_\sigma)_{\sigma\in A(\Omega/E(G,X))}$ is canonical, $f_{\sigma,K}(\sigma P) = P$ for all $P \in \Sigma_f$ and all $\sigma \in A(\mathbb{C}/L)$, and we may apply Corollary 1.2 to conclude that $(f_{\sigma,K})_{\sigma\in A(\mathbb{C}/E(G,X))}$ is effective. As this holds for all sufficiently small $K$, $(f_\sigma)_{\sigma\in A(\mathbb{C}/E(G,X))}$ is effective. □

REMARK 2.4. (a) If Langlands's Conjugacy Conjecture (Langlands 1979, p232, 233) is true for a Shimura variety $\mathrm{Sh}(G,X)$, then $\mathrm{Sh}(G,X)$ has a canonical descent system (ibid. §6; also Milne and Shih 1982, §7).
(b) Langlands's Conjugacy Conjecture is true for all Shimura varieties (Milne 1983). Hence canonical models exist for all Shimura varieties.

Another proof, based on different ideas, that the descent maps given by Langlands's conjecture are effective can be found in Moonen 1998. (I thank the referee for this reference.)

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