Compositio Mathematica 68: 59–102 (1988)

Motivic cohomology and values of zeta functions

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Received 27 October 1987; accepted 24 March 1988

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Introduction

In this paper we provide some complements to the paper [Lichtenbaum (1984)], which introduced the complexes $\mathbf{Z}(r)$ for the étale topology, and to [Milne (1986a)], which investigated the behaviour near integers of the zeta functions of projective varieties over finite fields.

Let X be a regular Noetherian scheme, and let r be a nonnegative integer. In the paper just cited, Lichtenbaum postulates the existence of a complex $\mathbf{Z}(r)$ of étale sheaves on X satisfying certain axioms. These axioms are reviewed in §1 below. The étale cohomology groups $H^i(X, \mathbf{Z}(r))$ are called the *motivic cohomology groups* of X. We note that $\mathbf{Z}(0) = \mathbf{Z}$ and $\mathbf{Z}(1) = \mathbf{G}_m[-1]$, that a compelling candidate for $\mathbf{Z}(2)$ is offered in [Lichtenbaum (1987a)], and that various definitions of $\mathbf{Z}(r)$ for $r > 2$ are suggested in [Bloch (1986)] and [Beilinson, MacPherson, and Schechtman (1987)].

One of Lichtenbaum's axioms for $\mathbf{Z}(r)$ relates its cohomology to the l-adic étale cohomology groups for l a prime different from the characteristic. In

[1] This work has been partially supported by the National Science Foundation; it was begun while the author was at MSRI.

§2 we consider an additional axiom, a "p-Kummer sequence", originally introduced in [Milne (1986a)], that relates the motivic cohomology groups to the p-adic cohomology groups, where p is the characteristic of X. We check that the axiom is consistent with the previous axioms and other known results. The complexes $\mathbf{Z}(0)$ and $\mathbf{Z}(1)$ obviously satisfy the axiom, and a recent theorem of Merkur'ev and Suslin allows it to be verified for $\mathbf{Z}(2)$.

Now assume that X is smooth of finite type over the ring of integers in a number field or an s-local field, $s \geqslant 0$. One of the axioms for the complexes $\mathbf{Z}(r)$ asserts that there are natural pairings

$$\mathbf{Z}(r) \otimes^{\mathrm{L}} \mathbf{Z}(r') \to \mathbf{Z}(r + r'),$$

and Lichtenbaum conjectures that these lead to nondegenerate pairings on the cohomology groups. In §3 we prove two results in the direction of this conjecture.

Next assume that X is a smooth projective variety over a finite field k. Theorems of Grothendieck and Deligne show that the zeta function of X takes the form

$$\zeta(X, s) = \frac{P_1(X, q^{-s}) \cdots P_{2d-1}(X, q^{-s})}{P_0(X, q^{-s})P_2(X, q^{-s}) \cdots P_{2d}(X, q^{-s})}$$

where $P_i(X, t) \in \mathbf{Z}[t]$ and has reciprocal roots of absolute value $q^{i/2}$. In particular, the order ϱ_r of the pole of $\zeta(X, s)$ at $s = r$ is the multiplicity of q^r as a reciprocal root of $P_{2r}(X, t)$. Let $\varrho'_{r,l}$ be the dimension of the subspace of $H^{2r}(\bar{X}, \mathbf{Q}_l(r))$ generated by algebraic cycles on X, and let $\varrho'_r = \max \varrho'_{r,l}$. It is known that $\varrho'_r \leqslant \varrho_r$, and so $\lim_{s\to r} \zeta(X, s)(1 - q^{s-r})^{\varrho'_r}$ is either a non-zero rational number or infinity; we denote it by $c_X(r)$. Define

$$\chi(X, \mathbf{Z}(r)) =$$

$$\prod_{i \neq 2r, 2r+2} [H^i(X, \mathbf{Z}(r))]^{(-1)^i}[H^{2r}(X, \mathbf{Z}(r))_{\mathrm{tors}}][H^{2r+2}(X, \mathbf{Z}(r))_{\mathrm{cotors}}]/R$$

when each of the terms on the right is finite. Here [$*$] denotes the order of $*$, and R is a certain regulator term (see §4, §5). Define

$$\chi(X, \mathcal{O}, r) = r\chi(X, \mathcal{O}_X) - (r - 1)\chi(X, \Omega^1_X) + \cdots \pm \chi(X, \Omega^{r-1}_X).$$

CONJECTURE 0.1.[2] *If $c_X(r) \neq \infty$, then $\chi(X, \mathbf{Z}(r))$ is defined, and*

$$c_X(r) = \pm\chi(X, \mathbf{Z}(r)) \cdot q^{\chi(X, \mathcal{O}, r)}.$$

The condition $c_X(r) \neq \infty$ is implied by Tate's conjecture on the existence of algebraic cycles (see §4). The conjecture was proved for $r = 0$ and 1 in [Milne (1986a)]. In §4 we prove that a slightly weakened form of the conjecture holds whenever a complex $\mathbf{Z}(r)$ exists satisfying certain of the axioms. Note that, for $r > \dim X$, $\zeta(X, s)$ does not have a pole at $s = r$, and so the conjecture simply states that

$$\chi(X, \mathbf{Z}(r)) = \zeta(X, r)q^{-\chi(X, \mathcal{O}, r)}.$$

In §5 we derive various alternative expressions for the regulator term R.

Now suppose that X is of even dimension $d = 2r$, and let $CH^r(X)$ be the Chow group of algebraic cycles of codimension r on X modulo rational equivalence. One of the axioms for $\mathbf{Z}(r)$ asserts that there is a cycle class map $CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$. In §6 we show that, whenever there exists a complex $\mathbf{Z}(r)$ satisfying certain conditions and the cycle map $CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$ is surjective, then Tate's conjecture implies that

$$P_{2r}(X, q^{-s}) \sim \pm \frac{[Br^r(X)] \det (D_i \cdot D_j)}{q^{\alpha_r(X)}[A^r(X)_{\text{tors}}]^2}(1 - q^{r-s})^{\rho_r} \quad \text{as } s \to r.$$

Here $Br^r(X) = H^{2r+1}(X, \mathbf{Z}(r))$, $A^r(X)$ is the image of $CH^r(X)$ in $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))$, $\{D_i\}$ is a basis for $A^r(X)$ modulo torsion, and $\alpha_r(X)$ is a certain integer defined in §6. In the case of a surface, this becomes the main theorem of [Milne (1975)] (then $\mathbf{Z}(1) = \mathbf{G}_m[-1]$, $Br(X)$ is the Brauer group, the cycle map is the isomorphism $CH^1(X) \to \operatorname{Pic}(X)$, and $A^1(X)$ is the Néron–Severi group of X).

Two of the axioms for $\mathbf{Z}(r)$ relate it to the K-sheaves. In §7 we exploit this relation and some results of [Quillen (1972)] and [Coombes (1987a)] to obtain further evidence for Conjecture 0.1.

Finally, I note that (happily) most of the hard work required for proving these results has already been carried out in [Milne (1986a)], and that previous results along these lines have been obtained by [Bayer and Neukirch (1978)],

[2] For X a surface and $r = 1$, this is essentially the conjecture of Artin and Tate [Tate (1965/66), Conjecture C]; for a general X and $r = 1$, it was stated in [Lichtenbaum (1983)]; for a general X and r, the term $\chi(X, \mathbf{Z}(r))$ was suggested by Lichtenbaum and the term $q^{\chi(X, \mathcal{O}, r)}$ by the author.

[Zarhin (1982)], [Schneider (1982)], and [Lichtenbaum (1983)]. Also I would like to thank S. Lichtenbaum for many conversations on these questions, and in particular for alerting me to [Merkur'ev and Suslin (1987)].

Notations and conventions

For M an abelian group, we use the following notations: M_m and $M^{(m)}$ are the kernel and cokernel respectively of m: $M \to M$; $\hat{M}$ is the profinite completion of M, $\hat{M} = \varprojlim M^{(m)}$ (limit over all integers m); M_{tors} is the torsion subgroup of M, $M_{\text{tors}} = \bigcup M_m$; M_{div} is the maximum divisible subgroup of M; and $M' = M/M_{\text{div}}$. Because M_{div} is a direct summand of M, $M'_{\text{tors}} \overset{\text{df}}{=} (M')_{\text{tors}} = (M_{\text{tors}})'$. When M is torsion, M' is also written M_{cotors}. Finally, $TM = \varprojlim M_m = \text{Hom}(\mathbf{Q}/\mathbf{Z}, M)$.

For a map α: $M \to N$ of abelian groups, we write

$$z(\alpha) = [\text{Ker}(\alpha)]/[\text{Coker}(\alpha)]$$

when both orders are finite. We define det (α) as in [Milne (1986a)]; thus

$$z(\alpha) = [M_{\text{tors}}]/[N_{\text{tors}}] \det(\alpha),$$

when all the terms are finite.

When X is a variety over a finite field k, $\bar{k}$ denotes the algebraic closure of k and $\bar{X} = X \otimes_k \bar{k}$.

Unless indicated otherwise, all cohomology groups are relative to the étale topology.

§1. Statement of the axioms

Let X be a regular Noetherian scheme, and let $\mathscr{S}(X_{\text{et}})$ be the category of étale sheaves on X. In [Lichtenbaum (1984), (1987b)] it is conjectured that there exist objects $\mathbf{Z}(r) = \mathbf{Z}(r)_X$ in the derived category of $\mathscr{S}(X_{\text{et}})$, commuting with restriction, and satisfying the following axioms.

(A0) $\mathbf{Z}(0) = \mathbf{Z}$; $\mathbf{Z}(1) = \mathbf{G}_m[-1]$; $\mathbf{Z}(2)$ is represented by the complex studied in [Lichtenbaum (1987a)].

(A1) For $r \neq 0$, $\mathbf{Z}(r)$ is acyclic outside $[1, r]$; i.e., $H^i(\mathbf{Z}(r)) = 0$ unless $1 \leqslant i \leqslant r$ or $r = 0$.

$(\text{A2})_m$ (Kummer m-sequence). For any integer m that is invertible on X, there is an exact triangle

$$\mathbf{Z}(r) \xrightarrow{m} \mathbf{Z}(r) \longrightarrow \mu_m^{\otimes r} \longrightarrow \mathbf{Z}(r)[1].$$

Note that $(A2)_m$ implies that there is an exact sequence of cohomology groups

$$\cdots \longrightarrow H^i(X, \mathbf{Z}(r)) \xrightarrow{m} H^i(X, \mathbf{Z}(r)) \longrightarrow H^i(X, \mu_m^{\otimes r})$$
$$\longrightarrow H^{i+1}(X, \mathbf{Z}(r)) \longrightarrow \cdots \tag{1.0.1}$$

(A3) There are natural pairings

$$\mathbf{Z}(r) \otimes^{\mathrm{L}} \mathbf{Z}(s) \to \mathbf{Z}(r+s).$$

These pairings should be compatible through $(A2)_m$ with the natural pairing

$$\mu_m^{\otimes r} \times \mu_m^{\otimes s} \to \mu_m^{\otimes r+s}.$$

(A4) There exist natural cycle maps $CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$. These cycle maps should be compatible through the maps in (1.0.1) with the cycle maps for the étale topology; they should also be compatible with the product structure in (A3).

(A5) (Hilbert Theorem 90). Let $f\colon X_{\mathrm{et}} \to X_{\mathrm{Zar}}$ be the morphism of sites defined by the indentity map. Then $R^{r+1}f_*\mathbf{Z}(r) = 0$. In particular, for a field F, $H^{r+1}(F, \mathbf{Z}(r)) = 0$.

The next two axioms relate the complexes $\mathbf{Z}(r)$ to the K-sheaves. Let $\mathscr{K}_r^M$ and $\mathscr{K}_r$ respectively be the sheaves on X_{et} associated with the presheaves,

$$U \mapsto K_r^M\Gamma(U, \mathcal{O}_X) \quad \text{(Milnor } K\text{-group)}$$

$$U \mapsto K_r\Gamma(U, \mathcal{O}_X) \quad \text{(Quillen } K\text{-group)}.$$

The stalk of $\mathscr{K}_r$ at a geometric point $\bar{x}$ of X is $K_r\mathcal{O}_{\bar{x}}$, and we define $F_\gamma^i\mathscr{K}_r$ to be the subsheaf of $\mathscr{K}_r$ with stalks $F_\gamma^iK_r\mathcal{O}_{\bar{x}}$ (ith filtered part for the γ-filtration, see [Soulé (1985)]). Finally, we set $gr_\gamma^i\mathscr{K}_r = F_\gamma^i\mathscr{K}_r/F_\gamma^{i+1}\mathscr{K}_r$.

(B1) The sheaf $R^rf_*\mathbf{Z}(r) = \mathscr{K}_r^M$. In particular, for a field F, $H^r(F, \mathbf{Z}(r)) = K_r^MF$.

(B2) The sheaf $H^i(\mathbf{Z}(r)) = gr_\gamma^r\mathscr{K}_{2r-i}$ up to torsion involving primes $<r$.

Note that Axiom B2 implies that there is a spectral sequence

$$H^i(X, gr_\gamma^r\mathscr{K}_{2r-j}) \Rightarrow H^{i+j}(X, \mathbf{Z}(r))$$

(up to torsion involving primes $<r$).

For a complex $A^\cdot$, we let $\tau_{\leqslant r}A^\cdot$ denote the truncated complex

$$\cdots \to A^{r-1} \to \operatorname{Ker}(d^r) \to 0 \to \cdots$$

Thus $H^i(\tau_{\leqslant r}A^\cdot) = H^i(A^\cdot)$ for $i \leqslant r$, and $H^i(\tau_{\leqslant r}A^\cdot) = 0$ for $i > r$. The functor $\tau_{\leqslant r}$ passes to the derived category.

The next two axioms are more tentative than the above.

(C1) (Gersten sequence). There is an exact sequence (in the derived category of sheaves on X_{et}):

$$0 \to \mathbf{Z}(r) \to \bigoplus_{x \in X_0} \tau_{\leqslant r} R\iota_* \mathbf{Z}(r)_{\kappa(x)} \to \bigoplus_{x \in X_1} (\tau_{\leqslant r-1} R\iota_* \mathbf{Z}(r-1)_{\kappa(x)})[-1] \to \cdots$$

Here X_c denotes the set of points x of codimension c, and ι denotes the inclusion map $x \hookrightarrow X$.

(C2) (Purity). Let $i\colon Z \hookrightarrow X$ be a closed immersion of a regular scheme Z into a regular scheme, and suppose that Z is of codimension c in X at each point of Z; then

$$\tau_{\leqslant r+c} Ri^! \mathbf{Z}(r) \xrightarrow{\approx} \mathbf{Z}(r-c)[-2c].$$

REMARK 1.1. (a) There seems to be no standard definition of exactness in a triangulated category. For the purposes of this paper, we adopt the following definition: a sequence

$$0 \to A \to B \to C \to 0$$

is *exact* if there is an exact triangle

$$A \to B \to C \to A[1],$$

and a sequence

$$0 \to A \to B \to C \to \cdots$$

is *exact* if there exist exact sequences

$$0 \to A \to B \to K_1 \to 0$$

$$0 \to K_1 \to C \to K_2 \to 0$$

$$\cdots$$

(b) The sequence in Axiom C1 gives rise to a map

$$\bigoplus_{x \in X_r} \mathbf{Z} = H^0\Big(X, \bigoplus_{x \in X_r} \iota_* \mathbf{Z}\Big) = H^r\Big(X, \bigoplus_{x \in X_r} \iota_* \mathbf{Z}[-r]\Big) \to H^{2r}(X, \mathbf{Z}(r))$$

factoring through $CH^r(X)$; it should be the cycle map in (A4).

(c) Axiom C1 implies that $H^i(X, \mathbf{Z}(r))$ is torsion for $i > 2r$. This can be proved by a standard argument using that all Galois cohomology groups except the zeroth are torsion. Note however that $H^{2r}(X, \mathbf{Z}(r))$ will not in general be torsion; for example, $H^0(X, \mathbf{Z}(0)) = \mathbf{Z}$ and $H^2(X, \mathbf{Z}(1)) = \operatorname{Pic}(X)$.

(d) When $c = 0$, Axiom C2 asserts that $\tau_{\leqslant r}\mathbf{Z}(r) = \mathbf{Z}(r)$, which is part of Axiom A1. When $c > 0$, Axiom C2 can be restated in terms of the open immersion $j\colon U \hookrightarrow X$, $U = X - Z$. As for any complex of sheaves on X, there is an exact triangle (cf. [Milne (1980), p. 242])

$$i_*Ri^!\mathbf{Z}(r)_X \to \mathbf{Z}(r)_X \to Rj_*\mathbf{Z}(r)_U \to i_*Ri^!\mathbf{Z}(r)_X[1]$$

or

$$\mathbf{Z}(r)_X \to Rj_*\mathbf{Z}(r)_U \to i_*Ri^!\mathbf{Z}(r)_X[1] \to \mathbf{Z}(r)_X[1].$$

As $H^{r+c}(\mathbf{Z}(r)) = 0$, this triangle remains exact after truncation:

$$\mathbf{Z}(r)_X \to \tau_{\leqslant r+c-1}Rj_*\mathbf{Z}(r)_U \to (\tau_{\leqslant r+c}i_*Ri^!\mathbf{Z}(r))[1] \to \mathbf{Z}(r)_X[1].$$

Therefore, Axiom C2 implies that there is an exact triangle

$$\mathbf{Z}(r)_X \to \tau_{\leqslant r+c-1}Rj_*\mathbf{Z}(r)_U \to i_*\mathbf{Z}(r-c)_Z[1-2c] \to \mathbf{Z}(r)_X[1] \qquad (1.1.1)$$

Conversely, the existence of such an exact triangle implies Axiom C2.

EXAMPLES 1.2. (a) For $r = 0$, Axiom C1 asserts the existence of an isomorphism

$$\mathbf{Z} \xrightarrow{\approx} \bigoplus_{x \in X_0} \iota_*\mathbf{Z}_{\kappa(x)}.$$

For $r = 1$, it asserts the existence of an exact sequence

$$0 \to \mathbf{G}_m \to \bigoplus_{x \in X_0} \iota_*\mathbf{G}_{m,\kappa(x)} \to \bigoplus_{x \in X_1} \iota_*\mathbf{Z}_{\kappa(x)} \to 0.$$

Both of these are known (see [Milne (1980), II.3.7, II.3.9]).

(b) For $r = 0$, Axiom C2 asserts that $R^s i^!\mathbf{Z} = 0$ for $0 \leqslant s \leqslant c$, $c > 0$. For $r = 1$, it asserts that

$$\begin{cases} i^!\mathbf{G}_m = 0, \quad R^1 i^!\mathbf{G} \xrightarrow{\approx} \mathbf{Z}, & \text{if } c = 1 \\ R^s i^!\mathbf{G}_m = 0 \quad \text{for } s \leqslant c, & \text{if } c > 1. \end{cases}$$

(c) For $r = 1$, $c = 1$, the variant (1.1.1) of Axiom C2 asserts that there is an exact sequence

$$0 \to \mathbf{G}_m \to j_*\mathbf{G}_{m,U} \to i_*\mathbf{Z} \to 0.$$

The next proposition summarizes what is known about the axioms when $r \leqslant 2$.

PROPOSITION 1.3. (a) *The objects* $\mathbf{Z}(0)$ *and* $\mathbf{Z}(1)$ *satisfy the Axioms* A, B, *and* C1; *when X and Z are smooth of finite type over a field of characteristic* $p \geqslant 0$, *they satisfy Axiom* C2 *up to p-torsion.*

(b) *The object* $\mathbf{Z}(2)$ *satisfies Axioms* A1 *and* A3; *when X is of finite type over a field, it satisfies Axiom* A2 *up to 2-torsion, Axiom* A5 *up to 2- and p-torsion, and Axiom* B2 *up to 2-torsion; when X is the spectrum of a field, it satisfies Axiom* B1 *up to p-torsion.*

Proof. (a) Except for (C2), the axioms are either obvious or follow from standard results in étale cohomology (for (C1), this has already been noted in (1.2a) above). The next lemma is a little stronger than (C2).

LEMMA 1.4. *Let* $i\colon Z \hookrightarrow X$ *be as in the statement of Axiom* C2, *but with X and Z smooth over a field.*

(a) *The complex* $\tau_{\leqslant 2c} Ri^!\mathbf{Z} = 0$ (*up to p-torsion*); *equivalently,* $\mathbf{Z} \xrightarrow{\approx} \tau_{\leqslant 2c-1} Rj_*\mathbf{Z}$ (*up to p-torsion*).

(b) *When* $c = 1$, *there is a canonical isomorphism* $Ri^!\mathbf{G}_m \xrightarrow{\approx} \mathbf{Z}[-1]$ (*up to p-torsion*); *for all* $c > 1$, $\tau_{\leqslant 2c-2} Ri^!\mathbf{G}_m = 0$ (*up to p-torsion*).

Proof. We shall need to use the purity theorem in étale cohomology. Recall [Milne (1980), VI.5, VI.6] that this states that, under the hypotheses of the lemma, $Ri^!\mu_m^{\otimes r}$ is canonically isomorphic to $\mu_m^{\otimes r-c}[-2c]$ provided m is prime to the characteristic of k; equivalently,

$$j_*\mu_m^{\otimes r} \xrightarrow{\approx} \mu_m^{\otimes r}, \quad R^{2c-1}j_*\mu_m^{\otimes r} \xrightarrow{\approx} i_*\mu_m^{\otimes r-c}, \quad R^s j_*\mu_m^{\otimes r} = 0 \quad \text{for } s \neq 0, 2c-1.$$

We now prove (a). It is obvious that $j_*\mathbf{Z} = \mathbf{Z}$, and that $R^s j_*\mathbf{Z}$ is torsion for $s > 0$ (cf. 1.1c above). The statement now follows from the exact sequences

$$\cdots \longrightarrow R^s j_*\mathbf{Z} \xrightarrow{m} R^s j_*\mathbf{Z} \longrightarrow R^s j_*(\mathbf{Z}/m\mathbf{Z}) \longrightarrow \cdots,$$

m prime to p, and the purity theorem for $\mathbf{Z}/m\mathbf{Z}$.

We now prove (b). First assume $c = 1$. It is obvious that there is an exact sequence

$$0 \to \mathbf{G}_m \to j_*\mathbf{G}_{m,U} \xrightarrow{\mathrm{ord}_Z} i_*\mathbf{Z} \to 0.$$

On comparing this with

$$0 \to i_* i^! \mathbf{G}_m \to \mathbf{G}_m \to j_* \mathbf{G}_{m,U} \to i_* R^1 i^! \mathbf{G}_m \to 0 \to \cdots$$

we see that $i^! \mathbf{G}_m = 0$ and $R^1 i^! \mathbf{G}_m \xrightarrow{\approx} \mathbf{Z}$.

For $s > 1$, $R^{s-1} j_* \mathbf{G}_m \xrightarrow{\approx} R^s i^! \mathbf{G}_m$, and so we must show that $R^s j_* \mathbf{G}_m = 0$ (up to p-torsion) for $s > 0$. But $R^1 j_* \mathbf{G}_m$ is the sheaf associated with the presheaf

$$V \mapsto \operatorname{Pic}(U \times_X V)$$

on X_{et}, and the open immersion $U \times_X V \hookrightarrow V$ induces a surjection $\operatorname{Pic}(V) \to \operatorname{Pic}(U \times_X V)$ (this is obvious from the point of view of Weil divisors). Any $\xi \in \operatorname{Pic}(V)$ becomes trivial on an open affine covering (V_i) of V, and so its image in $\operatorname{Pic}(U \times_X V)$ becomes trivial on the covering $(U \times_X V_i)$ of $U \times_X V$. Hence $R^1 j_* \mathbf{G}_m = 0$. For $s > 1$, $R^s j_* \mathbf{G}_m$ is torsion (because $H^s(U \times_X V, \mathbf{G}_m)$ is torsion; see (1.1c)), and so the exact sequence

$$\cdots \longrightarrow R^s j_* \mu_m \longrightarrow R^s j_* \mathbf{G}_m \xrightarrow{m} R^s j_* \mathbf{G}_m \longrightarrow \cdots$$

and the purity theorem for μ_m show that $R^s j_* \mathbf{G}_m$ is zero up to p-torsion, all $s > 1$.

Finally assume that $c > 1$. Let $z \in Z$. In some open neighbourhood X' of z in X, there will exist sections $f_1, \ldots, f_c \in \Gamma(X', \mathcal{O}_X)$ that generate the ideal of $Z \cap X'$ and which form part of a regular system of parameters at each point of X'. In particular, after replacing X by X', we can assume that i factors as $i = i_1 \circ i_0$ where i_1 and i_0 satisfy the conditions of the lemma but have codimensions 1 and $c - 1$. From the preceding paragraph we know that $Ri_1^! \mathbf{G}_m \xrightarrow{\approx} \mathbf{Z}[-1]$ (up to p-torsion). From (a) we know $\tau_{\leqslant 2c-1}(Ri_0^! \mathbf{Z}[-1]) = 0$ (up to p-torsion). Since $Ri_0^! \circ Ri_1^! = Ri^!$, this completes the proof of (b).

We now verify part (b) of the proposition. Axiom A1 is obviously true from the definition of $\mathbf{Z}(2)$ in [Lichtenbaum (1987a)]; Axiom A2 is proved in [ib., 8.3], Axiom A3 in [ib., 2.5], Axiom A5 in [ib., 9.7], Axiom B1 in [ib., 4.5], and Axiom B2 in [ib., 8.4].

REMARK 1.5. (a) We assumed that X is smooth over a field in (1.3a) only because we wished to use the form of the purity conjecture for étale cohomology proved in [Milne (1980)]. Recall [Grothendieck (1977), I.3.1.4] that this conjecture states that $Ri^! \mu_m(r) \xrightarrow{\approx} \mu_m(r - c)[-2c]$ whenever

$i\colon Z \hookrightarrow X$ satisfies the conditions in (C1) and m is invertible on X. For a list of situations where the conjecture is known, see [Thomason (1984), p. 398].

(b) Let Z be a closed subscheme of codimension 1 of X. When X has dimension 1 and its residue fields are perfect, then it is possible to prove that $Ri^!\mathbf{G}_m = \mathbf{Z}[-1]$ (including the p-part); see for example [Milne (1986b), 1.7b]. In contrast, when X has dimension 2, $Ri^!\mathbf{G}_m \neq \mathbf{Z}[-1]$; see (2.4) below.

CONJECTURE 1.6. [Lichtenbaum (1984), §7] *If X is a complete smooth variety over a finite field, then $H^i(X, \mathbf{Z}(r))$ is torsion for $i \neq 2r$, and $H^i(X, \mathbf{Z}(r))$ is finitely generated for $i \neq 2r+1, 2r+2$; therefore $H^i(X, \mathbf{Z}(r))$ is finite for $i \neq 2r, 2r+1, 2r+2$.*

It is also conjectured that $H^{2r+1}(X, \mathbf{Z}(r))$ is finite, but this is expected to lie much deeper than the above statements because it implies Tate's conjecture; see §4.

§2. An additional axiom: the Kummer p-sequence

Now assume X is a smooth scheme of finite type over a perfect field k of characteristic $p \neq 0$. Let $(W_n\Omega^{\cdot}_{X/k})_{n\geqslant 1}$ be the de Rham–Witt pro-complex of Bloch–Deligne–Illusie [Illusie (1979)]. As usual, we define $\nu_n(r)$ to be the additive subsheaf of $W_n\Omega^r_{X/k}$ on X_{et} generated by 1 if $r = 0$ and by the differentials of the form

$$d\mathbf{f}_1/\mathbf{f}_1 \wedge \cdots \wedge d\mathbf{f}_r/\mathbf{f}_r, \quad \mathbf{f}_i = (f_i, 0, \ldots, 0), f_i \text{ a section of } \mathcal{O}_X,$$

if $r > 0$, and we write

$$H^i(X, (\mathbf{Z}/p^n\mathbf{Z})(r)) = H^{i-r}(X, \nu_n(r)),$$

$$H^i(X, \mathbf{Z}_p(r)) = \varprojlim_n H^{i-r}(X, \nu_n(r)).$$

In view of the philosophy embodied in the above notation (see [Milne (1982), §7]), it is natural to adjoin the following axiom to those in the last section.

$(\text{A2})_p$ (Kummer p-sequence). For all integers n, there is an exact triangle

$$\mathbf{Z}(r) \xrightarrow{p^n} \mathbf{Z}(r) \longrightarrow \nu_n(r)[-r] \longrightarrow \mathbf{Z}(r)[1].$$

Note that the axiom gives an exact sequence

$$\cdots \longrightarrow H^i(X, \mathbf{Z}(r)) \xrightarrow{p^n} H^i(X, \mathbf{Z}(r)) \longrightarrow H^{i-r}(X, \nu_n(r)) \longrightarrow \cdots$$

or

$$\cdots \longrightarrow H^i(X, \mathbf{Z}(r)) \xrightarrow{p^n} H^i(X, \mathbf{Z}(r)) \longrightarrow H^i(X, (\mathbf{Z}/p^n\mathbf{Z})(r)) \longrightarrow \cdots$$

We write (A2) for the conjunction of $(\mathrm{A2})_m$ with $(\mathrm{A2})_p$; it implies that there exists an exact sequence

$$\cdots \longrightarrow H^i(X, \mathbf{Z}(r)) \xrightarrow{m} H^i(X, \mathbf{Z}(r)) \longrightarrow H^i(X, (\mathbf{Z}/m\mathbf{Z})(r)) \longrightarrow \cdots \tag{2.0.1}$$

where m is now allowed to be any integer and $H^i(X, (\mathbf{Z}/m\mathbf{Z})(r))$ is defined to be

$$H^i(X, \mu_{m_0}^{\otimes r}) \times H^i(X, (\mathbf{Z}/p^n\mathbf{Z})(r)) \quad m = m_0 p^n, \quad (m_0, p) = 1.$$

Write

$$H^i(X, \hat{\mathbf{Z}}(r)) \;=\; \varprojlim H^i(X, (\mathbf{Z}/m\mathbf{Z})(r))$$

and

$$H^i(X, (\mathbf{Q}/\mathbf{Z})(r)) \;=\; \varinjlim H^i(X, (\mathbf{Z}/m\mathbf{Z})(r)).$$

In the remainder of this section, we study the compatibility of $(\mathrm{A2})_p$ with the other axioms. Our first result demonstrates its compatibility with Axiom A0.

PROPOSITION 2.1. *Axiom* $(\mathrm{A2})_p$ *is satisfied by* $\mathbf{Z}(r)$ *for* $r \leqslant 2$, *except possibly when* $p = 2$ *and* $r = 2$.

Proof. Clearly $\nu_n(0)$ is the subsheaf $\mathbf{Z}/p^n\mathbf{Z}$ of $W_n\mathcal{O}_X$. The exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{p^n} \mathbf{Z} \longrightarrow \mathbf{Z}/p^n\mathbf{Z} \longrightarrow 0$$

can be written

$$0 \longrightarrow \mathbf{Z}(0) \xrightarrow{p^n} \mathbf{Z}(0) \longrightarrow \nu_n(0) \longrightarrow 0.$$

For $r = 1$, we have to show there is an exact sequence

$$0 \longrightarrow \mathbf{G}_m \xrightarrow{p^n} \mathbf{G}_m \longrightarrow \nu_n(1) \longrightarrow 0,$$

but this follows easily from the definition of $\nu_n(1)$ (the map $\mathbf{G}_m \to \nu_n(1)$ sends g to $\mathrm{dlog}\,(g) \stackrel{\mathrm{df}}{=} \mathrm{d}g/g$).

Recall that the complex $C^{\cdot}$ representing $\mathbf{Z}(2)$ is of length two, $C^{\cdot} = C^1 \xrightarrow{d} C^2$, and that the kernel and cokernal of d are $gr_\gamma^2\mathscr{K}_3$ and $gr_\gamma^2\mathscr{K}_2 = \mathscr{K}_2$ respectively (up to 2-torsion) (see [Lichtenbaum (1987a), 8.4]). Thus the statement for $r = 2$ is a consequence of the next lemma.

LEMMA 2.2. (a) *The sheaf* $gr_\gamma^2\mathscr{K}_3$ *is uniquely divisible by* p *provided* $p \neq 2$.
(b) *There is an exact sequence*

$$0 \longrightarrow \mathscr{K}_2 \xrightarrow{p^n} \mathscr{K}_2 \xrightarrow{\mathrm{d}\log^2} \nu_n(2) \longrightarrow 0.$$

Proof. (a) Let $F = k(X)$. It follows ffrom [Soulé (1985), Thm 5] that, up to 2-torsion, there is an exact sequence of sheaves on X,

$$0 \to gr_\gamma^2\mathscr{K}_3 \to \iota_* gr_\gamma^2 K_3 F \to \bigoplus_{x \in X_1} \iota_* gr_\gamma^2 K_2(k(x)) \to \cdots$$

Since $gr_\gamma^1 K_2(k(x)) = 0$ [ib., Pptn 1], $gr_\gamma^2\mathscr{K}_3 \xrightarrow{\approx} \iota_* gr_\gamma^2 K_3 F$ (up to 2-torsion). But $gr_2^\gamma K_3 F$ is the indecomposable part $K_3F/K_3^M F$ of $K_3 F$, and this group is shown to be uniquely divisible by p in [Merkur'ev and Suslin (1987), 4.5]. (The absence of p-torsion in $K_3F/K_3^M F$ is also proved in [Levine (1987), 4.4].)

(b) This has already been noted in [Milne (1986a), 7.3, and p. 300] (the exactness at the left is due to [Suslin (1983)], at the middle to [Bloch and Kato (1986), 2.8], and at the right to several people.)

AXIOM A1. Take $r > 0$; in the presence of (A1), Axiom $(A2)_p$ can be restated as follows:

for $1 \leqslant i \leqslant r - 1$, $H^i(\mathbf{Z}(r))$ is uniquely divisible by p, and
there is an exact sequence

$$0 \longrightarrow H^r(\mathbf{Z}(r)) \xrightarrow{p^n} H^r(\mathbf{Z}(r)) \longrightarrow \nu_n(r) \longrightarrow 0.$$

On comparing this last sequence with that given by $(A2)_m$,

$$0 \longrightarrow \mu_m^{\otimes r} \longrightarrow H^1(\mathbf{Z}(r)) \xrightarrow{m} H^1(\mathbf{Z}(r)) \longrightarrow 0,$$

we see that Axiom A1 postulates that $\mathbf{Z}(r)$ has the minimum possible length consistent with the existence of the exact sequence (2.0.1).

Write $\mathbf{Z}(r)^{(m)}$ for $\mathbf{Z}(r) \otimes^{\mathrm{L}} \mathbf{Z}/m\mathbf{Z} = (\mathbf{Z}(r) \xrightarrow{m} \mathbf{Z}(r))$. Let $m = m_0 p^n$ with m_0 prime to p. Then, in the presence of (A1), Axiom A2 asserts that for $r > 0$,

$$H^i(\mathbf{Z}(r)^{(m)}) = \begin{cases} \mu_{m_0}^{\otimes r} & \text{for } i = 0 \\ \nu_n(r) & \text{for } i = r \\ 0 & \text{for } i \neq 0, r. \end{cases}$$

Axiom A3. In the presence of $(\mathrm{A2})_p$, one should adjoin to (A3) that the pairing in that axiom is compatible with the pairing

$$\nu_n(r) \times \nu_n(s) \to \nu_n(r+s)$$

defined by the wedge product. Note that the numberings are consistent:

$$\begin{array}{ccccc} \mathbf{Z}(r) & \times & \mathbf{Z}(s) & \to & \mathbf{Z}(r+s) \\ \downarrow & & \downarrow & & \downarrow \\ \nu_n(r)[-r] & \times & \nu_n(s)[-s] & \to & \nu_n(r+s)[-r-s]. \end{array}$$

Axiom A4. In the presence of $(\mathrm{A2})_p$, one should adjoin to (A4) that the cycle map is compatible with the p-adic cycle map defined in [Milne (1986a), §2].

Axioms A5 and B1. The two axioms, when combined with $(\mathrm{A2})_p$, assert that there is an exact sequence

$$K_r^M(F) \xrightarrow{p^n} K_r^M(F) \longrightarrow \nu_n(r) \longrightarrow 0$$

for any field F of characteristic $p \neq 0$. Such an exact sequence is proved in [Bloch and Kato (1986), 2.8] (it is also shown there that the p-torsion subgroup in $K_r^M(F)$ is p-divisible). More generally, the three axioms imply that there is an exact sequence of sheaves on X_{Zar},

$$\mathscr{K}_r^M \xrightarrow{p^n} \mathscr{K}_r^M \longrightarrow \nu_n(r) \longrightarrow 0.$$

This is known for $r \leqslant 2$, and it makes a plausible conjecture in general.

Axiom B2. In the presence of (B2), $(\mathrm{A2})_p$ becomes the conjecture: $gr_\gamma^r K_s$ is uniquely divisible by p for $r < s \leqslant 2r - 1$, and there is an exact sequence

$$0 \longrightarrow gr_\gamma^r \mathscr{K}_r \xrightarrow{p^n} gr_\gamma^r \mathscr{K}_r \longrightarrow \nu_n(r) \longrightarrow 0,$$

except possibly when $p \leqslant r-1$. (Note that $gr^r_\gamma \mathscr{K}_r$ can be replaced by $\mathscr{K}^M_r$ in this statement.) This is known for $s \leqslant 2$, and for $s = 3$, $r \neq 3$ (see Lemma 2.2).

Axiom C1. Let $\imath\colon x \hookrightarrow X$ be the inclusion of a point of codimension c into X. Because the functor $- \otimes^{\mathrm{L}} \mathbf{Z}/p^n\mathbf{Z}$ commutes with any functor on the derived category, $(R\imath_*\mathbf{Z}(r-c))^{(p^n)} = R\imath_*(\mathbf{Z}(r-c)^{(p^n)})$.

Note that for any complex $C^\cdot$, $\tau_{\leqslant i}(C^\cdot \otimes^{\mathrm{L}} \mathbf{Z}/p^n\mathbf{Z}) = (\tau_{\leqslant i} C^\cdot) \otimes^{\mathrm{L}} \mathbf{Z}/p^n\mathbf{Z}$ provided $H^{i+1}(C^\cdot)$ has no p-torsion. Hence

$$\tau_{\leqslant r-c}(R\imath_*\mathbf{Z}(r-c)^{(p^n)}) = \tau_{\leqslant r-c} R\imath_*\mathbf{Z}(r-c) \otimes^{\mathrm{L}} \mathbf{Z}/p^n\mathbf{Z} \qquad (*)$$

provided $R^{r-c+1}\imath_*\mathbf{Z}(r-c)$ has no p-torsion. But $(\mathrm{A2})_p$ provides us with an exact sequence

$$\cdots \longrightarrow R^{-c}\imath_*\nu_1(r-c) \longrightarrow R^{r-c+1}\imath_*\mathbf{Z}(r) \xrightarrow{p} R^{r-c+1}\imath_*\mathbf{Z}(r) \longrightarrow \cdots.$$

Since $R^{-c}\imath_*\nu_1(r-c)$ is zero if $c > 0$, $(*)$ holds when $c > 0$. When $c = 0$, x is a generic point, and Axiom A5 applied to the extension fields of $k(X)$ shows that $R^{r+1}\imath_*\mathbf{Z}(r) = 0$. Therefore $(*)$ holds for all $c \geqslant 0$.

Consequently, on applying $- \otimes^L \mathbf{Z}/p^n\mathbf{Z}$ to the sequence in (C1), we obtain an exact sequence

$$0 \to \mathbf{Z}(r)^{(p^n)} \to \tau_{\leqslant r} \bigoplus_{x \in X_0} R\imath_*(\mathbf{Z}(r)^{(p^n)})_{\kappa(x)} \to \cdots$$

$$\to \bigoplus_{x \in X_c} (\tau_{\leqslant r-c} R\imath_*(\mathbf{Z}(r-c)^{(p^n)})_{\kappa(x)})[-c] \to \cdots.$$

According to $(\mathrm{A2})_p$

$$\tau_{\leqslant r-c} R\imath_*\mathbf{Z}(r-c)^{(p^n)})_{\kappa(x)} = \tau_{\leqslant r-c}(R\imath_*\nu_n(r-c)[r-c]),$$

and the right hand term can be rewritten as

$$(\tau_{\leqslant 0} R\imath_*\nu_n(r-c))[r-c] = \imath_*\nu_n(r-c)[r-c].$$

Therefore the axioms predict the existence of an exact sequence

$$0 \to \nu_n(r) \to \bigoplus_{x \in X_0} \imath_*\nu_n(r)_{\kappa(x)} \to \bigoplus_{x \in X_1} \imath_*\nu_n(r-1)_{\kappa(x)} \to \cdots.$$

Such a sequence was shown to exist in [Milne (1982), 4.3] for $r = 2$, conjectured to exist for all r in [Milne (1986a), 2.12], and proven to exist (even for the Zariski topology) in [Gros and Suwa (1987)].

REMARK 2.3. We have noted in (1.1c) that Axiom C1 implies that $H^i(X, \mathbf{Z}(r))$ is torsion for $i > 2r$. Therefore, in the presence of this axiom, passage to the direct limit over all m in the sequence (2.0.1) gives an isomorphism

$$H^i(X, (\mathbf{Q}/\mathbf{Z})(r)) \xrightarrow{\approx} H^{i+1}(X, \mathbf{Z}(r))$$

for $i \geqslant 2r + 1$.

AXIOM C2. It is known [Milne (1986a), §2] that $R^s i^! \nu_n(r) = 0$ for $s \neq c$, $c + 1$, and that $R^c i^! \nu_n(r) = \nu_n(r - c)$ (if $r \geqslant c$). On the other hand, (C2) states that $R^s i^! \mathbf{Z}(r) = H^{s-2c}(\mathbf{Z}(r - c))$ for $s \leqslant r + c$. These two statements are consistent with $(A2)_p$, as the following diagram shows

$$\begin{array}{ccccccccc}
\cdots R^{r+c-1} i^! \mathbf{Z}(r) & \to 0 \to & R^{r+c} i^! \mathbf{Z}(r) & \xrightarrow{p^n} & R^{r+c} i^! \mathbf{Z}(r) & \to & R^c i^! \nu_n(r) \\
\downarrow \approx & \| & \downarrow \approx & & \downarrow \approx & & \downarrow \approx \\
\cdots H^{r-c-1}(\mathbf{Z}(r-c)) & \to 0 \to & H^{r-c}(\mathbf{Z}(r-c)) & \xrightarrow{p^n} & H^{r-c}(\mathbf{Z}(r-c)) & \to & \nu_n(r-c).
\end{array}$$

REMARK 2.4. (a) In general $R^{c+1} i^! \nu_n(r) \neq 0$. Therefore

$$\tau_{\leqslant c} R i^! \nu_n(r) = \nu_n(r - c)[-c]$$

would be false without the truncation on the left, and it follows that Axiom C2 would also be false without the truncation. We shall give an example of this phenomenon for $r = 1 = c$, which will show that $R i^! \mathbf{G}_m \neq \mathbf{Z}[-1]$ even for the inclusion of a smooth divisor into a smooth variety.

Let $X = \operatorname{Spec} k[S, T] = \mathbf{A}_k^2$ with k algebraically closed of characteristic $p \neq 0$, and let $Z = \operatorname{Spec} k[S]$ be the subscheme defined by $T = 0$. Let A be the strictly local ring at the origin o of $\mathbf{A}_k^2$, let $X' = \operatorname{Spec} A$, and let Z' be the subscheme of X' defined by $T = 0$. Then the exact sequence [Milne (1976), §1]

$$0 \to \nu_1(r) \to \Omega^1_{X/k, d=0} \xrightarrow{C-1} \Omega^1_{X/k} \to 0$$

gives rise to an exact sequence

$$H^1_{Z'}(X', \Omega^1_{X'/k, d=0}) \xrightarrow{C-1} H^1_{Z'}(X', \Omega^1_{X'/k}) \to H^2_{Z'}(X', \nu_1(1)),$$

and $H^2_{Z'}(X', \nu_1(1))$ is the stalk of $R^2 i^! \nu_1(1)$ at 0. The exact sequences

$$\Omega^1_{A/k,d=0} \to \Omega^1_{A[T^{-1}]/k,d=0} \to H^1_{Z'}(X', \Omega^1_{X'/k,d=0}) \to 0$$

$$\Omega^1_{A/k} \to \Omega^1_{A[T^{-1}]/k} \to H^1_{Z'}(X', \Omega^1_{X'/k}) \to 0$$

allow us to give a more explicit presentation of $H^2_{Z'}(X', \nu_1(1))$:

$$\Omega^1_{A[T^{-1}]/k,d=0} \xrightarrow{C-1} \Omega^1_{A[T^{-1}]/k}/\Omega^1_{A/k} \to H^2_{Z'}(X', \nu_1(1)) \to 0.$$

Nothing essential changes when we pass to the completion of A. Hence we may suppose $A = k[[S, T]]$, and we let $B = A/(T) = k[[S]]$. A differential in $\Omega^1_{A[T^{-1}]/k}$ can be written uniquely as a *finite* sum

$$\eta = \sum_{i \geqslant 1} \frac{\alpha_i}{T^i} + \sum_{i \geqslant 1} \frac{\beta_i}{T^i} \mathrm{d}T + \omega \tag{2.4.1}$$

with $\alpha_i \in \Omega^1_{B/k}$, $\beta_i \in B$, and $\omega \in \Omega^1_{A/k}$. When η is closed, β_{pi+1} is a pth power, and $-i\alpha_i = d\beta_{i+1}$ for $(i, p) = 1$; moreover

$$C(\eta) = \sum_{i \geqslant 1} \frac{C(\alpha_{pi})}{T^i} + \sum_{i \geqslant 0} \frac{\beta^{1/p}_{pi+1}}{T^{i+1}} \mathrm{d}T + C(\omega).$$

Consider $\eta' = a/T$, $a \neq 0$, $a \in \Omega^1_{B/k}$, and suppose $C(\eta) - \eta = \eta' \pmod{\Omega^1_{A/k}}$ for some $\eta \in \Omega^1_{A[T^{-1}]/k,d=0}$ as in (2.4.1). Then $\beta_{pi+1} = \beta^p_{i+1}$ for all $i \geqslant 0$, and since only finitely many β_i's can be nonzero, this implies that $\beta_1 \in \mathbf{F}_p$ and that $\beta_i = 0$ for $i > 1$. Therefore $\alpha_i = 0$ for $(i, p) = 1$. Now $C(\alpha_p) = a$, and $C(\alpha_{pi}) = \alpha_i$ for $i > 1$. Hence $C(\alpha_{p^2}) = \alpha_p \neq 0$, $C(\alpha_{p^3}) = \alpha_{p^2} \neq 0, \ldots,$ contradicting the finiteness of the sum in (2.4.1).

(b) Let X' and Z' be as in (a). Then the above calculation shows that $H^2_{Z'}(X', \nu_n(1)) \neq 0$, and this implies that $H^2_{Z'}(X', \mathbf{G}_m) \neq 0$ or $H^3_{Z'}(X', \mathbf{G}_m) \neq 0$. In fact $H^2_{Z'}(X', \mathbf{G}_m) = 0$ because Pic $(X') \to$ Pic $(X' \to Z')$ is surjective and Br $(X') = 0$ [Milne (1980), III.3.11 or IV.1.7].

(c) We have seen that $Ri^!\mathbf{Z}(r)$ can fail to be isomorphic to $\mathbf{Z}(r - c)[-2c]$ for two distinct reasons: firstly, if $c > r$, then the Kummer sequence and the purity theorem in étale cohomology show that $Ri^!\mathbf{Z}(r) \neq 0$ (even when $r = 0$); secondly, there are problems with the p-torsion in characteristic p. The first of these suggests that $\mathbf{Z}(r)$ should also be defined for $r < 0$.

Note that when $r \geqslant c$, none of our examples includes the possibility that $Ri^!\mathbf{Z}(r) \xrightarrow{\approx} \mathbf{Z}(r - c)[-2c]$ up to torsion connected with the residue characteristics of X.

REMARK 2.5. Since the groups $H^i(X, \mathbf{Z}_p(r))$ map into the crystalline groups, Axiom A2$_p$ provides maps from the motivic cohomology groups into the crystalline cohomology groups.

REMARK 2.6. Let $\mathbf{Z}_B(r)$ denote the complex conjectured to exist in [Beilinson (1982)] (see also [Beilinson (1987)]); thus $\mathbf{Z}_B(r)$ is an object in $D(\mathscr{S}(X_{\mathrm{Zar}}))$ expected to satisfy axioms similar to those of $\mathbf{Z}(r)$. In particular, there should be an exact triangle

$$\mathbf{Z}_B(r) \xrightarrow{m} \mathbf{Z}_B(r) \longrightarrow \tau_{\leqslant r} Rf_* \mu_m^{\otimes r} \longrightarrow \mathbf{Z}_B(r)[1]$$

for m invertible on X.

From $(\mathrm{A2})_p$ we get an exact triangle

$$Rf_*\mathbf{Z}(r) \xrightarrow{p^n} Rf_*\mathbf{Z}(r) \longrightarrow (Rf_*\nu_n(r))[-r] \longrightarrow Rf_*\mathbf{Z}(r)[1].$$

If we assume (A5), then this triangle remains exact when we truncate at r. But $\tau_{\leqslant r}(Rf_*\nu_n(r)[-r]) = (\tau_{\leqslant 0} Rf_*\nu_n(r))[-r] = f_*\nu_n(r)[-r]$, and so the triangle is

$$\tau_{\leqslant r} Rf_*\mathbf{Z}(r) \xrightarrow{p^n} \tau_{\leqslant r} Rf_*\mathbf{Z}(r) \longrightarrow f_*\nu_n(r)[-r] \longrightarrow \tau_{\leqslant r} Rf_*\mathbf{Z}(r)[1].$$

Since the best guess for the relation between the two complexes is that $\mathbf{Z}_B(r) = \tau_{\leqslant r} Rf_*\mathbf{Z}(r)$, this suggests that we should adjoin to Beilinson's axioms the requirement that (when X is smooth of finite type over a perfect field of characteristic p) there is an exact triangle

$$\mathbf{Z}_B(r) \xrightarrow{p^n} \mathbf{Z}_B(r) \longrightarrow f_*\nu_n(r)[-r] \longrightarrow \mathbf{Z}_B(r)[1].$$

This is consistent with the remainder of Beilinson's axioms and the known results for $\nu_n(r)$.

REMARK 2.7. Axiom A2 asserts that $\mathbf{Z}(r)^{(m)}$ should be isomorphic to $\mu_m^{\otimes r}$ when m is invertible on X and to $\nu_n(r)[-r]$ when X is of characteristic $p \neq 0$ and $m = p^n$. This leaves open the (most interesting) case where X is of mixed characteristic. Clearly, $\mathbf{Z}(r)^{(m)}$ should not then be concentrated in a single degree. For a smooth proper scheme X over $\mathbf{Z}_p$, the most promising candidate appears to be $Rf_*\mathscr{S}_n^r$ where $\mathscr{S}_n^r$ is the complex of sheaves for the syntomic site on X defined in [Fontaine and Messing (1987)] and f is the morphism $X_{\mathrm{syn}} \to X_{\mathrm{et}}$ defined by the identity map. Some recent results of Kato and Kurihara lend plausibility to this. If $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$ are the inclusions of the open and closed fibres of $X/\mathbf{Z}_p$ into X, then the exact triangle (1.1.1) suggests there should be an exact triangle

$$i^*\mathbf{Z}(r)^{(p^n)} \to \tau_{\leqslant r} i^* Rj_* \mu_{p^n}^{\otimes r} \to \nu_n(r-1)[-r] \to i^*\mathbf{Z}(r)^{(p^n)}[1].$$

When $r < p$, Kato and Kurihara have proved a result of this form with $\mathbf{Z}(r)^{(p^n)}$ replaced by $Rf_*\mathscr{S}_n^r$.

§3. Duality

Recall that a 0-local field is a finite field, and an s-local field is a field complete with respect to a discrete valuation whose residue field is an $(s-1)$-local field. In [Lichtenbaum (1984), §6] it is conjectured that, when X is a smooth projective variety of dimension d over an s-local field, then

(3.1a) there is a canonical isomorphism $H^{2d+s+2}(X, \mathbf{Z}(d+s)) \xrightarrow{\approx} \mathbf{Q}/\mathbf{Z}$;

(3.1b) the pairings

$$H^{2d+s+2-i}(X, \mathbf{Z}(d+s-r)) \times H^i(X, \mathbf{Z}(r))$$
$$\to H^{2d+s+2}(X, \mathbf{Z}(d+s)) \approx \mathbf{Q}/\mathbf{Z}$$

defined by the products in (A3) are nondegenerate in an appropriate sense.

We prove two results in this direction to illustrate the methods that are available. Recall that M' denotes M/M_{div}.

THEOREM 3.2. *Let X be a projective variety over a finite field k. Assume that there exist complexes* $\mathbf{Z}(r)$ *on X satisfying Axioms* A2 *and* A3, *and that* $H^{2d+1}(X, \mathbf{Z}(d))$ *and* $H^{2d+2}(X, \mathbf{Z}(d))$ *are torsion.*

(a) *There is a canonical isomorphism*

$$H^{2d+2}(X, \mathbf{Z}(d)) \xrightarrow{\approx} \mathbf{Q}/\mathbf{Z},$$

and the products in (A3) *define nondegenerate pairings*

$$H^{2d+2-i}(X, Z(d-r))' \times H^i(X, \mathbf{Z}(r))' \to H^{2d+2}(X, \mathbf{Z}(d)) \approx \mathbf{Q}/\mathbf{Z}$$

of finite groups for $i \neq 2r, 2r+1, 2r+2$.

(b) *If* $H^{2r+1}(X, \mathbf{Z}(r))$ *and* $H^{2d-2r-1}(X, \mathbf{Z}(d-r))$ *are torsion and* $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}}$ *and* $H^{2d-2r+1}(X, \hat{\mathbf{Z}}(d-r))_{\mathrm{tors}}$ *are finite, then*

$$H^{2d-2r-1}(X, \mathbf{Z}(d-r))' \times H^{2r+1}(X, \mathbf{Z}(r))' \to H^{2d+2}(X, \mathbf{Z}(d)) \approx \mathbf{Q}/\mathbf{Z}$$

is a nondegenerate pairing of finite groups.

(c) *The products in Axiom* A3 *define nondegenerate pairing of finite groups*

$$H^{2d-2r}(X, \mathbf{Z}(d-r))'_{\mathrm{tors}} \times H^{2r+2}(X, \mathbf{Z}(r))' \to H^{2d+2}(X, \mathbf{Z}(d)) \approx \mathbf{Q}/\mathbf{Z}$$

(d) *The pairing*

$$H^{2d-2r+2}(X, \mathbf{Z}(d-r)) \times H^{2r}(X, \mathbf{Z}(r)) \to \mathbf{Q}/\mathbf{Z}$$

induces an isomorphism

$$H^{2d-2r+2}(X, \mathbf{Z}(d-r))_{\mathrm{tors}} \xrightarrow{\approx} \mathrm{Hom}_{\mathrm{conts}}(H^{2r}(X, \mathbf{Z}(r))^{\wedge}, \mathbf{Q}/\mathbf{Z})$$

if $H^{2d-2r-1}(X, \mathbf{Z}(d-r))$ *is torsion and* $H^{2r+1}(X, \mathbf{Z}(r))_{\mathrm{div}} = 0$.

Proof. We shall use the result (see [Milne (1986a), §1] that, for all m, there is a canonical isomorphism $H^{2d+1}(X, (\mathbf{Z}/m\mathbf{Z})(d)) \xrightarrow{\approx} m^{-1}\mathbf{Z}/\mathbf{Z}$ and a nondegenerate pairing of finite groups

$$H^{2d+1-i}(X, \mathbf{Z}/m\mathbf{Z})(d-r)) \times H^{i}(X, (\mathbf{Z}/m\mathbf{Z})(r))$$
$$\to H^{2d+1}(X, (\mathbf{Z}/m\mathbf{Z})(d)) \approx m^{-1}\mathbf{Z}/\mathbf{Z}.$$

After passage to the limit, this becomes a duality

$$H^{2d+1-i}(X, \hat{\mathbf{Z}}(d-r)) \times H^{i}(X, (\mathbf{Q}/\mathbf{Z})(r)) \to H^{2d+1}(X, (\mathbf{Q}/\mathbf{Z})(d)) \approx \mathbf{Q}/Z.$$

The left group is profinite, and the right group is torsion. The next lemma allows us to deduce the existence of a nondegenerate pairing

$$H^{2d+1-i}(X, \hat{\mathbf{Z}}(d-r))_{\mathrm{tors}} \times H^{i}(X, (\mathbf{Q}/\mathbf{Z})(r))' \to \mathbf{Q}/\mathbf{Z}.$$

LEMMA 3.3. *Let N be an abelian group, and let $N^1 \stackrel{\mathrm{df}}{=} \bigcap mN$ be its first Ulm subgroup.*

(a) *If $N^{(m)}$ is finite for all integers m, then N^1 is divisible; if in addition $\hat{N}$ is finite, then N' is finite and equals $\hat{N}$.*

(b) *Assume N is torsion, and let M be the Pontryagin dual of N (regarding N as a discrete group). Then the pairing*

$$N' \times M_{\mathrm{tors}} \to \mathbf{Q}/\mathbf{Z}$$

is nondegenerate, and so the Pontryagin dual of N' is the closure of M_{tors} in M.

Proof. (a) If N^1 is not divisible, then there is a prime l and an element $x \in N^1$ that is not divisible (in N^1) by all powers l. Consequently, N/N^1 will contain torsion elements of arbitrarily high l-power order. As N/N^1 contains no nonzero element that is divisible by m for all m, every finite subset of $(N/N^1)_{\text{tors}}$ is contained in a finite direct summand of it [Fuchs (1973), 65.1]. Therefore $((N/N^1)_{\text{tors}})^{(l)}$ contains subgroups of arbitrarily high order, and so is infinite. Since

$$((N/N^1)_{\text{tors}})^{(l)} \hookrightarrow (N/N^1)^{(l)} = N^{(l)}$$

we see that this contradicts our hypothesis.

For the second part of the statement, note that the diagrams

$$\begin{tikzcd} N \arrow[r] \arrow[dr] & \hat{N} \arrow[d] \\ & N^{(m)} \end{tikzcd}$$

show that the kernel of $N \to \hat{N}$ is N^1, which we have just shown to equal N_{div}. Therefore $N' \to \hat{N}$ is injective, and it follows that N' is finite and $N' = \hat{N}$.

(b) For each m, the subgroups mN and M_m of N and M are closed and are exact annihilators. Therefore $N^1 = \bigcap mN$ is the exact annihilator of $M_{\text{tor}} = \bigcup M_m$.

Let $\Gamma = \text{Gal}(\bar{k}/k)$, let σ be the Frobenius element of Γ, and write M^Γ and M_Γ respectively for the kernel and cokernel of $\sigma - 1: M \to M$.

LEMMA 3.4. *There is an exact sequence*

$$0 \to H^{i-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma \to H^i(X, \hat{\mathbf{Z}}(r)) \to H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma \to 0.$$

Proof. So far as I know, the literature does not contain a correct proof of this, and so I include one here. For each m, the Hochschild–Serre spectral sequence for $\bar{X}/X$ gives an exact sequence

$$0 \to H^{i-1}(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(r))_\Gamma \to H^i(X, (\mathbf{Z}/m\mathbf{Z})(r)) \to H^i(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(r))^\Gamma \to 0.$$

Because all of the terms in the sequence are finite, it stays exact in the limit,

$$0 \to \varprojlim_m H^{i-1}(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(r))_\Gamma \to H^i(X, \hat{\mathbf{Z}}(r)) \to \varprojlim_m H^i(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(r))^\Gamma \to 0,$$

and it remains for us to interpret the end terms.

Write $M(m)$ for $H^i(\bar{X}, \mathbf{Z}/m\mathbf{Z})(r))$. On breaking the sequence

$$0 \to M(m)^\Gamma \to M(m) \xrightarrow{\sigma-1} M(m) \to M(m)_\Gamma \to 0$$

in two and passing to the inverse limits, we obtain exact sequences

$$0 \to \varprojlim M(m)^\Gamma \to \varprojlim M(m) \xrightarrow{\sigma-1} \varprojlim Q(m) \to \varprojlim^{(1)} M(m)^\Gamma \tag{3.4.1}$$

$$0 \to \varprojlim Q(m) \to \varprojlim M(m) \to \varprojlim M(m)_\Gamma \to \varprojlim^{(1)} Q(m) \tag{3.4.2}$$

with $Q(m) = (\sigma - 1)M(m)$. There is an exact sequence

$$0 \to U_m(\bar{k}) \to M(m) \to D_m(\bar{k}) \to 0$$

with U_m a connected algebraic perfect group scheme over k and D_m a finite perfect group scheme over k [Milne (1986a), §1]. Therefore $M(m)^\Gamma$ is finite, and this implies that $\varprojlim^{(1)} M(m)^\Gamma = 0$. Thus (3.4.1) becomes

$$0 \to \varprojlim M(m)^\Gamma \to \varprojlim M(m) \xrightarrow{\sigma-1} \varprojlim Q(m) \to 0.$$

Because $\sigma - 1\colon U_m(\bar{k}) \to U_m(\bar{k})$ is surjective, there is an exact sequence

$$0 \to U_m(\bar{k}) \to Q(m) \to F_m \to 0$$

with F_m is finite. As U_m is constant for large m, $\varprojlim^{(1)} U_m(\bar{k}) = 0$, and this implies that $\varprojlim^{(1)} Q(m) = 0$. Thus (3.4.2) becomes

$$0 \to \varprojlim Q(m) \to \varprojlim M(m) \to \varprojlim M(m)_\Gamma \to 0.$$

On splicing the two sequences, we obtain a sequence

$$0 \to \varprojlim M(m)^\Gamma \to \varprojlim M(m) \xrightarrow{\sigma-1} \varprojlim M(m) \to \varprojlim M(m)_\Gamma \to 0,$$

which shows that

$$H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma = \varprojlim H^i(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(r))^\Gamma$$

and

$$H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma = \varprojlim H^i(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(r))_\Gamma.$$

Since this holds for all i, the second sequence in the proof is the required one.

In [Milne (1986a), 6.4] it is shown that $H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma$ and $H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma$ are finite for $i \neq 2r$. Therefore (3.4) shows that $H^i(X, \hat{\mathbf{Z}}(r))$ is finite for $i \neq 2r$, $2r + 1$.

We now prove the theorem. Because the groups $H^i(X, (\mathbf{Z}/m\mathbf{Z}))(r)$ are finite, on passing to the inverse and direct limits in the sequences (2.0.1), we obtain exact sequences

$$0 \to H^i(X, \mathbf{Z}(r))^\wedge \to H^i(X, \hat{\mathbf{Z}}(r)) \to TH^{i+1}(X, \mathbf{Z}(r)) \to 0 \tag{3.4.3}$$

and

$$0 \to H^i(X, \mathbf{Z}(r)) \otimes \mathbf{Q}/\mathbf{Z} \to H^i(X, (\mathbf{Q}/\mathbf{Z})(r)) \to H^{i+1}(X, \mathbf{Z}(r))_{\text{tors}} \to 0 \tag{3.4.4}$$

As TH^{i+1} is torsion free, the first sequence gives an isomorphism

$$(H^i(X, \mathbf{Z}(r))^\wedge)_{\text{tors}} \xrightarrow{\approx} H^i(X, \hat{\mathbf{Z}}(r))_{\text{tors}}, \tag{3.4.5}$$

and, as $H^i(X, \mathbf{Z}(r)) \otimes \mathbf{Q}/\mathbf{Z}$ is divisible, the second sequence gives an isomorphism

$$H^i(X, (\mathbf{Q}/\mathbf{Z})(r))' \xrightarrow{\approx} H^{i+1}(X, \mathbf{Z}(r))'_{\text{tors}}. \tag{3.4.6}$$

When $H^i(X, \mathbf{Z}(r))$ is torsion, $H^i(X, \mathbf{Z}(r)) \otimes \mathbf{Q}/\mathbf{Z} = 0$, and the primes may be dropped in the last isomorphism. In particular, if $H^{2d+1}(X, \mathbf{Z}(d))$ is torsion, then

$$H^{2d+2}(X, \mathbf{Z}(d))_{\text{tors}} \xleftarrow{\approx} H^{2d+1}(X, (\mathbf{Q}/\mathbf{Z})(d)) \xrightarrow{\approx} \mathbf{Q}/\mathbf{Z},$$

and so this gives us the isomorphism $H^{2d+2}(X, \mathbf{Z}(d)) \xrightarrow{\approx} \mathbf{Q}/\mathbf{Z}$ required in (a).

Note that the results already proved show that there is a nondegenerate pairing

$$(H^{2d+2-i}(X, \mathbf{Z}(d - r))^\wedge)_{\text{tors}} \times H^i(X, \mathbf{Z}(r))'_{\text{tors}} \to H^{2d+2}(X, \mathbf{Z}(d)) \approx \mathbf{Q}/\mathbf{Z};$$

also that (2.0.1) shows that $H^i(X, \mathbf{Z}(r))^{(m)}$ is finite for all i and m.

If $i \neq 2r$, $2r + 1$, then from (3.4.3) and (3.3a) we see that $H^i(X, \mathbf{Z}(r))'$ is finite and equals $H^i(X, \mathbf{Z}(r))^\wedge$; in particular, $H^i(X, \mathbf{Z}(r))' = H^i(X, \mathbf{Z}(r))'_{\text{tors}}$. If $i \neq 2r + 1$, $2r + 2$, so that $2d + 2 - i \neq 2(d - r)$, $2(d - r) + 1$, then the same argument shows that $H^{2d+2-i}(X, \mathbf{Z}(d - r))'$ is finite and equals $H^{2d+2-i}(X, \mathbf{Z}(d - r))^\wedge$. This completes the proof of (a) of the theorem.

For $i \neq 2r$, we obtain a duality of finite groups

$$(H^{2d+2-i}(X, \mathbf{Z}(d-r)))' \times H^i(X, \mathbf{Z}(r))'_{\text{tors}} \to H^{2d+2}(X, \mathbf{Z}(d)) \approx \mathbf{Q}/\mathbf{Z},$$

which proves (c).

Assume that $H^{2r+1}(X, \mathbf{Z}(r))$ is torsion and that $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tors}}$ is finite. Then (3.4.5) shows that the torsion subgroup of $H^{2r+1}(X, \mathbf{Z}(r))^\wedge$ is finite, and so the image of $H^{2r+1}(X, \mathbf{Z}(r))$ in $H^{2r+1}(X, (\mathbf{Z}(r))^\wedge$ is finite. Since it is also dense, this implies that $H^{2r+1}(X, \mathbf{Z}(r))^\wedge$ is finite, and (3.3a) shows that $H^{2r+1}(X, \mathbf{Z}(r))'$ is finite, and therefore equals $H^{2r+1}(X, \mathbf{Z}(r))'_{\text{tors}}$. The same argument shows that $H^{2d-2r+1}(X, \mathbf{Z}(r))'$ is finite, and therefore equals $H^{2d-2r+1}(X, \mathbf{Z}(r))^\wedge$. This proves (b).

Finally, if $H^{2d-2r+1}(X, \mathbf{Z}(d-r))$ is torsion, then (3.4.4) shows that

$$H^{2d-2r+2}(X, \mathbf{Z}(d-r))_{\text{tors}} \approx H^{2d-2r+1}(X, (\mathbf{Q}/\mathbf{Z})(d-r)),$$

and we know that

$$H^{2d-2r+1}(X, (\mathbf{Q}/\mathbf{Z})(r)) \xrightarrow{\approx} \operatorname{Hom}_{\text{conts}}(H^{2r}(X, \hat{\mathbf{Z}}(r)), \mathbf{Q}/\mathbf{Z}).$$

If in addition $H^{2r+1}(X, \mathbf{Z}(r))_{\text{div}}$ is zero, then $TH^{2r+1}(X, \mathbf{Z}(r)) = 0$ and so $H^{2r}(X, \hat{\mathbf{Z}}(r)) \approx H^{2r}(X, \mathbf{Z}(r))^\wedge$. This completes the proof of (d).

REMARK 3.5. (a) If the condition that $H^{2d+2}(X, \mathbf{Z}(d))$ is torsion is dropped, then $H^{2d+2}(X, \mathbf{Z}(d))$ must be replaced by $H^{2d+2}(X, \mathbf{Z}(d))_{\text{tors}}$ in the statement of the theorem. If the condition that $H^{2d+1}(X, \mathbf{Z}(d))$ is torsion is also dropped, then the pairing must map into $H^{2d+1}(X, (\mathbf{Q}/\mathbf{Z})(d))$.

(b) Concerning the various other hypotheses made in the theorem, it is known that $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tors}}$ is finite when, for all primes l, 1 is not a multiple root of the minimal polynomial of the Frobenius element acting on $H^{2r}(X, \mathbf{Q}_l(r))$ [Milne (1986a), 6.6]. The Tate conjecture implies that $H^{2r+1}(X, \mathbf{Z}(r))_{\text{div}} = 0$ (see §4 below), and Conjecture 1.6 predicts that the remaining conditions always hold, and that $H^{2r}(X, \mathbf{Z}(r))$ is finitely generated and $H^{2d-2r+2}(X, \mathbf{Z}(d-r))$ is torsion. When this is so, part (d) of the theorem shows that

$$H^{2d-2r+2}(X, \mathbf{Z}(r)) \xrightarrow{\approx} \operatorname{Hom}(H^{2r}(X, \mathbf{Z}(r)), \mathbf{Q}/\mathbf{Z}).$$

(c) Since $H^i(X, \hat{\mathbf{Z}}(r))$ is finite for $i \neq 2r, 2r+1$, (3.4.3) shows that $TH^i(X, \mathbf{Z}(r))$ is zero for $i \neq 2r+1, 2r+2$. Hence the divisible subgroups of $H^i(X, \mathbf{Z}(r))$ are in fact uniquely divisible (and presumably zero) for

$i \neq 2r + 1, 2r + 2$. On the other hand, $H^{2r+2}(X, \mathbf{Z}(r))_{\text{div}}$ is not uniquely divisible (the last remark predicts it equals $\operatorname{Hom}(H^{2d-2r}(X, \mathbf{Z}(r)), \mathbf{Z})$), and to prove that $H^{2r+1}(X, \mathbf{Z}(r))_{\text{div}}$ is uniquely divisible requires the Tate conjecture.

THEOREM 3.6. *Let K be an s-local field such that the* 1*-local field in its inductive definition has characteristic zero, and let X be a smooth projective variety of dimension d over K. Assume that there exist complexes $\mathbf{Z}(r)$ on X satisfying Axioms* A2 *and* A3, *and that $H^{2d+s+1}(X, \mathbf{Z}(d + s))$ and $H^{2d+s+2}(X, \mathbf{Z}(d + s))$ are torsion. Then there is a canonical isomorphism*

$$H^{2d+s+2}(X, \mathbf{Z}(d + s)) \xrightarrow{\approx} \mathbf{Q}/\mathbf{Z},$$

and the products in (A3) *define nondegenerate pairings of groups*

$$H^{2d+s+2-i}(X, \mathbf{Z}(d + s - r))'_{\text{tors}} \times (H^i(X, \mathbf{Z}(r))_{\text{tors}})^{\wedge}$$

$$\to H^{2d+s+2}(X, \mathbf{Z}(d + s)) \approx \mathbf{Q}/\mathbf{Z}$$

for all i.

Proof. We first need to prove the corresponding result for finite sheaves.

LEMMA 3.7. *Let X be as in the theorem, and let m be an integer. Then there is a canonical isomorphism*

$$H^{2d+s+1}(X, \mu_m^{\otimes d+s}) \xrightarrow{\approx} m^{-1}\mathbf{Z}/\mathbf{Z},$$

and there are canonical nondegenerate pairings of finite groups

$$H^{2d+s+1-i}(X, \mu_m^{\otimes d+s-r}) \times H^i(X, \mu_m^{\otimes r}) \to H^{2d+s+1}(X, \mu_m^{\otimes d+s}) = m^{-1}\mathbf{Z}/\mathbf{Z}$$

for all i.

Proof. It is proved in [Deninger and Wingberg (1986)] (see also [Milne (1986b), I.2.17]) that $H^{s+1}(K, \mu_m^{\otimes s}) \xrightarrow{\approx} m^{-1}\mathbf{Z}/\mathbf{Z}$ and that for any finite $\operatorname{Gal}(\bar{K}/K)$-module M,

$$H^{2s+1-i}(K, M^*(s - r)) \times H^i(K, M(r)) \to H^{2s+1}(K, \mu_m^{\otimes d}) \xrightarrow{\approx} m^{-1}\mathbf{Z}/\mathbf{Z}$$

is a nondegenerate pairing of finite groups. Here $M^* = \operatorname{Hom}(M, \mathbf{Z}/m\mathbf{Z})$ and $M(r) = M \otimes \mu_m^{\otimes r}$. A standard argument extends this result from M to a bounded-below complex $M^\cdot$ of modules such that $H^i(M^\cdot)$ is finite for all i and is zero for all sufficiently large i; $M^{\cdot *}$ is replaced by $\mathbf{R}\ \mathrm{Hom}\ (M^\cdot, \mathbf{Z}/m\mathbf{Z})$ (cf. [Milne (1980), p. 280]).

Let $\pi\colon X \to \mathrm{Spec}\ K$ be the structure map. Then the fundamental theorems in étale cohomology show:

(a) $R^{2d}\pi_*\mu_m^{\otimes d+s} \xrightarrow{\approx} \mu_m^{\otimes s}$, and $R^{2i}\pi_*\mu_m^{\otimes d+2} = 0$ for $i > 2d$;

(b) $M^\cdot \stackrel{\mathrm{df}}{=} R\pi_*\mathbf{Z}/m\mathbf{Z}$ satisfies the conditions in the last paragraph;

(c) $M^{\cdot *} \xrightarrow{\approx} R\pi_*\mu_m^{\otimes d}[2d]$.

Therefore

$$H^{2d+s+1}(X, \mu_m^{\otimes d+s}) \approx H^{s+1}(K, \mu_m^{\otimes s}) \xrightarrow{\approx} m^{-1}\mathbf{Z}/\mathbf{Z},$$

and, as the group

$$H^i(X, \mu_m^{\otimes r}) = \mathbf{H}^i(K, R\pi_*\mu_m^{\otimes r}) = \mathbf{H}^i(K, M^\cdot(r)),$$

it is finite and dual to $\mathbf{H}^{s+1-i}(K, M^{\cdot *}(s-r))$, which equals

$$H^{s+1-i}(K, R\pi_*\mu_m^{\otimes d+s-r}[2d]) = H^{2d+s+1-i}(X, \mu_m^{\otimes d+s-r}).$$

We now prove the theorem. After passing to the limit in (3.7), we obtain a duality

$$H^{2d+s+1-i}(X, (\mathbf{Q}/\mathbf{Z})(d+s-r)) \times H^i(X, \hat{\mathbf{Z}}(r))$$
$$\to H^{2d+s+1}(X, (\mathbf{Q}/\mathbf{Z})(d+s)) \approx \mathbf{Q}/\mathbf{Z}.$$

Lemma 3.3b shows that this gives a duality

$$H^{2d+s+1-i}(X, (\mathbf{Q}/\mathbf{Z})(d+s-r))' \times (H^i(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}})^-$$
$$\to H^{2d+s+1}(X, (\mathbf{Q}/\mathbf{Z})(d+s)),$$

and the isomorphisms (3.4.5) and (3.4.6) show that

$$H^{2d+s+1-i}(X, (\mathbf{Q}/\mathbf{Z})(d+s-r))' \xrightarrow{\approx} H^{2d+s+2-i}(X, \mathbf{Z}(d+s-r))'_{\mathrm{tors}},$$

$$(H^i(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}})^\wedge \xrightarrow{\approx} (H^i(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}})^-,$$

and

$$H^{2d+s+1}(X, (\mathbf{Q}/\mathbf{Z})(d+s)) \xrightarrow{\approx} H^{2d+s+2}(X, \mathbf{Z}(r)).$$

REMARK 3.8. (a) If the 1-local field in the statement of (3.6) has characteristic $p \neq 0$, then the proposition still holds up to p-torsion.

(b) Our results are limited by the fact that we are essentially using only Axiom A2, which clearly does not allow us to detect uniquely divisible subgroups in $H^i(X, \mathbf{Z}(r))$. Presumably to go further, one will have to use the Axioms B and results about the cohomology of the K-sheaves. As Lichtenbaum pointed out to me, there is an air of compatibility between Bass's conjecture on the finite generation of the K-groups of finitely generated $\mathbf{Z}$-algebras and the conjectured finiteness properties of the motivic cohomology groups.

(c) It is to be hoped that the duality will extend to a pairing between cohomology groups and Ext groups. Unfortunately the discussion in [Milne (1986b), pp. 263–266] is based on an earlier form of the purity axiom C2 in which the truncation is omitted – as we noted in §2, this cannot be correct – and so is inaccurate. In particular, the Conjecture 7.16 there is unrealistic if the Ext group is meant to be computed in the category of étale sheaves on X.

(d) In [Saito (1987)], some of the results of higher dimensional class field theory are interpreted in the language of motivic cohomology. For example, if K is an s-local field, then [Kato (1980)] proves that there is a canonical pairing

$$H^{r+1}(K, (\mathbf{Q}/\mathbf{Z})(r)) \times K^M_{s-r}(K) \to H^{s+1}(K, (\mathbf{Q}/\mathbf{Z})(d)) \approx \mathbf{Q}/\mathbf{Z},$$

and the induced homomorphism

$$H^{r+1}(K, (\mathbf{Q}/\mathbf{Z})(r)) \to \mathrm{Hom}\,(K^M_{s-r}(K), \mathbf{Q}/\mathbf{Z})$$

is injective. But (see (1.6) and (2.3)), $H^{r+2}(\dot{K}, \mathbf{Z}(r))$ should be isomorphic to $H^{r+1}(K, (\mathbf{Q}/\mathbf{Z})(r))$, and (see B1) $H^{s-r}(K, \mathbf{Z}(s - r))$ should equal $K^M_{s-r}K$, and so the pairing should be able to be identified with a pairing

$$H^{r+2}(X, \mathbf{Z}(r)) \times H^{s-r}(K, \mathbf{Z}(s - r)) \to H^{s+2}(K, \mathbf{Z}(s)) \approx \mathbf{Q}/\mathbf{Z}.$$

Similar, but more complicated, interpretations are also given in the global case.

§4. Values of zeta functions

Throughout this section, X will be a smooth projective variety over a finite field k.

The homomorphism Gal $(\bar{k}/k) \to \hat{\mathbf{Z}}$ sending the Frobenius element to 1 defines a canonical element of $H^1(k, \hat{\mathbf{Z}}) \subset H^1(X, \hat{\mathbf{Z}})$, and we define

$$\varepsilon^{2r}\colon H^{2r}(X, \hat{\mathbf{Z}}(r)) \to H^{2r+1}(X, \hat{\mathbf{Z}}(r))$$

to be the map taking a cohomology class to its cup-product with this element. The map δ^r is defined to be the composite of the remaining maps in the diagram

$$\begin{array}{ccc} H^{2r}(X, \mathbf{Z}(r))^\wedge & \xrightarrow{\delta^r} & TH^{2r+2}(X, \mathbf{Z}(r)) \\ \downarrow{\scriptstyle (3.4.3)} & & \uparrow{\scriptstyle (3.4.3)} \\ H^{2r}(X, \hat{\mathbf{Z}}(r)) & \xrightarrow{\varepsilon^{2r}} & H^{2r+1}(X, \hat{\mathbf{Z}}(r)). \end{array}$$

Recall (3.2) that $H^i(X, \mathbf{Z}(r))'$ is finite for $i \neq 2r, 2r+1$. Set

$$\chi'(X, \mathbf{Z}(r)) = \prod_{i \neq 2r} [H^i(X, \mathbf{Z}(r))']^{(-1)^i} [H^{2r}(X, \mathbf{Z}(r))_{\mathrm{tors}}] \det(\delta^r)^{-1}$$

when $[H^{2r}(X, \mathbf{Z}(r))_{\mathrm{tors}}]$ and $[H^{2r+1}(X, \mathbf{Z}(r))']$ are finite and $\det(\delta^r)$ is defined and nonzero.

The strong form of Tate's conjecture for an integer r and prime l states that the dimension $\varrho'_{r,l}$ of the subspace of $H^{2r}(\bar{X}, \mathbf{Q}_l(r))$ generated by algebraic cycles on X is equal to the order ϱ_r of the pole of $\zeta(X, s)$ at $s = r$.

PROPOSITION 4.1. *Let* $\mathbf{Z}(r)$ *be a complex on X satisfying Axioms* A2 *and* A4. *If* $H^{2r}(X, \mathbf{Z}(r))$ *is finitely generated, then the strong form of Tate's conjecture holds (for r) and all primes l when it holds for one prime l.*

Proof. The condition implies that $H^{2r}(X, \mathbf{Z}(r))^\wedge = H^{2r}(X, \mathbf{Z}(r)) \otimes \hat{\mathbf{Z}}$, and we know that this injects into $H^{2r}(X, \hat{\mathbf{Z}}(r))$. According to (3.4) the kernel of $H^{2r}(X, \mathbf{Z}_l(r)) \to H^{2r}(\bar{X}, \mathbf{Z}_l(r))$ is $H^{2r-1}(\bar{X}, \mathbf{Z}_l(r))_\Gamma$, which is finite by [Milne (1986a), 6.4], and so the composite map $H^{2r}(X, \mathbf{Z}(r)) \otimes \mathbf{Q}_l \to H^{2r}(\bar{X}, \mathbf{Q}_l(r))$ is injective for all l. Let B^r be the subspace of $H^{2r}(X, \mathbf{Z}(r)) \otimes \mathbf{Q}$ generated by algebraic cycles. Then the subspace of $H^{2r}(\bar{X}, \mathbf{Q}_l(r))$ generated by algebraic cycles is isomorphic to $B^r \otimes \mathbf{Q}_l$. In particular, its dimension is independent of l, and the lemma is now obvious.

LEMMA 4.2. *Let* $\mathbf{Z}(r)$ *be a complex on X satisfying Axioms* A2 *and* A4, *and assume that* $H^{2r+1}(X, \mathbf{Z}(r))'$ *is torsion. If the strong form of Tate's conjecture holds for all l and a given r, then* $H^{2r}(X, \mathbf{Z}(r))_{\mathrm{tors}}$ *and* $H^{2r+1}(X, \mathbf{Z}(r))'$ *are finite and* $\det(\delta^r)$ *is defined and nonzero; moreover,*

$$\det(\delta^r) = z(\varepsilon^{2r})^{-1}[H^{2r}(X, \mathbf{Z}(r))_{\mathrm{tors}}][H^{2r+1}(X, \mathbf{Z}(r))']^{-1} = \det(\varepsilon^{2r}).$$

Proof. In [Milne (1986a), 8.2], it is shown that the strong form of Tate's conjecture for r and l implies than 1 is not a multiple root of the minimal polynomial of the Frobenius element acting on $H^{2r}(\bar{X}, \mathbf{Q}_l(r))$. Therefore $z(\varepsilon^{2r})$ is defined and $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tors}}$ is finite [ib., 6.6]. Consider

$$\begin{array}{ccc} CH^r(X)^\wedge & \xrightarrow{c^r} & H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma \\ \downarrow \hat{c} & & \uparrow d \\ H^{2r}(X, \mathbf{Z}(r))^\wedge & \xrightarrow{b} & H^{2r}(X, \hat{\mathbf{Z}}(r)). \end{array}$$

Here c^r is the product of the l-adic cycle maps,[3] all primes l, and $c\colon CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$ is the cycle map given by Axiom A4. The same axiom shows that the diagram commutes.

The map b is injective with cokernel $TH^{2r+1}(X, \mathbf{Z}(r))$ (see 3.4.3), and d is surjective with kernel $H^{2r-1}(X, \hat{\mathbf{Z}}(r))_\Gamma$ (see 3.4), which is finite [ib., 6.4]. Tate's conjecture implies that the cokernel of c^r is torsion, and so the exact sequence

$$\text{Ker}\,(d) \to \text{Coker}\,(b \circ \hat{c}) \to \text{Coker}\,(c^r)$$

shows that $TH^{2r+1}(X, \mathbf{Z}(r))$ is torsion and hence zero. Consequently $H^{2r+1}(X, \mathbf{Z}(r))_{\text{div}}$ has no torsion elements, and (3.3a) shows that $H^{2r+1}(X, \mathbf{Z}(r))_{\text{tors}} \to H^{2r+1}(X, \mathbf{Z}(r))^\wedge$ is injective. As $H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tors}}$ is finite, (3.4.3) now shows that $H^{2r+1}(X, \mathbf{Z}(r))_{\text{tors}}$ is finite. Our assumption allows us to conclude that $H^{2r+1}(X, \mathbf{Z}(r))'$ is finite, and therefore

$$H^{2r+1}(X, \mathbf{Z}(r))' = H^{2r+1}(X, \mathbf{Z}(r))^\wedge = H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\text{tors}}.$$

Now consider the diagram

$$\begin{array}{ccc} H^{2r}(X, \mathbf{Z}(r))^\wedge & \xrightarrow{\delta^r} & TH^{2r+2}(X, \mathbf{Z}(r))^\wedge \\ \downarrow \approx & & \uparrow \text{surj} \\ H^{2r}(X, \hat{\mathbf{Z}}(r)) & \xrightarrow{\varepsilon^{2r}} & H^{2r+1}(X, \hat{\mathbf{Z}}(r)). \end{array}$$

The isomorphism is the map in (3.4.3); its cokernel is $TH^{2r+1}(X, \mathbf{Z}(r))$, which we have just shown to be zero. The second vertical map is also the map in (3.4.3); its kernel is $H^{2r+1}(X, \mathbf{Z}(r))^\wedge$, which we know to be the finite group $H^{2r+1}(X, \mathbf{Z}(r))'$. Since the z's of all the maps except δ^r are defined, $z(\delta^r)$ must

[3] In the absence of a published proof that the cycle map into the integral group $H^{2r}(\bar{X}, \mathbf{Z}_p(r))$ factors through the Chow group, the reader may prefer to take c^r to be the map making the diagram commute. Then Axiom A4 will ensure that $c^r \otimes \mathbf{Q}$ is the product of the l-adic cycle maps, which is all that is needed.

also be defined, and it must equal the product of the z's of the remaining maps: $z(\delta^r) = z(\varepsilon^{2r})[H^{2r+1}(X, \mathbf{Z}(r))']$.

Note that, because $TH^{2r+2}(X, \mathbf{Z}(r))$ is torsion free,

$$z(\delta^r) = [(H^{2r}(X, \mathbf{Z}(r))^\wedge)_{\text{tors}}] \det(\delta^r)^{-1},$$

and both terms on the right are defined and finite. Thus, for the first equality, it remains to show that the torsion subgroup of $H^{2r}(X, \mathbf{Z}(r))$ is unchanged when the group is completed.

Note that $H^{2r}(X, \hat{\mathbf{Z}}(r))_{\text{tors}}$ is finite because $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\text{tors}}$ and $H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma$ are finite. From the maps

$$H^{2r}(X, \mathbf{Z}(r))' \hookrightarrow H^{2r}(X, \mathbf{Z}(r))^\wedge \hookrightarrow H^{2r}(X, \hat{\mathbf{Z}}(r))$$

we see that $H^{2r}(X, \mathbf{Z}(r))'_{\text{tors}}$ is finite. The finiteness of $H^{2r-1}(X, \hat{\mathbf{Z}}(r))$ shows that $H^{2r}(X, \mathbf{Z}(r))_{\text{div}}$ has no torsion elements, and therefore $H^{2r}(X, \mathbf{Z}(r))_{\text{tors}} \xrightarrow{\approx} H^{2r}(X, \mathbf{Z}(r))'_{\text{tors}}$. Consequently

$$H^{2r}(X, \mathbf{Z}(r))'_{\text{tors}} = (H^{2r}(X, \mathbf{Z}(r))^\wedge)_{\text{tors}} = (H^{2r}(X, \mathbf{Z}(r))_{\text{tors}})^\wedge.$$

This completes the proof of the first equality, and the second follows from the definitions.

THEOREM 4.3. *Let $\mathbf{Z}(r)$ be a complex on X satisfying Axioms* A2 *and* A4, *and assume that $H^{2r+1}(X, \mathbf{Z}(r))'$ is torsion. If the strong form of Tate's conjecture holds for r and for all l, then $\chi'(X, \mathbf{Z}(r))$ is defined, and*

$$\zeta(X, s) \sim \pm\chi'(X, \mathbf{Z}(r)) q^{\chi(X, \mathcal{O}, r)}(1 - q^{r-s})^{-\rho_r} \quad \textit{as } s \to r.$$

Proof. After [Milne (1986a), 0.1] it suffices to show that $\chi'(X, \mathbf{Z}(r))$ is defined and equals $\chi(X, \hat{\mathbf{Z}}(r))$.

First note that for $i \neq 2r, 2r + 1$, $H^i(X, \hat{\mathbf{Z}}(r))$ is finite, and so, for these values of i, (3.4.3) gives an isomorphism $H^i(X, \mathbf{Z}(r))' \xrightarrow{\approx} H^i(X, \hat{\mathbf{Z}}(r))$. Lemma 4.2 now shows that $\chi'(X, \mathbf{Z}(r))$ is defined and that

$$\begin{aligned} \chi(X, \hat{\mathbf{Z}}(r)) &\stackrel{\text{df}}{=} \prod_{i \neq 2r, 2r+1} [H^i(X, \hat{\mathbf{Z}}(r))]^{(-1)^i} z(\varepsilon^{2r}) \\ &= \prod_{i \neq 2r, 2r+1} [H^i(X, \mathbf{Z}(r))']^{(-1)^i} [H^{2r}(X, \mathbf{Z}(r))_{\text{tors}}] \\ &\qquad \times [H^{2r+1}(X, \mathbf{Z}(r))']^{-1} \det(\delta^r)^{-1} \\ &= \chi'(X, \mathbf{Z}(r)), \end{aligned}$$

which completes the proof.

COROLLARY 4.4. *Assume that there is a complex* $\mathbf{Z}(r)$ *on* X *satisfying Axiom* A2 *and such that* $H^{2r+1}(X, \mathbf{Z}(r))'$ *is torsion; then*

$$\chi'(X, \mathbf{Z}(r)) = \zeta(X, r)q^{-\chi(X, \mathcal{O}_X, r)}$$

for all $r > d$.

Proof. For $r > d$, $\varrho_r = 0$ and Tate's conjecture is obvious. Also the cycle maps (Axiom A4) play no role.

REMARK 4.5. (a) Define $\chi(X, \mathbf{Z}(r))$ as in the introduction, taking $R = \det(\delta^r)$. If $\chi'(X, \mathbf{Z}(r))$ is defined, then $\chi(X, \mathbf{Z}(r))$ is defined if and only if the groups $H^i(X, \mathbf{Z}(r))$ have no uniquely divisible subgroups and $H^{2r+1}(X, \mathbf{Z}(r))$ has no divisible subgroup, in which case it equals $\chi'(X, \mathbf{Z}(r))$. Therefore Conjecture 1.6 and the conjecture that $H^{2r+1}(X, \mathbf{Z}(r))$ is finite imply that $\chi(X, \mathbf{Z}(r)) = \chi'(X, \mathbf{Z}(r))$.

(b) When $r = 0$ or 1, stronger results are known (see [Milne (1986a)]).

(c) When $r = 2$, we know Axiom A2 but still lack Axiom A4. Note however, that without any assumptions whatever, we have shown that $\chi'(X, \mathbf{Z}(2)) = \zeta(X, 2)q^{3g-3}$ when X is a curve.

(d) Tate's conjecture is often true for trivial reasons. For example, when X is a complete intersection of dimension d, then the conjecture is obvious exccept when d is even and $r = d/2$.

(e) In the case that X is an elliptic surface with base curve C and $r = 0$, S. Turner has shown that (4.3) can be written in the form $\chi(X, \mathbf{Z}) = \pm q \cdot \zeta(C, 2) \cdot \mu$, with μ the measure of a certain adèle group. This, as he points out, can be regarded as an arithmetic analogue of the Gauss–Bonnet formula.

(f) It would be interesting to consider a generalization of Conjecture 0.1 in which the zeta function is replaced by the L-series of a representation of the fundamental group of X.

(g) It is clear from the sequence (3.4.3) that Tate's conjecture for a given r and l is equivalent to the nullity of the divisible subgroup of $H^{2r+1}(X, \mathbf{Z}(r))$. In [Thomason (1985)] it is shown that Tate's conjecture is equivalent to the nullity of the divisible subgroup of a certain group defined using K-theory. It would be interesting to find a direct relation between this group and $H^{2r+1}(X, \mathbf{Z}(r))$.

§5. Re-interpretation of the regulator term

We show that the term $\det(\delta^r)$ occurring in the definition of $\chi'(X, \mathbf{Z}(r))$ can (conjecturally) be replaced by the discriminant of a pairing between two finitely generated $\mathbf{Z}$-modules, as in [Lichtenbaum (1984), §7]. Throughout the section, X will be a smooth projective variety of dimension d over a finite field k.

As noted in §3, for all m, there is a trace map

$$H^{2d}(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(d)) \xrightarrow{\approx} \mathbf{Z}/m\mathbf{Z}.$$

We shall need to assume:

(∗) there exists a map *deg*: $H^{2d}(\bar{X}, \mathbf{Z}(d)) \to \mathbf{Z}$ such that

$$\begin{array}{ccc} H^{2d}(\bar{X}, \mathbf{Z}(d)) & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ H^{2d}(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(d)) & \xrightarrow{\approx} & \mathbf{Z}/m\mathbf{Z} \end{array}$$

commutes for all m.

We define the degree map *deg*: $H^{2d}(X, \mathbf{Z}(d)) \to \mathbf{Z}$ to be the composite $H^{2d}(X, \mathbf{Z}(d)) \to H^{2d}(\bar{X}, \mathbf{Z}(d)) \to \mathbf{Z}$.

REMARK 5.1 (a) Obviously, there can be at most one map *deg* satisfying (∗). If we assume Axiom A4, then *deg*: $H^{2d}(X, \mathbf{Z}(d)) \to \mathbf{Z}$ sends the class of a zero cycle to the degree of the cycle, and this property characterizes *deg*.

(b) If $H^{2d+1}(X, \mathbf{Z}(r))_{\mathrm{div}} = 0$, then (3.4.3) gives an isomorphism $H^{2d}(X, \mathbf{Z}(d))^{\wedge} \xrightarrow{\approx} H^{2d}(X, \hat{\mathbf{Z}}(d))$, and it is easy to see by using duality that $H^{2d}(X, \hat{\mathbf{Z}}(d))$ modulo its torsion subgroup is isomorphic to $\hat{\mathbf{Z}}$. Therefore, if $H^{2d}(X, \mathbf{Z}(d))$ is finitely generated, it is of rank one. Consequently, there will be exactly two epimorphisms $H^{2d}(X, \mathbf{Z}(d)) \to \mathbf{Z}$, only one of which will send the class of a zero cycle to its degree. In [Lichtenbaum (1984)] the degree map is defined to be (either) one of the maps.

Assume in addition to the above:

(∗∗) Axiom A3 holds, and the groups $H^{2r}(X, \mathbf{Z}(r))$ and $H^{2d-2r}(X, \mathbf{Z}(d-r))$ are finitely generated.

Then there is a pairing

$$H^{2r}(X, \mathbf{Z}(r)) \times H^{2d-2r}(X, \mathbf{Z}(d-r)) \to H^{2d}(X, \mathbf{Z}(d)) \xrightarrow{\deg} \mathbf{Z},$$

and, following Lichtenbaum [ib., §7], we define the regulator $\mathrm{Reg}^r(X)$ to be the discriminant of this pairing when the two groups have the same rank, i.e., we set $\mathrm{Reg}^r(X) = \deg(\langle a_i, b_j\rangle)$ where $\{a_i\}$ and $\{b_j\}$ are bases for $H^{2r}(X, \mathbf{Z}(r))$ and $H^{2d-2r}(X, \mathbf{Z}(d-r))$ modulo torsion.

THEOREM 5.2. *Assume there exist complexes* $\mathbf{Z}(r)$, $\mathbf{Z}(d-r)$, *and* $\mathbf{Z}(d)$ *satisfying Axiom* A2 *and conditions* $(*)$ *and* $(**)$. *If* $H^{2d-2r-1}(X, \mathbf{Z}(d-r))_{\mathrm{div}} = 0$, $H^{2r+1}(X, \mathbf{Z}(r))^{\wedge}$ *is torsion, and* $\det(\delta^r)$ *is defined, then* $\mathrm{Reg}^r(X)$ *is defined and equals* $\pm\deg(\delta^r)$.

Proof. We first need a lemma.

LEMMA 5.3. *In the limit the pairings*

$$H^{2d-2r}(X, (\mathbf{Z}/m\mathbf{Z})(d-r)) \times H^{2r+1}(X, (\mathbf{Z}/m\mathbf{Z})(r))$$

$$\to H^{2d+1}(X, (\mathbf{Z}/m\mathbf{Z})(d)) = \mathbf{Z}/m\mathbf{Z}$$

define a surjective map

$$H^{2r+1}(X, \hat{\mathbf{Z}}(r)) \to \mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r)), \hat{\mathbf{Z}})$$

whose kernel is the torsion subgroup of $H^{2r+1}(X, \hat{\mathbf{Z}}(r))$.

Proof. Fix an m and choose n such that

(i) the image of $H^{2r+1}(X, (\mathbf{Z}/mn\mathbf{Z})(r))$ in $H^{2r+1}(X, (\mathbf{Z}/m\mathbf{Z})(r))$ is equal to the image of $H^{2r+1}(X, \hat{\mathbf{Z}}(r))$, and

(ii) the image of $H^{2d-2r}(X, \mathbf{Z}/mn\mathbf{Z})(d-r))$ in $H^{2d-2r}(X, (\mathbf{Z}/m\mathbf{Z})(d-r))$ is equal to the image of $H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r))$.

Let $f \in \mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r)), \hat{\mathbf{Z}})$, and let f_m be the composite of f with $\hat{\mathbf{Z}} \twoheadrightarrow \mathbf{Z}/m\mathbf{Z}$. Then f_m factors through $H^{2d-2r}(X, (\mathbf{Z}/mn\mathbf{Z})(d-r))$, and so there exists an $a_m \in H^{2r+1}(X, (\mathbf{Z}/mn\mathbf{Z})(r))$ such that $f_m(x) = \langle x, a_m\rangle$ for all $x \in H^{2d-2r}(X, (\mathbf{Z}/mn\mathbf{Z})(d-r))$. By assumption, a_m lifts to an element $a \in H^{2r+1}(X, \hat{\mathbf{Z}}(r))$, and we have that

$$f(x) \equiv \langle x, a\rangle \bmod m$$

for all $x \in H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r))$. Since m is arbitrary, this shows that the image of $H^{2r+1}(X, \hat{\mathbf{Z}}(r))$ is dense in $\mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r)), \hat{\mathbf{Z}})$. But $H^{2r+1}(X, \hat{\mathbf{Z}}(r))$ is compact, and so this proves the map is surjective. The kernel obviously contains the torsion subgroup of $H^{2r+1}(X, \hat{\mathbf{Z}}(r))$; that it contains nothing more can be seen by tensoring with $\mathbf{Z}_l$ and counting ranks.

We now prove the theorem. Note that $|\mathrm{Reg}^r(X)|$ is the determinant of a map

$$H^{2r}(X, \mathbf{Z}(r))^{\wedge} \xrightarrow{a} \mathrm{Hom}\,(H^{2d-2r}(X, \mathbf{Z}(d-r)), \mathbf{Z})^{\wedge}$$

and that, for any finitely generated abelian group M,

$$\mathrm{Hom}\,(M, \mathbf{Z})^{\wedge} \;=\; \mathrm{Hom}_{\mathrm{conts}}(\hat{M}, \hat{\mathbf{Z}}) \;=\; \mathrm{Hom}\,(M, \hat{\mathbf{Z}}).$$

Consider the diagram

$$\begin{tikzcd}
H^{2r}(X, \mathbf{Z}(r))^{\wedge} \arrow[r, "a"] \arrow[d, "\delta^r"] & \mathrm{Hom}\,(H^{2d-2r}(X, \mathbf{Z}(d-r)), \hat{\mathbf{Z}}) \arrow[d, equal] \\
TH^{2r+2}(X, \mathbf{Z}(r)) & \mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(X, \mathbf{Z}(d-r))^{\wedge}, \hat{\mathbf{Z}}) \\
H^{2r+1}(X, \hat{\mathbf{Z}}(r))/\mathrm{torsion} \arrow[u, "\approx"] \arrow[r, "\approx"] & \mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r)), \hat{\mathbf{Z}}). \arrow[u, "\approx"]
\end{tikzcd}$$

The isomorphism at left arises from (3.4.3), and uses that $H^{2r+1}(X, \mathbf{Z}(r))^{\wedge}$ is torsion; the isomorphism at the centre is that in the lemma; the isomorphism at right arises from (3.4.3), and uses that $H^{2d-2r+1}(X, \mathbf{Z}(r))_{\mathrm{div}} = 0$. In the next lemma we shall show that the diagram commutes, and it follows immediately that $\pm\det(\delta^r) = \mathrm{Reg}^r(X)$.

LEMMA 5.4. *The above diagram commutes.*

Proof. According to [Milne (1986a), 6.5], ε^{2r} is the composite of the maps

$$H^{2r}(X, \mathbf{Z}(r)) \to H^{2r}(\bar{X}, \mathbf{Z}(r)) \to H^{2r}(\bar{X}, \mathbf{Z}(r))_{\Gamma} \to H^{2r+1}(X, \mathbf{Z}(r))$$

the last of which is given by the Hochschild–Serre spectral sequence. Therefore, there is a commutative diagram

$$\begin{tikzcd}
H^{2r}(X, \mathbf{Z}(r))^{\wedge} \arrow[r] \arrow[d, "\delta^r"] & H^{2r}(X, \hat{\mathbf{Z}}(r)) \arrow[dd] \\
TH^{2r+2}(X, \mathbf{Z}(r)) & \\
H^{2r+1}(X, \hat{\mathbf{Z}}(r)) \arrow[u] & H^{2r}(X, \hat{\mathbf{Z}}(r))_{\Gamma}, \arrow[l]
\end{tikzcd}$$

and so proving the lemma is equivalent to showing that

$$\begin{tikzcd}
H^{2r}(X, \mathbf{Z}(r)) \arrow[r] \arrow[d] & \mathrm{Hom}\,(H^{2d-2r}(X, \mathbf{Z}(d-r)), \hat{\mathbf{Z}}) \arrow[d, equal] \\
H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\Gamma} \arrow[d] & \mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(X, \mathbf{Z}(d-r))^{\wedge}, \hat{\mathbf{Z}}) \\
H^{2r+1}(X, \hat{\mathbf{Z}}(r))/\mathrm{torsion} \arrow[r, "\approx"] & \mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r)), \hat{\mathbf{Z}}) \arrow[u, "\approx"]
\end{tikzcd}$$

commutes. But the pairings in the duality theorems for X and $\bar{X}$ with finite coefficients are compatible, and so this is equivalent to showing that the following diagram commutes,

$$\begin{array}{ccc} H^{2r}(X, \mathbf{Z}(r)) & \to & \mathrm{Hom}\,(H^{2d-2r}(X, \mathbf{Z}(d-r)), \hat{\mathbf{Z}}) \\ \downarrow & & \uparrow \\ H^{2r}(\bar{X}, \mathbf{Z}(r)) & \to & \mathrm{Hom}_{\mathrm{conts}}(H^{2d-2r}(\bar{X}, \hat{\mathbf{Z}}(d-r)), \hat{\mathbf{Z}}), \end{array}$$

in which the map at the right is induced by

$$H^{2d-2r}(X, \mathbf{Z}(d-r)) \to H^{2d-2r}(X, \hat{\mathbf{Z}}(d-r)) \to H^{2d-2r}(\bar{X}, \hat{\mathbf{Z}}(d-r)).$$

Axiom (A3) and (*) imply that the following diagram commutes,

$$\begin{array}{ccccccc} H^{2r}(X, \mathbf{Z}(r)) & \times & H^{2d-2r}(X, \mathbf{Z}(d-r)) & \to & H^{2d}(X, \mathbf{Z}(d)) & \xrightarrow{\deg} & \mathbf{Z} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{2r}(\bar{X}, (\mathbf{Z}/m\mathbf{Z}(r)) & \times & H^{2d-2r}(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(d-r)) & \to & H^{2d}(\bar{X}, (\mathbf{Z}/m\mathbf{Z})(d)) & \longrightarrow & \mathbf{Z}/m\mathbf{Z}, \end{array}$$

and this implies that the preceding diagram commutes.

Tate's conjecture implies that the image of $\mathrm{CH}^r(X)$ in $H^{2r}(X, \mathbf{Z}(r))$ has the same rank as $H^{2r}(X, \mathbf{Z}(r))$, and therefore it is of finite index. We write $t(X, r)$ for the order of the cokernel of

$$CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))/H^{2r}(X, \mathbf{Z}(r))_{\mathrm{tors}}.$$

Let $N^r(X)$ be the image of $CH^r(X)$ in $H^{2r}(X, \mathbf{Z}(r))$.

Corollary 5.5. *Under the hypotheses of* (5.2),

$$\det(\delta^r) = \pm \mathrm{Reg}^r(X) = \pm \det(D_i \cdot E_j) \cdot t(X, r) \cdot t(X, d-r),$$

where $\{D_i\}$ and $\{E_j\}$ are bases for $N^r(X)$ and $N^{d-r}(X)$ modulo torsion, provided $t(X, r)$ and $t(X, d-r)$ are finite.

Proof. The second equality is obvious, and the first has just been proved.

Remark 5.6. (a) It is possible that $CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$ might always be surjective. This combined with Tate's conjecture would imply that $CH^r(X) \otimes \hat{\mathbf{Z}} \to H^{2r}(X, \hat{\mathbf{Z}}(r))$ is surjective, which constitutes an integral form of Tate's conjecture.

(b) Clearly it is more attractive to take the regulator term in the definition of $\chi(X, \mathbf{Z}(r))$ to be Reg^r rather than $\det(\delta^r)$. However, for the present, $\det(\delta^r)$ is the more approachable.

(c) As the kernel of $H^{2r}(X, \mathbf{Z}(r)) \to H^{2r}(\bar{X}, \mathbf{Z}(r))$ is torsion, $N^r(X) \to A^r(X)$ is surjective with a torsion kernel. It follows that the groups $N^r(X)$ and $N^{d-r}(X)$ in the above corollary can be replaced by $A^r(X)$ and $A^{d-r}(X)$.

§6. Values of partial zeta functions

In this section, we consider the asymptotic behaviour of $P_i(X, q^{-s})$ as s approaches r. We restrict ourselves to the most interesting case where $i = 2r = \dim X$. Our goal then is to show that, under enough assumptions on the complexes $\mathbf{Z}(r)$, it is possible to state a direct generalization of the conjecture of Artin and Tate, and to prove that it is implied by Tate's conjecture. Again X is a smooth projective variety over a finite field.

Recall [Milne (1986a), §2] that $H^i(X, \mathbf{Z}_p(r))$, when regarded as a functor on the base ring, acquires a natural structure as a perfect affine group scheme whose identity component is algebraic; as in that reference, we write $s^i(r)$ for the dimension of the identity component. Let $P_i(X, t) = \Pi_j(1 - a_{ij}t)$ and

$$e^i(r) = s^i(r) - \sum_{\mathrm{ord}_q(a_{ij}) < r} (r - \mathrm{ord}_q(a_{ij})),$$

where ord_q is the p-adic valuation such that $\mathrm{ord}_q(q) = 1$.

PROPOSITION 6.1. *If $i \neq 2r$, then $H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma$ and $H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma$ are both finite, and*

$$P_i(X, q^{-r}) = \pm \frac{[H^i(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]}{[H^i(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]} q^{e^i(r)}.$$

Proof. This is [ib., 6.4].

LEMMA 6.2. *Assume that for all primes l (including $l = p$), 1 is not a multiple root of the minimal polynomial of the Frobenius element acting on $H^{2r}(\bar{X}, \mathbf{Q}_l(r))$. Then $z(\varepsilon^{2r})$ is defined and*

$$P_{2r}(X, q^{-s}) \sim \pm z(\varepsilon^{2r})^{-1} \frac{[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_\Gamma]}{[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma]} q^{e^{2r}(r)} (1 - q^{r-s})^{\rho_r} \quad \text{as } s \to r.$$

Proof. Combine [ib., 6.3] with [ib., 6.6].

LEMMA 6.3. *If $d = 2r$, then*

$$\frac{1}{[H^{2r}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}}]}\,\frac{[H^{2r-1}(\bar{X}, \hat{\mathbf{Z}}(r))_{\Gamma}]}{[H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}]} = \frac{q^{s^{2r}(r)-s^{2r+1}(r)}}{[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}_{\mathrm{tors}}]^2}$$

Proof. This is [ib., Lemma 6.7], except that there a term has been dropped.[4] The statement p. 337, line 1, that $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}}$ finite is incorrect (except for a surface). In fact it is an extension of a finite group by a unipotent group of dimension $s^{2r}(r)$. Therefore, the second line of p. 337 should be

$$\begin{aligned}[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}_{\mathrm{tor}}] &= [(H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))_{\mathrm{tor}})_{\Gamma}]q^{s^{2r}(r)} \\ &= [H^{2r+1}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}]q^{s^{2r}(r)-s^{2r+1}(r)}.\end{aligned}$$

Thus the term $q^{-s^{2r+1}(r)}$ in [ib., Lemma 6.7], should be replaced by $q^{s^{2r}(r)-s^{2r+1}(r)}$, as in the above statement.

The erroneous Lemma 6.7 of [Milne (1986a)] is used only in the proof of [ib., (0.2), (0.6)] (not (0.1) or (0.4)). In the definition of $\alpha_r(X)$ on p. 299, the term $-s^{2r}(r)$ must be replaced by $-2s^{2r}(r)$. The correct definition is as follows:

$$\alpha_r(X) = s^{2r+1}(r) - 2s^{2r}(r) + \sum_{\mathrm{ord}_q(a_{2r,j})<r} (r - \mathrm{ord}_q(a_{2r,j})).$$

PROPOSITION 6.4. *Let $d = 2r$, and assume for all l that* 1 *is not a multiple root of the minimal polynomial of the Frobenius element acting on $H^{2r}(\bar{X}, \mathbf{Q}_l(r))$. Then*

$$P_{2r}(X, q^{-s}) \sim \pm\frac{[H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}}]\det(\varepsilon^{2r})}{q^{\alpha_r(X)}[H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^{\Gamma}_{\mathrm{tors}}]^2}(1 - q^{r-s})^{\varrho_r} \quad \text{as } s \to r.$$

Proof. This is [ib., 0.2], except that we have corrected the definition of $\alpha_r(X)$.

REMARK 6.5. When $d = 2$, $\alpha_1(X) = \chi(X, \mathcal{O}_X) - 1 + \dim \mathrm{Pic\,Var}(X)$. (As $s^2(1) = 0$, the calculation for $\alpha_1(X)$ in [ib., 6.9] is unchanged; the calculation of $\alpha_2(X)$ is incorrect.)

THEOREM 6.6. *Let X be of even dimension $d = 2r$. Assume that there exists a complex $\mathbf{Z}(r)$ satisfying Axioms* A2, A3, A4 *and conditions* (*) *and* (**) *of* §5,

[4] I am indebted to J.-Y. Etesse for pointing this error out to me.

and suppose that $H^{2r+1}(X, \mathbf{Z}(r))$ is torsion. If the strong form of Tate's conjecture holds for r and all l, and the cycle map $CH^r(X) \to H^{2r}(X, \mathbf{Z}(r))$ is surjective, then

$$P_{2r}(X, q^{-s}) \sim \pm \frac{[Br^r(X)] \det (D_i \cdot D_j)}{q^{\chi_r(X)}[A^r(X)_{\mathrm{tors}}]^2}(1 - q^{r-s})^{\varrho_r} \quad \text{as } s \to r,$$

where $Br^r(X) = H^{2r+1}(X, \mathbf{Z}(r))$, $A^r(X)$ is the image of $CH^r(X)$ in $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))$, and *$\{D_i\}$ is a basis for $A^r(X)$ modulo torsion.*

Proof. From (4.2) we know that $H^{2r+1}(X, \mathbf{Z}(r))'$ is finite, and that $\det(\varepsilon^{2r}) = \det(\delta^r)$. Tate's conjecture implies that the divisible subgroup of $H^{2r+1}(X, \mathbf{Z}(r))$ is zero, and so $H^{2r+1}(X, \mathbf{Z}(r))$ is finite. Thus we can replace the term $\det(\varepsilon^{2r})$ in (6.4) with $\det(\delta^r)$, and even (see 5.5) with $\det(D_i \cdot D_j)$. By assumption, the map $CH^r(X) \to H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma$ has dense image, and so $H^{2r}(\bar{X}, \hat{\mathbf{Z}}(r))^\Gamma_{\mathrm{tors}} = A^r(X)_{\mathrm{tors}}$. Finally, $H^{2r+1}(X, \mathbf{Z}(r)) \xrightarrow{\approx} H^{2r+1}(X, \hat{\mathbf{Z}}(r))_{\mathrm{tors}}$ because $H^{2r+2}(X, \mathbf{Z}(r))_{\mathrm{tors}}$ is finite.

REMARK 6.7. (a) Note that $Br^1(X) = H^3(X, \mathbf{Z}(1)) = H^2(X, \mathbf{G}_m)$, which is the cohomological Brauer group of X. This explains our notation: $Br^r(X)$ behaves as a higher (cohomological) Brauer group of X.

(b) By definition, $A^1(X)$ is the image of $\mathrm{Pic}(X)$ in $H^2(\bar{X}, \hat{\mathbf{Z}}(1))$. Since the kernel of $\mathrm{Pic}(\bar{X}) \to H^2(\bar{X}, \hat{\mathbf{Z}}(1))$ is precisely its divisible subgroup $\mathrm{PicVar}(\bar{X})$, we see that $A^1(X)$ is the image of $\mathrm{Pic}(X)$ in $NS(\bar{X})$ (the group of divisors $\bar{X}$ modulo algebraic equivalence). According to the definition (see [Tate (1965/66), §4]), this is $NS(X)$.

(c) The above remarks, together with (6.5), show that when X is a surface the statement in (6.6) is precisely the original conjecture of Artin and Tate [ib., Conjecture C].

(d) The hypotheses in the statement of the theorem can be weakened at the cost of obtaining a less pleasant formula.

REMARK 6.8. Let $\{\pi^i\}$ be the Künneth components of the diagonal of $X \times X$. According to [Katz and Messing (1974)] the π^i are algebraic, and so $X^i \stackrel{\mathrm{df}}{=} (X, \pi^i)$ is a ($\mathbf{Q}$-linear) motive. Moreover, $\zeta(X^i, s) = P_i(X, q^{-s})^{\pm 1}$. Is it possible to state a conjecture for the values of $\zeta(X^i, s)$ purely in terms of X^i? For $\mathbf{Z}$-linear motives, we discuss this in the next section.

§7. Examples; motives

In this section we study Conjecture 0.1 using the axioms relating $\mathbf{Z}(r)$ to the K-sheaves rather than to the sheaves $\mu_m^{\otimes r}$ and $\nu_n(r)$. First recall the calculations made in [Milne (1986a), §10].

PROPOSITION 7.1. *Let X be a smooth projective variety of dimension d over a finite field.*

(a) $\zeta(X, s)q^{\chi(X,\mathcal{O},d-r)} \sim \zeta(X, d - s)q^{\chi(X,\mathcal{O},r)}$ as $s \to r$.

(b) *For $r > d$, $\zeta(X, r) = u \cdot q^{\chi(X,\mathcal{O},r)}$ where u is a p-adic unit.*

Proof. See [ib., 10.2, 10.4].

When combined with the Conjecture, (a) and (b) respectively predict that $\chi(X, \mathbf{Z}(r)) = \chi(X, \mathbf{Z}(d - r))$ and, when $r > d$, $|\chi(X, \mathbf{Z}(r))|_p = 1$. The first equality is also predicted by the conjectured duality between the groups $H^r(X, \mathbf{Z}(r))$ and $H^{d-r}(X, \mathbf{Z}(d - r))$, and the second by Axiom B2 and the conjecture that $\mathscr{K}_r, \ldots, \mathscr{K}_{2r-1}$ should be uniquely divisible by p when $r > d$.

Finite fields

When $d = 0$, we use the calculation [Quillen (1972)] of the K-groups of a finite field and its algebraic closure to verify Conjecture 0.1 up to small primes.

THEOREM 7.2. *Let $X =$ Spec k, k a finite field. If $\mathbf{Z}(r)$ is a complex on X_{et} satisfying Axioms* A1 *and* B2, *then*

$$\zeta(k, s) \sim \chi(X, \mathbf{Z}(0)) \cdot q^{\chi(X,\mathcal{O},0)}(1 - q^{-s})^{-1} \quad \textit{as } s \to 0,$$

and

$$\zeta(k, r) = u \cdot \chi(X, \mathbf{Z}(r)) \cdot q^{\chi(X,\mathcal{O},r)} \quad \textit{for } r > 0,$$

where u is a rational number involving only primes $l < r$.

Proof. For $r = 0$, Axiom B2 implies that $\mathbf{Z}(r)$ is quasi-isomorphic to $\mathbf{Z}$. Therefore the first assertion is easy to verify, and in fact is a very special case of [Milne (1986a), 0.4a]. Henceforth, we assume that $r > 0$.

The zeta function of k is $1/(1 - q^{-s})$; therefore

$$\zeta(k, r) = q^r/(q^r - 1)$$

for $r \neq 0$.

The term $\chi(X, \mathcal{O}, r) = r \cdot \dim_k H^0(X, \mathcal{O}_X) = r$.

From [Quillen (1972)] we find that

$$K_{2r}\bar{k} = 0, \qquad K_{2r-1}\bar{k} = (\mathbf{Q}/\mathbf{Z})_{\text{non-}p},$$

and that Frob acts on $K_{2r-1}\bar{k}$ as q^r. An easy calculation now shows that $H^0(k, \mathscr{K}_{2r-1})$ has order $q^r - 1$, and $H^i(k, \mathscr{K}_{2r-1}) = 0$ for $i > 0$. Thus

$$\zeta(k, r) = q^r/(q^r - 1) = q^{\chi(X, \mathscr{O}, r)}/\chi(X, \mathscr{K}_{2r-1}),$$

and it remains to relate $\chi(X, \mathscr{K}_{2r-1})$ to $\chi(X, \mathbf{Z}(r))$. Note that Axiom B2 implies that

$$\chi(X, \mathbf{Z}(r)) = \chi(X, gr_\gamma^r \mathscr{K}_{2r-1})^{-1} \cdot \chi(X, gr_\gamma^r \mathscr{K}_{2r-2}) \cdots \chi(X, gr_\gamma^r \mathscr{K}_{2r})^{\pm 1}$$

up to factors involving primes $< r$.

Consider $gr_\gamma^r K_{2r-i}\bar{k}$. According to [Soulé (1985), p. 493]), the sth Adams operator ψ^s acts on $gr_\gamma^r K_{2r-i}\bar{k}$ as s^r. But (see [Kratzer (1970), 7.2]), ψ^s acts on $K_{2r-i}\bar{k}$ as s^i for all k. Therefore $(s^r - s^i)gr_\gamma^r \mathscr{K}_{2r-i} = 0$, and so $gr_\gamma^r \mathscr{K}_{2r-i} = 0$ for $i \neq 1$ and $gr_\gamma^r \mathscr{K}_{2r-1} = \mathscr{K}_{2r-1}$ (up to torsion involving primes $< r$). The result is now obvious.

Motives

Let $V(k)$ be the category of smooth projective varieties over k, and let $M(k)$ be the category of motives over k constructed using the integral Chow theory (see for example [Soulé (1984), §1]); thus $M(k)$ is a $\mathbf{Z}$-linear category. We write h: $V(k)^{\text{opp}} \to M(k)$ for the functor taking a variety to its associated motive, and we write L for the Lefschetz motive. Thus $h(\mathbf{P}^1) = 1 + L$, and $M \otimes L = M(-1)$.

We now assume that the definition of the motivic cohomology groups extends to $\mathbf{Z}$-linear motives, and that the resulting groups have the expected properties. For example, we should have $H^i(M \otimes L, \mathbf{Z}(r)) = H^{i-2}(M, \mathbf{Z}(r - 1))$.

The conjecture in the introduction has a natural extension to motives. Define $\chi(M, \mathbf{Z}(r))$ as in the case of a variety, taking the regulator to be the determinant of the map

$$\delta^r: H^{2r}(M, \mathbf{Z}(r)) \to TH^{2r+2}(M, \mathbf{Z}(r)),$$

considered in §4. Let ϱ_r be the order of the pole of $\zeta(M, s)$ at $s = r$, and define $\chi(M, \mathscr{O}, r)$ as for varieties. The definition of $\chi(X, \mathscr{O}, r)$ extends to motives.

CONJECTURE 7.3. *For any motive M over a finite field, $\chi(M, \mathbf{Z}(r))$ is defined and*

$$\zeta(M, s) \sim \pm\chi(M, \mathbf{Z}(r))q^{\chi(X, \mathscr{O}, r)}(1 - q^{r-s})^{\varrho_r} \quad \textit{as } s \to r.$$

Note that $\zeta(M \otimes L, s) = \zeta(M, s-1)$, $\chi(M \otimes L, \mathbf{Z}(r)) = \chi(M, \mathbf{Z}(r-1))$, and $\chi(M \otimes L, \mathcal{O}, r) = \chi(M, \mathcal{O}, r-1)$, and so Conjecture 7.3 is true for M if and only if it is true for $M \otimes L$.

Motives of weight zero

Let ϱ: Gal $(\bar{k}/k) \to$ Aut (M) be a representation (always assumed to be continuous) of Gal $(\bar{k}/k)$ on a free $\mathbf{Z}$-module M of finite rank. Write $\bar{k}^M$ for the k-algebra Hom $(M, \bar{k})$. Then Spec $\bar{k}^M$ is a finite disjoint union of copies of Spec $\bar{k}$ on which Gal $(\bar{k}/k)$ acts through its action on $\bar{k}$ and M, and so $\mathbf{Z}^M(r) \overset{\text{df}}{=} \Gamma(\text{Spec}\,(\bar{k}^M), \mathbf{Z}(r))$ is an object in the derived category of Gal $(\bar{k}/k)$-modules. Define

$$H^i(\varrho, \mathbf{Z}(r)) \;=\; H^i(\text{Gal}\,(\bar{k}/k), \mathbf{Z}^M(r)).$$

From Axiom B2 we obtain a spectral sequence

$$H^i(\text{Gal}\,(\bar{k}/k), gr^r_\gamma K_{2r-j}\bar{k}) \Rightarrow H^{i+j}(\varrho, \mathbf{Z}(r)).$$

Following [Coombes (1987a)], we define a *continuous pure motive of weight* 0 to be a motive M such that

(a) $H^i(\bar{M}, \mathbf{Q}_l) = 0$ for $i \neq 0, l \neq p$;

(b) the natural map End $(\bar{M}) \to$ End $(H^0(\bar{M}, \mathbf{Q}_l))$ is injective on $\mathbf{Z}$[Frob], $l \neq p$:

(c) the representation of Gal $(\bar{k}/k)$ on $H^i(\bar{M}, \mathbf{Q}_l)$ arises from a representation ϱ of Gal $(\bar{k}/k)$ on a $\mathbf{Z}$-module of finite rank, and ϱ is independent of l.

(d) Frob has finite order.

PROPOSITION 7.4. *Let M be a continuous pure motive of weight* 0, *and let ϱ be the associated integral representation. Under the above assumptions, $H^i(M, \mathbf{Z}(r)) = H^i(\varrho, \mathbf{Z}(r))$ up to torsion by primes $l < r$ and by p.*

Proof. In [Coombes (1987a), 2.7] it is shown that $K_r(M) \approx K_r(\varrho)$, and so this follows from the spectral sequences relating the motivic cohomology to the K-cohomology.

REMARK 7.5. If M is split by an extension of degree prime to p, then the proposition also holds for the p-torsion.

Rational surfaces, projective bundles, and blow-ups

PROPOSITION 7.6. *Under the above assumptions, the following hold.*

(a) *If X is a rational surface, then*

$$\zeta(k, s) \sim u \cdot \chi(X, \mathbf{Z}) \cdot q^{\chi(X, \mathcal{O}_X, r)} \cdot (1 - q^{r-s})^{\varrho_r} \quad \text{as } s \to r$$

where u is a rational number divisible only by primes $<r$ and by p; if the representation of Gal $(\bar{k}/k)$ *on $NS(\bar{X})$ becomes trivial after an extension of degree prime to p, then u is a p-adic unit.*

(b) *Let $Y = \mathbf{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X; if* (0.1) *is true for X, then it is true for Y.*

(c) *Let Y be the blow up of X along a smooth subvariety Z; if* (0.1) *is true for X and Z, then it is true for Z.*

Proof. Use (7.2) and (7.4), plus the decompositions:

(a) $h(X) = 1 + M \otimes L + L^2$, M a continuous pure motive of weight 0;

(b) $h(Y) = h(X) + h(X)(-1) + \cdots + h(X)(-m)$;

(c) $h(Y) = h(X) + \bigoplus_{r=1}^{c-1} h(Z)(-r)$.

COROLLARY 7.7. *Conjecture* 0.1 *is true for $\mathbf{P}^n$ up to small torsion.*

REMARK 7.8. (a) By using [Coombes (1987b), §3] one can prove a similar result to (a) for Enriques surfaces.

(b) Quillen has conjectured that for an affine curve U over a finite field k,

$$K_{2r-2} \otimes \mathbf{Z}_l = H^2(U, \mathbf{Z}_l(r))$$

$$K_{2r-1} \otimes \mathbf{Z}_l = H^1(U, \mathbf{Z}_l(r)).$$

As is explained in [Coombes (1987a)], this conjecture implies that

$$\zeta(X, r) = \chi(X_{\mathrm{Zar}}, \mathscr{K}_{2r-1})^{-1}\chi(X_{\mathrm{Zar}}, \mathscr{K}_{2r-2})$$

for $r > 1$ up to a power of p. I do not see how to pass from there to the conjecture

$$\begin{aligned}\zeta(X, r) &= \chi(X, \mathbf{Z}(r)) \\ &= \chi(X_{\mathrm{et}}, gr_\gamma^r\mathscr{K}_{2r-1})^{-1}\chi(X_{\mathrm{et}}, gr_\gamma^r\mathscr{K}_{2r-2}) \cdots \chi(X_{\mathrm{et}}, gr_\gamma^r\mathscr{K}_r)^{\pm 1}\end{aligned}$$

up to a power of p. By looking at the actions of the Adams operators on the sheaves $gr_\gamma^r\mathscr{K}_{2r-i}$ one can see that the terms $gr_\gamma^r\mathscr{K}_{2r-i}$, $i > 2$, only contribute powers of small primes, but this leaves the problem of passing from the étale cohomology groups to the Zariski cohomology groups. It has been suggested that

$$\tau_{\leqslant r} Rf_*\mathbf{Z}(r) = \mathbf{Z}_B(r) = f_*\mathbf{Z}(r)$$

(see 2.6). If this is so, then

$$\begin{aligned}\chi(X_{\mathrm{et}}, \mathbf{Z}(r)) &= \chi(X_{\mathrm{Zar}}, Rf_*\mathbf{Z}(r))\\ &= \chi(X_{\mathrm{Zar}}, \mathbf{Z}(r)|X_{\mathrm{Zar}})\cdot\chi(X_{\mathrm{Zar}}, \tau_{>r}Rf_*\mathbf{Z}(r))\end{aligned}$$

where $\tau_{>r}Rf_*\mathbf{Z}(r)$ is the mapping cone of $\tau_{\leqslant r}Rf_*\mathbf{Z}(r) \to Rf_*\mathbf{Z}(r)$. Perhaps the contribution of $\chi(X_{\mathrm{Zar}}, \tau_{>r}Rf_*\mathbf{Z}(r))$ is small.

§8. Corrections to [Milne (1986a)]

Apart from that noted in §6 above, there are the following minor errors in [Milne (1986a)].

p. 298: in the statement of $SS(X, r, l)$, the group should be $H^{2r}(\bar{X}, \mathbf{Q}_l(r))$.

p. 299: $\beta_i = \dim_{\mathbf{Q}_l} H^i(\bar{X}, \mathbf{Q}_l)$.

p. 301: in the statement of Theorem 0.4a, the last term should be $(1 - q^{-s})^{-1}$.

p. 310: A step has been dropped from the proof of Proposition 1.15. The degeneracy of the spectral sequence shows only that

$$H^i_{\mathrm{crys}}(X/W) \otimes \mathbf{Q} \approx \bigoplus_{s+t=i} H^s(X, W\Omega^t_X) \otimes \mathbf{Q}_p.$$

The action of F on the left corresponds to $p^t F$ on $H^s(X, W\Omega^t_X) \otimes \mathbf{Q}_p$. We need to look at

$$p^t F - p^r\colon H^s(X, W\Omega^t_X) \otimes \mathbf{Q}_p \to H^s(X, W\Omega^t_X) \otimes \mathbf{Q}_p.$$

As we note on p. 311, $1 - p^j F$ is an automorphism of $W\Omega^t$ for all $j \geqslant 1$, and so the terms with $r > t$ contribute nothing to the kernel or cokernel. If $r < t$, then we need to consider

$$1 - p^{t-r}F\colon H^s(X, W\Omega^t_X) \otimes \mathbf{Q}_p \to H^s(X, W\Omega^t_X) \otimes \mathbf{Q}_p,$$

but $H^s(X, W\Omega^t_X)$/torsion is a free $\mathbf{Z}_p$-submodule of finite rank stable under F, and therefore it is obvious that $1 - p^{t-r}F$ defines an automorphism of it. Therefore the only term that contributes to either the kernel or cokernel is $H^{i-r}(X, W\Omega^r_X) \otimes \mathbf{Q}_p$, and so the rest of the proof applies as before.

p. 324: The alternative proof of the equality $s^3(1) = d^2(0)$ presented in (3.4), while essentially correct, is a little too slick.

p. 339: In the first exact sequence, an underline has been omitted.

Note (added March, 1988). In a recent preprint (New results on weight-two motivic cohomology), S. Lichtenbaum makes important improvements to his earlier results on $\mathbf{Z}(2)$. In particular he proves (up to 2-torsion):

(i) Axiom C1 (the Gersten sequence) for $\mathbf{Z}(2)$ when X is a regular scheme over a field;

(ii) Axiom C2 (purity) for $\mathbf{Z}(2)$ under the same condition on X.

These two results allow him to show that there exist cycle maps $CH^2(X) \to H^4(X, \mathbf{Z}(2))$ compatible with the cycle maps into $H^4(X, \mathbf{Z}_l(2))$ for all l (including $l = p$). Thus Theorem 4.3 of this paper now shows that if $\chi(X, \mathbf{Z}(2))$ is defined, then

$$\zeta(X, 2) \sim \chi(Z, \mathbf{Z}(2))q^{\chi(X, \mathscr{O}, 2)}(1 - q^{2-s})^{-\varrho_2} \quad \text{as } s \to 2,$$

(up to 2-torsion) when X is a smooth projective variety over a finite field. Moreover, $\chi(X, \mathbf{Z}(2))$ is defined whenever the groups $H^i(X, \mathbf{Z}(2))$ have no uniquely divisible subgroups and $Br^2(X)$ ($\overset{\text{df}}{=} H^5(X, \mathbf{Z}(2))$) is finite.

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