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The action of complex conjugation on a Shimura variety

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In the case that a Shimura variety has a real canonical model, complex conjugation defines an involution of the set of complex points of the variety. It is necessary to have an explicit description of this involution in order, for example, to compute the zeta function of the variety. Langlands conjectured such a description in [1, p. 417-18] and our purpose is to prove his conjecture for all Shimura varieties of abelian type. (This class of Shimura varieties is defined in § 1 of this paper; it contains all those whose canonical model is known, at the time of writing, to exist.)

Recall that a Shimura variety $\mathrm{Sh}(G, X)$ is defined by a $\mathbf{Q}$-rational reductive group G and a family X of homomorphisms $\mathbf{C}^{\times} \to G(\mathbf{R})$ satisfying certain conditions. Initially $\mathrm{Sh}(G, X)$ is defined as a complex variety but is expected to have a model over a certain number field $E(G, X)$ called the reflex field. A canonical model for $\mathrm{Sh}(G, X)$ is a variety $M(G, X)$ over $E(G, X)$ satisfying certain conditions sufficient to determine it uniquely. Assume that $E(G, X)$ is real and that the canonical model exists so that complex conjugation defines an involution θ of $\mathrm{Sh}(G, X)$. When the canonical model is a moduli variety, and so has a direct description, the proof of the conjecture is straightforward. This is not usually the case, and just as the construction of the canonical model is intricate in general, so must be the proof of the conjecture. In particular, it must involve an analogous assertion for connected Shimura varieties. Such an assertion is proved in Shih [1] for connected Shimura varieties that are of primitive abelian type C, and this result is the starting point of our proof of the conjecture for all Shimura varieties of abelian type. (Since θ does not preserve the connected component the analogous assertion takes on quite a different form from the original; it becomes rather a statement about the action of a "negative" element of $G(\mathbf{Q})$.)

To see how our result relates to the zeta function, consider an arbitrary

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variety V over a number field E. For a complex infinite prime $v: E \hookrightarrow \mathbf{C}$ of E the Hodge structure on $H^i(V \otimes_{E,v} \mathbf{C}, \mathbf{Q})$ defines a representation ρ^i of $\mathbf{C}^\times$, which we can regard as a representation of the Weil group $W_{\mathbf{C}}$. For a real prime v the involution of $H^i(V \otimes \mathbf{C}, \mathbf{Q})$ induced by complex conjugation enables one to define a representation of the Weil group $W_{\mathbf{R}}$. In either case the factor $Z_v(V, s)$ of the zeta function corresponding to v is defined to be the alternating product of the L-series $L(s, \rho^i)$. Thus in order to compute the factors at infinity of the zeta function of a Shimura variety one must compute its cohomology and also the involution induced by complex conjugation (in the case of a real prime). The first of these is a problem in continuous cohomology, and our theorem reduces the second also to a problem in continuous cohomology when v is the prime corresponding to the canonical embedding of $E(G, X)$ in $\mathbf{C}$. For the other infinite primes one needs a second conjecture of Langlands concerning the conjugate of a Shimura variety, which we prove in another paper [2]. See Langlands [3, § 7] where the assumption is made that both conjectures are true.

In Section 1 of the paper we review the basic definitions concerning Shimura varieties and introduce the notion of a Shimura variety of abelian type. The following section is largely concerned with the statement of a conjecture (conjecture CM) that describes how any automorphism of $\mathbf{C}$ acts on an abelian variety of CM-type and its points of finite order; it is therefore a strengthening of the main theorem of complex multiplication. (Conjecture CM is discussed at greater length in Milne-Shih [2].) We state in Section 3 the conjecture of Langlands (conjecture B) which it is our purpose to prove, and also begin the proof by finding an explicit description of the action of θ on the set $\pi_0(\mathrm{Sh}(G, X))$ of connected components of $\mathrm{Sh}(G, X)$. This allows us in Sections 4 and 5 to formulate a conjecture (conjecture B^0) for connected Shimura varieties and to show it to be equivalent to conjecture B. In Section 6 we show that conjecture B^0 is equivalent to a special case of conjecture CM. We begin, in Section 7, by reviewing the proof in Shih [1] of conjecture B^0 for Shimura varieties that are primitive of type C. From this result we are able to deduce the special case of conjecture CM, and hence also conjecture B^0 for all Shimura varieties of abelian type. From this, conjecture B follows. In an appendix we provide a brief, but explicit, description of the Shimura varieties of primitive abelian type.

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Notations and conventions

For Shimura varieties and algebraic groups we generally follow the notations of Deligne [2]. Thus a reductive algebraic group G is always connected, with derived group G^{der}, adjoint group G^{ad}, and center $Z = Z(G)$. A central extension is an epimorphism $G \to G'$ whose kernel is contained in $Z(G)$, and a covering is a central extension such that G is connected and the kernel is finite. If G is reductive, then $\rho: \tilde{G} \to G^{\mathrm{der}}$ is the universal covering of G^{der}.

A superscript $+$ refers to a topological connected component; for example $G(\mathbf{R})^+$ is the identity connected component of $G(\mathbf{R})$ relative to the real topology, and $G(\mathbf{Q})^+ = G(\mathbf{Q}) \cap G(\mathbf{R})^+$. For G reductive, $G(\mathbf{R})_+$ is the inverse image of $G^{\mathrm{ad}}(\mathbf{R})^+$ in $G(\mathbf{R})$ and $G(\mathbf{Q})_+ = G(\mathbf{Q}) \cap G(\mathbf{R})_+$. In contrast to Deligne [3], we use the superscript $\wedge$ to denote both completions and closures since we wish to reserve the superscript $-$ for certain negative components.

We write $\mathrm{Sh}(G, X)$ for the Shimura variety defined by a pair (G, X) and $\mathrm{Sh}^0(G, G', X^+)$ for the connected Shimura variety defined by a triple (G, G', X^+). The canonical model of $\mathrm{Sh}(G, X)$ is denoted by $M(G, X)$.

Vector spaces are finite-dimensional, number fields are of finite degree over $\mathbf{Q}$ (and usually contained in $\mathbf{C}$), and $\bar{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. If V is a vector space over $\mathbf{Q}$ and R is a $\mathbf{Q}$-algebra, we often write $V(R)$ for $V \otimes R$.

We write $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/m\mathbf{Z}$, $\mathbf{A}^f = \mathbf{Q} \otimes \hat{\mathbf{Z}}$ for the ring of finite adèles of $\mathbf{Q}$, and $\mathbf{A} = \mathbf{R} \times \mathbf{A}^f$ for the ring of adèles of $\mathbf{Q}$. For E a number field, $\mathbf{A}_E^f$ and $\mathbf{A}_E$ denote $E \otimes_{\mathbf{Q}} \mathbf{A}^f$ and $E \otimes \mathbf{A}$. The group of idèles of E is $\mathbf{A}_E^\times$ and the idèle class group is $C_E = \mathbf{A}_E^\times / E^\times$.

If A is an abelian variety, $A_n = \ker(n: A \to A)$, $TA = \varprojlim A_n$, and $V^f(A) = \mathbf{Q} \otimes TA$. Throughout the paper an abelian variety A will be systematically confused with its isogeny class; thus only $V^f(A)$ (not TA), $H_r(A, \mathbf{Q})$ (not $H_r(A, Z)$), and $H^r(A_{\mathrm{et}}, \mathbf{Q}_l)$ (not $H^r(A_{\mathrm{et}}, \mathbf{Z}_l)$) are defined, and $\mathrm{Hom}(A, B)$ means $\mathrm{Hom}(A, B) \otimes \mathbf{Q}$.

Complex conjugation is denoted by $z \mapsto \iota z$.

We use $[*]$ to denote an equivalence class containing $*$; for example, if $x \in X$ and $g \in G(\mathbf{A}^f)$ then $[x, g]$ denotes the element of $\mathrm{Sh}(G, X) = G(\mathbf{Q}) \backslash X \times G(\mathbf{A}^f)/Z(\mathbf{Q})^\wedge$ containing (x, g). The Hecke operator $[x, g] \mapsto [x, gg']$ is denoted by $\mathcal{T}(g')$.

We normalize the reciprocity isomorphism of class field theory so that a uniformizing parameter corresponds to the reciprocal of the (arithmetic) Frobenius element; we thus agree with Deligne [2] and Tate [1], but disagree with Langlands [3].

We write S for the Weil restriction of scalars of $\mathbf{G}_m$ from C to $\mathbf{R}$. Thus $S_{\mathbf{C}} \approx \mathbf{G}_m \times \mathbf{G}_m$, and we associate with any homomorphism h from S into an algebraic group G over $\mathbf{R}$ the cocharacter μ_h of $G_{\mathbf{C}}$ that is the restriction of h to the first factor. For notations on Hodge structures we follow Deligne [2]; in particular a $\mathbf{Q}$-rational Hodge structure on a vector space V over $\mathbf{Q}$ is a homomorphism $h\colon S \to GL(V_{\mathbf{R}})$.

1. Shimura varieties of abelian type

A *Shimura variety* $\mathrm{Sh}(G, X)$ is defined by a pair (G, X), comprising a reductive group G over $\mathbf{Q}$ and a $G(\mathbf{R})$-conjugacy class X of homomorphisms $S \to G_{\mathbf{R}}$, that satisfies the following axioms:

(1.1a) The Hodge structure defined on $\mathrm{Lie}(G_{\mathbf{R}})$ by any $h \in X$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$.

(1.1b) For any $h \in X$, $\mathbf{ad}\, h(i)$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbf{R}}$.

(1.1c) The group G^{ad} has no factor defined over $\mathbf{Q}$ whose real points form a compact group.

Then $\mathrm{Sh}(G, X)$ has complex points $G(\mathbf{Q})\backslash X \times G(\mathbf{A}^f)/Z(\mathbf{Q})^{\wedge}$, where Z is the center of G and $Z(\mathbf{Q})^{\wedge}$ the closure of $Z(\mathbf{Q})$ in $Z(\mathbf{A}^f)$.

A *connected Shimura variety* $\mathrm{Sh}^0(G, G', X^+)$ is defined by a triple (G, G', X^+) comprising an adjoint group G over $\mathbf{Q}$, a covering G' of G, and a $G(\mathbf{R})^+$-conjugacy class of homomophisms $S \to G_{\mathbf{R}}$ such that G and the $G(\mathbf{R})$-conjugacy class of X containing X^+ satisfy (1.1). The topology $\tau(G')$ on $G(\mathbf{Q})$ is that for which the images of the congruence subgroups of $G'(\mathbf{Q})$ form a fundamental system of neighborhoods of the identity, and $\mathrm{Sh}^0(G, G', X^+)$ has complex points $\varprojlim \Gamma\backslash X^+$ where Γ runs over the arithmetic subgroups of $G(\mathbf{Q})^+$ that are open relative to the topology $\tau(G')$ (Deligne [2, 2.1.8]).

The relation between the two notions of Shimura variety is as follows: let (G, X) be as in the first paragraph and let X^+ be some connected component of X; then X^+ can be regarded as a $G^{\mathrm{ad}}(\mathbf{R})^+$-conjugacy class of maps $S \to G^{\mathrm{ad}}_{\mathbf{R}}$ and $\mathrm{Sh}^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ can be identified with the connected component of $\mathrm{Sh}(G, X)$ that contains the image of $X^+ \times \{1\}$.

We recall that the reflex field $E(G, X)$ of (G, X) is the subfield of C that is the field of definition of the $G(\mathbf{C})$-conjugacy class of μ_h, any $h \in X$, and that $E(G, X^+)$ is defined to equal $E(G, X)$ if X^+ is a connected component of X (Deligne [2, 2.2.1]).

The following easy lemma will be needed in comparing the Shimura varieties defined by (G, X) and $(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$.

LEMMA 1.2. *Let $G_1 \to G$ be a central extension of reductive groups over* $\mathbf{C}$; *let M be a $G(\mathbf{C})$-conjugacy class of homomorphisms $\mathbf{G}_m \to G$ and let M_1 be a $G_1(\mathbf{C})$-conjugacy class lifting M. Then $M_1 \to M$ is bijective.*

Let (G, X) be as in (1.1) with G adjoint and $\mathbf{Q}$-simple; if every $\mathbf{R}$-simple factor of $G_{\mathbf{R}}$ is of one of the types A, B, C, $D^{\mathbf{R}}$, $D^{\mathbf{H}}$, or E (in the sense of Deligne [2, 2.3.8]; see also the appendix), then G will be said to be of that type. When G' is a covering of G, we say that (G, G') (or (G, G', X)) is of *primitive abelian type* if G is of type A, B, C, or $D^{\mathbf{R}}$ and G' is the universal covering of G, or if G is of type $D^{\mathbf{H}}$ and G' is the double covering described in Deligne [2, 2.3.8] (see also the appendix).

Notations 1.3. If (G, X) satisfies (1.1) and G is adjoint and $\mathbf{Q}$-simple, then there are a totally real number field F_0 and absolutely simple group G^s over F_0 such that $G = \mathrm{Res}_{F_0/\mathbf{Q}} G^s$. For any embedding $v: F_0 \hookrightarrow \mathbf{R}$, let $G_v = G^s \otimes_{F_0, v} \mathbf{R}$, and write I_c and I_{nc} for the sets of embeddings for which $G_v(\mathbf{R})$ is compact and noncompact. Let F be a quadratic totally imaginary extension of F_0 and let $\Sigma = (\sigma_v)_{v \in I_c}$ be a set of embeddings $\sigma_v: F \hookrightarrow \mathbf{C}$ such that $\sigma_v | F_0 = v$; we define h_Σ to be the Hodge structure on F (regarded as a vector space over $\mathbf{Q}$) such that $(F \otimes_{\mathbf{Q}} \mathbf{C})^{-1,0}$, $(F \otimes_{\mathbf{Q}} \mathbf{C})^{0,-1}$, and $(F \otimes_{\mathbf{Q}} \mathbf{C})^{0,0}$ are the direct summands of $F \otimes_{\mathbf{Q}} \mathbf{C} = \mathbf{C}^{\mathrm{Hom}(F,\mathbf{C})}$ corresponding to Σ, $\iota\Sigma$, and $\{\sigma: F \hookrightarrow \mathbf{C} \mid \sigma | F_0 \in I_{nc}\}$.

PROPOSITION 1.4. *Let G be a $\mathbf{Q}$-simple adjoint group and assume that (G, G', X) is of primitive abelian type. For any pair (F, Σ) as above there exists a diagram*

$$(G, X) \longleftarrow (G_1, X_1) \hookrightarrow (C\mathrm{Sp}(V), S^{\pm})$$

such that $G_1^{\mathrm{ad}} = G$, $G_1^{\mathrm{der}} = G'$, and $E(G_1, X_1) = E(G, X) E(F^\times, h_\Sigma)$.

Proof. This is Deligne's [2, 2.3.10].

Remark 1.5(a). We shall need a supplement to the proposition. Consider an h in X that is special, and so factors through $T_{\mathbf{R}}$ where T is a $\mathbf{Q}$-rational maximal torus in G. The inverse image of T in G_1 is a $\mathbf{Q}$-rational maximal torus $T_1 \subset G_1$, and h lifts to an h_1 in X_1 factoring through T_1. We claim that $E(T_1, h_1) = E(T, h) E(F^\times, h_\Sigma)$.

To see this, first note that it is obvious that $E(T_1, h_1) \supset E(T, h)$ and $E(T_1, h_1) \supset E(G_1, X_1)$. As the proposition shows that $E(G_1, X_1) \supset E(F^\times, h_\Sigma)$, we see that $E(T_1, h_1) \supset E(T, h) E(F^\times, h_\Sigma)$. For the reverse inclusion, let σ be an automorphism of $\bar{\mathbf{Q}}$ fixing $E(T, h)$ and $E(F^\times, h_\Sigma)$, and let μ_1, μ, and μ_Σ be the cocharacters of T_1, T, and $F^\times$ corresponding to h_1, h, and h_Σ. Since σ fixes $E(T, h)$, it fixes $E(G, X)$, and the proposition shows that it fixes

$E(G_1, X_1)$; thus $\sigma\mu_1$ is $G_1(\mathbf{C})$-conjugate to μ_1. But $\sigma\mu = \mu$; that is, $\sigma\mu_1$ and μ_1 map to the same cocharacter of G. Thus (1.2) shows they are equal, and σ fixes $E(T_1, h_1)$.

(b) The vector space V constructed in (1.4) has the structure of a vector space over F, and so h_Σ can be regarded as a map $\mathrm{S} \to \mathrm{GL}(V)$.

Let (G, X) satisfy (1.1) with G adjoint, and let G' be a covering of G. We say that (G, G') or (G, G', X) is of *abelian type* if there exist pairs $(G_i, G_i')_i$ of primitive abelian type such that $G = \prod G_i$ and G' is a quotient of the covering $\prod G_i'$ of $\prod G_i$. If (G, X) satisfies (1.1), we say that G or (G, X) is of *abelian type* if $(G^{\mathrm{ad}}, G^{\mathrm{der}})$ is of this type. Finally, we say that a Shimura variety $\mathrm{Sh}^0(G, G', X^+)$ or $\mathrm{Sh}(G, X)$ is of *abelian type* if (G, G') or G is.

Remark 1.6(a). Let (G, G', X) be of primitive abelian type; then the diagram in (1.4) induces a diagram

$$\mathrm{Sh}^0(G, G', X^+) \xleftarrow{\approx} \mathrm{Sh}^0(G_1^{\mathrm{ad}}, G_1^{\mathrm{der}}, X_1^+) \hookrightarrow \mathrm{Sh}^0(\mathrm{Sp}(V)^{\mathrm{ad}}, \mathrm{Sp}(V), S^+)\,.$$

Thus $\mathrm{Sh}^0(G, G', X^+)$ can be regarded (in many different ways) as the parameter space for a family of abelian varieties.

(b) Essentially, the Shimura varieties of abelian type are those that are accessible to study by Shimura's original methods. They exclude those for which G^{ad} has factors over $\mathbf{Q}$ that are of exceptional type or of mixed type D and, when G^{ad} has factors of type D^{H}, those for which G^{der} is too large. To say that $\mathrm{Sh}(G, X)$ is of abelian type means that its connected component has a covering that is a product of connected Shimura varieties, each of which carries (non-canonically) a family of abelian varieties.

2. Conjecture CM

We shall need certain properties of the Taniyama group, for which we refer to Langlands [3; § 5, § 6] (or Milne-Shih [1]). The Taniyama group is a projective system of extensions of $\mathbf{Q}$-rational pro-algebraic groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^{L'} & \longrightarrow & \mathbf{T}^{L'} & \xrightarrow{\pi} & \mathrm{Gal}(L'^{\mathrm{ab}}/\mathbf{Q}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N_{L'/L}} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & S^{L} & \longrightarrow & \mathbf{T}^{L} & \xrightarrow{\pi} & \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q}) & \longrightarrow & 1 \end{array}$$

in which $L \subset L'$ are finite Galois extensions of $\mathbf{Q}$ contained in $\bar{\mathbf{Q}}$ and S^L is the Serre group of L; there is a canonical family of splittings $\mathrm{sp}^L\colon \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q}) \to \mathbf{T}^L(\mathbf{A}^f)$, $\pi \circ \mathrm{sp}^L = \mathrm{id}$.

Now fix a large L and drop the superscript on sp. There will exist a section $\tau \mapsto a(\tau)$ to $\mathbf{T}^L(L) \to \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q})$ that is a morphism of pro-algebraic groups and, after choosing such a section, we can define a continuous map

$\beta\colon \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q}) \to S^L(\mathbf{A}_L^f)$ and 1-cocycles $(\gamma_\sigma(\tau))$ for $\mathrm{Gal}(L/\mathbf{Q})$ with values in $S^L(L)$ by the formulas:

$$\mathrm{sp}(\tau)\beta(\tau) = a(\tau),\ \tau \in \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q})\ , \tag{2.1a}$$

$$\sigma a(\tau) = a(\tau)\gamma_\sigma(\tau),\ \tau \in \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q}),\ \sigma \in \mathrm{Gal}(L/\mathbf{Q})\ . \tag{2.1b}$$

Note that

$$\gamma_\sigma(\tau) = \beta(\tau)^{-1}\cdot\sigma\beta(\tau)\ . \tag{2.1c}$$

The composite $\bar\beta\colon \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q}) \to S^L(\mathbf{A}_L^f)/S^L(L)$ is independent of the choice of the section a, and satisfies

$$\bar\beta(\tau_1\tau_2) = \tau_2^{-1}\bar\beta(\tau_1)\cdot\bar\beta(\tau_2)\ . \tag{2.1d}$$

It is a consequence of the construction of the Taniyama group that

$$\bar\beta(\iota) = 1\ . \tag{2.1e}$$

The Serre group S^L and its canonical cocharacter μ^L are universal in the following sense: for any $\mathbf{Q}$-rational torus T split over L and any cocharacter μ of T for which

$$(\sigma - 1)(\iota + 1)\mu = 0 = (\iota + 1)(\sigma - 1)\mu\ , \qquad \text{for all} \quad \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})\ , \tag{2.2}$$

there is a unique $\mathbf{Q}$-rational homomorphism $\rho_\mu\colon S^L \to T$ such that $\rho_\mu\circ\mu^L = \mu$. We then write $\beta(\tau, \mu)$, $\bar\beta(\tau, \mu)$, and $\gamma_\sigma(\tau, \mu)$ for $\rho_\mu(\beta(\tau))$, $\rho_\mu(\bar\beta(\tau))$, and $\rho_\mu(\gamma_\sigma(\tau))$.

More generally, let T be a $\mathbf{Q}$-rational torus split by L and μ a cocharacter such that

$$(1 + \iota)(\tau - 1)\mu = 0 \tag{2.3}$$

for some $\tau \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Then there need not be a homomorphism $\rho_\mu\colon S^L \to T$ but it is still possible to define $\beta(\tau, \mu) \in T(\mathbf{A}_L^f)$, $\bar\beta(\tau, \mu) \in T(\mathbf{A}_L^f)/T(L)\,T(\mathbf{Q})^\wedge$, and $(\gamma_\sigma(\tau, \mu))$ having the properties predicted by the formulas (2.1):

$$\gamma_\sigma(\tau, \mu) = \beta(\tau, \mu)^{-1}\cdot\sigma\beta(\tau, \mu)\ , \tag{2.4a}$$

$$\bar\beta(\tau_1\tau_2, \mu) = \tau_2^{-1}\bar\beta(\tau_1, \mu)\cdot\bar\beta(\tau_2, \mu)\ . \tag{2.4b}$$

Equation (2.3) is always satisfied by $\tau = \iota$, and

$$\bar\beta(\iota, \mu) = 1 \tag{2.4c}$$

(see Langlands [3, p. 234], Milne-Shih [1, 3.9]).

Let T be a $\mathbf{Q}$-rational torus and $h\colon \mathbf{S} \to T_{\mathbf{R}}$ a homomorphism with associated cocharacter μ. For any τ fixing the field E of definition of μ, (2.3) holds and so $\bar\beta(\tau, \mu)$ is defined. In [2, 2.2.3] Deligne defines a reciprocity morphism $r_E(T, h)$, and the definition of $\bar\beta$ is such as to make

(2.5)
$$\begin{array}{ccc} \mathrm{Gal}(L^{\mathrm{ab}}/E) & \xrightarrow{\bar\beta(-,\mu)} & T(\mathbf{A}_L^f)/T(L)\,T(\mathbf{Q})^\wedge \\ \downarrow & & \cup\!\!\uparrow \\ \mathrm{Gal}(E^{\mathrm{ab}}/E) & \xrightarrow{r_E(T,h)^{-1}} & T(\mathbf{A}^f)/T(\mathbf{Q})^\wedge \end{array}$$

commute.

Assume that there exists an embedding

$$(T, \{h\}) \hookrightarrow (C\mathrm{Sp}(V), S^\pm)$$

where $C\mathrm{Sp}(V)$ is the group of symplectic similitudes corresponding to some non-degenerate skew-symmetric form ψ on a vector space V over $\mathbf{Q}$, and $S^\pm$ is the Siegel double space in the sense of Deligne [2, 1.3.1]. (We regard ψ as being defined only up to multiplication by a non-zero element of $\mathbf{Q}$.) In a well-known way (Deligne [1, § 4]), $\mathrm{Sh}(C\mathrm{Sp}(V), S^\pm)$ can be identified with a family $\mathfrak{A}(V, \psi)$ of isomorphism classes of triples (A, t, k) in which A is an abelian variety over C, t is a homogeneous polarization on A, and k is an isomorphism $k\colon V^f(A) \overset{\approx}{\to} V(\mathbf{A}^f)$. An element g of $G(\mathbf{A}^f)$ acts on a class $[A, t, k] \in \mathfrak{A}(V, \psi)$ as follows: $[A, t, k]g = [A, t, g^{-1}k]$.

The assumption on $(T, \{h\})$ implies that μ satisfies (2.2). We have therefore homomorphisms $(S^L \overset{\rho}{\to} T \hookrightarrow C\mathrm{Sp}(V))$. The inverse image in $\mathbf{T}^L$ of a $\tau \in \mathrm{Gal}(L^{\mathrm{ab}}/\mathbf{Q})$ is a right S^L-torsor ${}^\tau S^L$ corresponding in $H^1(L/\mathbf{Q}, S^L)$ to the class $\gamma(\tau, \mu)$ of $(\gamma_\sigma(\tau, \mu))$. The map ρ defines an action of S^L on (V, ψ) and, with the notations of Serre [1, I.5], we define $({}^\tau V, {}^\tau\psi) = {}^\tau S \times^S (V, \psi)$. The element $\mathrm{sp}(\tau) \in {}^\tau S(\mathbf{A}^f)$ defines an isomorphism $v \mapsto \mathrm{sp}(\tau)\cdot v\colon V(\mathbf{A}^f) \overset{\approx}{\to} {}^\tau V(\mathbf{A}^f)$ which we shall again denote by $\mathrm{sp}(\tau)$. Clearly ${}^\tau S \times^S T = T$, and so there is a canonical embedding $T \hookrightarrow C\mathrm{Sp}({}^\tau V)$. It sends ${}^\tau h$, the map $\mathbf{S} \to T$ with cocharacter $\tau\mu$, into the Siegel double space for $({}^\tau V, {}^\tau\psi)$. Let $[A, t, k] \in \mathfrak{A}(V, \psi)$ and let ${}^\tau k$ be the composite:

$$V^f(\tau A) \xrightarrow{\tau^{-1}} V^f(A) \xrightarrow{k} V(\mathbf{A}^f) \xrightarrow{\mathrm{sp}(\tau)} {}^\tau V(\mathbf{A}^f)\ .$$

Then $[\tau A, \tau t, {}^\tau k] \in \mathfrak{A}({}^\tau V, {}^\tau\psi)$ and we write χ_τ for the map

$$[A, t, k] \longmapsto [\tau A, \tau t, {}^\tau k]\colon \mathfrak{A}(V, \psi) \longrightarrow \mathfrak{A}({}^\tau V, {}^\tau\psi)\ .$$

Conjecture CM: the following diagram commutes:

$$\begin{array}{cccccc} [h, g] & \mathrm{Sh}(T, \{h\}) & \hookrightarrow & \mathrm{Sh}(C\mathrm{Sp}(V), S^\pm) & \xrightarrow{\approx} & \mathfrak{A}(V, \psi) \\ \downarrow & \downarrow{\scriptstyle\approx} & & & & {\scriptstyle\approx}\downarrow{\scriptstyle\chi_\tau} \\ [{}^\tau h, g] & \mathrm{Sh}(T, \{{}^\tau h\}) & \hookrightarrow & \mathrm{Sh}(C\mathrm{Sp}({}^\tau V), S^\pm) & \xrightarrow{\approx} & \mathfrak{A}({}^\tau V, {}^\tau\psi)\ . \end{array}$$

Let A be an abelian variety of CM-type; we shall say that conjecture

CM holds for A if it holds with T the Mumford-Tate group of A, $V = H_1(A, \mathbf{Q})$, and ψ a Riemann form for A.

Example 2.6(a). Suppose in the above that $\gamma(\tau, \mu)$ is trivial in $H^1(\mathbf{Q}, T)$. Then $\bar{\beta}(\tau, \mu)$ lies in $T(\mathbf{A}^f)/T(\mathbf{Q})^\wedge$ and there is an element $\beta(\tau, \mu) \in T(\mathbf{A}^f)$ representing $\bar{\beta}(\tau, \mu)$ and an isomorphism $a(\tau): (V, \psi) \to ({}^\tau V, {}^\tau\psi)$ such that $\mathrm{sp}(\tau)\beta(\tau, \mu) = a(\tau)$ (as maps $V(\mathbf{A}^f) \to {}^\tau V(\mathbf{A}^f)$). When $a(\tau)$ is used to identify $\mathcal{A}(V, \psi)$ with $\mathcal{A}({}^\tau V, {}^\tau\psi)$ then χ_τ becomes identified with the map

$$[A, t, k] \longmapsto [\tau A, \tau t, \beta(\tau, \mu)^{-1} \circ k \circ \tau^{-1}] ,$$

i.e., with the composite

$$\mathcal{A}(V, \psi) \xrightarrow{\tau} \mathcal{A}(V, \psi) \xrightarrow{\beta(\tau,\mu)} \mathcal{A}(V, \psi) .$$

Thus, in this case, the conjecture asserts that

$$\begin{array}{ccc} [h, g] & \mathrm{Sh}(T, \{h\}) \hookrightarrow & \mathcal{A}(V, \psi) \\ \downarrow & \downarrow & \downarrow \tau \\ [{}^\tau h, g\beta(\tau, \mu)^{-1}] & \mathrm{Sh}(T, \{{}^\tau h\}) \hookrightarrow & \mathcal{A}(V, \psi) \end{array}$$

commutes.

(b) Suppose that τ fixes the reflex field $E(T, \{h\})$; then (2.5) shows that $\bar{\beta}(\tau, \mu) = r_E(T, h)(\tau)^{-1} \in T(A^f)/T(\mathbf{Q})^\wedge$. It follows that $\gamma(\tau, \mu)$ is trivial and so, once $({}^\tau V, {}^\tau\psi)$ has been identified as in (a) with (V, ψ), conjecture CM becomes the statement that the action of τ on the image of $\mathrm{Sh}(T, \{h\})$ in $\mathcal{A}(V, \psi)$ corresponds to the right action of $\tilde{r}(\tau)$ on $\mathrm{Sh}(T, \{h\})$, where $\tilde{r}(\tau) \in T(A^f)$ represents $r_E(T, h)(\tau)$. This is essentially the statement of the main theorem of complex multiplication to be found, for example, in Deligne [1, 4.19]. Thus the conjecture is a generalization of that theorem.

Example 2.7. Let F_0 be a totally real number field, let F_1 and F_2 be distinct, totally imaginary, quadratic extensions of F_0, and let $F = F_1F_2$. For each $\sigma \in I \overset{\mathrm{df}}{=} \mathrm{Hom}(F_0, \mathbf{C})$ choose an extension σ_1 of σ to F_1 and an extension σ_2 of σ to F_2. Write σ' and σ'' for the elements of $\mathrm{Hom}(F, \mathbf{C})$ such that

$$\begin{aligned} \sigma' &= \sigma_1 \text{ on } F_1 , \qquad & \sigma'' &= \iota\sigma_1 \text{ on } F_1 \\ \sigma' &= \sigma_2 \text{ on } F_2 , \qquad & \sigma' &= \sigma_2 \text{ on } F_2 . \end{aligned}$$

Let Σ_0 be a subset of $\mathrm{Hom}(F_0, \mathbf{C})$ and define

$$\Sigma = \{\sigma' \mid \sigma \in I\} \cup \{\sigma'' \in \Sigma_0\} \cup \{\iota\sigma'' \mid \sigma \notin \Sigma_0\} .$$

Then Σ is a CM-type for F. The sets of complex embeddings Σ_0, $\Sigma_1 = \{\sigma_1 \mid \sigma \in \Sigma_0\}$, $\Sigma_2 = \{\sigma_2 \mid \sigma \notin \Sigma_0\}$, and Σ define $\mathbf{Q}$-rational Hodge structures on the vector spaces F_0, F_1, F_2, and F, and hence homomorphisms $h_j: \mathbf{S} \to (F_j \otimes \mathbf{R})^\times$

for $j = 0, \cdots$. Let μ_0, μ_1, μ_2, and $\mu = \mu_1\mu_2$ be the corresponding cocharacters, and let E_0, E_1, and E_2 be the reflex fields $E(F_0^\times, h_0)$, $E(F_1^\times, h_1)$, and $E(F_2^\times, h_2)$. In the following we assume that E_1 and E_2 are linearly disjoint over E_0. As E_0 is totally real, this assumption allows us to consider an automorphism τ of C over E_0 such that

$$\begin{aligned}\tau &= \text{id on } E_1 \\ &= \iota \quad \text{on } E_2 .\end{aligned}$$

Note that $(\iota + 1)(\tau - 1)\mu_1 = 0 = (\iota + 1)(\tau - 1)\mu_2$, and so $(\iota + 1)(\tau - 1)\mu = 0$. If we let $T = \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m$ then μ_0, μ_1, μ_2, and μ can be regarded as elements of $X_*(T)$. Thus, if L is large enough to split T, there are defined elements $\bar\beta(\tau, \mu_j) \in \big(T(\mathbf{A}_L^f)/T(L)\big)^{\mathrm{Gal}(L/\mathbf{Q})}$ for $j = 0, \cdots$. As $\mu = \mu_1\mu_2$, $\bar\beta(\tau, \mu) = \bar\beta(\tau, \mu_1)\bar\beta(\tau, \mu_2)$. Since $\tau = \text{id}$ on E_1 and $\iota\tau = \text{id}$ on E_2, (2.5) shows that

$$\begin{aligned}\bar\beta(\tau, \mu_1) &= r_{E_1}\big(\tau \,|\, E_1^{\mathrm{ab}}\big)^{-1} \in T(\mathbf{A}^f)/T(\mathbf{Q})^\wedge , \\ \bar\beta(\iota\tau, \mu_2) &= r_{E_2}\big(\iota\tau \,|\, E_2^{\mathrm{ab}}\big)^{-1} \in T(\mathbf{A}^f)/T(\mathbf{Q})^\wedge .\end{aligned}$$

From (2.4b) we know $\bar\beta(\tau, \mu_2) = (\iota\tau)^{-1}\bar\beta(\iota, \mu_2) \cdot \bar\beta(\iota\tau, \mu_2)$, and (2.4c) shows $\bar\beta(\iota, \mu_2) = 1$. Thus

$$\bar\beta(\tau, \mu) = r_{E_1}\big(\tau \,|\, E_1^{\mathrm{ab}}\big)^{-1} r_{E_2}\big(\iota\tau \,|\, E_2^{\mathrm{ab}}\big)^{-1} \in T(\mathbf{A}^f)/T(\mathbf{Q})^\wedge .$$

Let A be an abelian variety over C with complex multiplication by F and of CM-type Σ. Choose identifications of $H_1(A, \mathbf{Q})$ and $H_1(\tau A, \mathbf{Q})$ with F. Then $V^f(A)$ and $V^f(\tau A)$ are identified with $\mathbf{A}_F^f$ and $\mathrm{sp}(\tau)$: $V^f(A) \to V^f(\tau A)$ is multiplication by an element $\beta(\tau, \mu)^{-1} \in (\mathbf{A}_F^f)^\times = T(\mathbf{A}^f)$. This $\beta(\tau, \mu)$ lifts $\bar\beta(\tau, \mu) = r_{E_1}(\tau)^{-1} r_{E_2}(\iota\tau)^{-1}$. Conjecture CM asserts in this case that the two maps

$$\begin{aligned}&V^f(A) \xrightarrow{\ \tau\ } V^f(\tau A) , \\ &V^f(A) \xrightarrow{\beta(\tau,\mu)^{-1}} V^f(\tau A)\end{aligned}$$

are equal.

This last statement is, apart from notation, Theorem 9 of Shih [1] (see also § 7 below).

Let A be an abelian variety of CM-type, let T be the Mumford-Tate group of A, and let $h: \mathbf{S} \to T$ define the (natural) Hodge structure on $V = H_1(A, \mathbf{Q})$. Choose a Riemann form ψ for A and let (G, X) be a pair satisfying (1.1) and such that there are maps

$$\big(T, \{h\}\big) \hookrightarrow (G, X) \hookrightarrow \big(C\mathrm{Sp}(V), S^\pm\big) ;$$

for example, we could take $(G, X) = \big(C\mathrm{Sp}(V), S^\pm\big)$. We know $\mathrm{Sh}(G, X)$ has a canonical model M over its reflex field $E(G, X)$, and if we identify $\mathrm{Sh}(G, X)$

with $M_{\mathbf{C}}$ then any automorphism τ of $\mathbf{C}$ fixing E acts on the points of $\mathrm{Sh}(G, X)$. We recall a result from Milne-Shih [2] that relates the action of such a τ on $[h, 1] \in \mathrm{Sh}(G, X)$ to conjecture CM for A.

Let μ be the cocharacter of h. Then μ satisfies (2.2), and so $\gamma(\tau, \mu)$ is defined.

LEMMA 2.8. *The image of* $\gamma(\tau, \mu)$ *in* $H^1(\mathbf{Q}, G)$ *is trivial.*

Proof. Langlands [3, p. 230], Milne-Shih [2, 7.2].

There is therefore a $v \in G(\bar{\mathbf{Q}})$ such that $\gamma_\sigma(\tau, \mu) = v^{-1} \cdot \sigma v$. We define

$$\beta_1(\tau, \mu) = \beta(\tau, \mu)v^{-1} \in G(\mathbf{A}^f) . \tag{2.9}$$

PROPOSITION 2.10. *Conjecture* CM *is true for* A *and a* τ *fixing* $E(G, X)$ *if and only if*

$$\tau[h, 1] = [\mathbf{ad}\, v \circ {}^\tau h, \beta_1(\tau, \mu)^{-1}] . \tag{2.11}$$

Proof. Milne-Shih [2, 7.16].

Remark 2.12. It is shown in Milne-Shih [2, 7] that $\mathbf{ad}\, v \circ {}^\tau h \in X$, and that $[\mathbf{ad}\, v \circ {}^\tau h, \beta_1(\tau, \mu)^{-1}]$ is independent of the choice of $\beta(\tau, \mu)$ and v. Thus (2.11) makes sense.

3. Conjecture B and the action of ι on $\pi_0(\mathrm{Sh}(G, X))$

Let (G, X) satisfy (1.1) and assume $E(G, X) \subset \mathbf{R}$. The adjoint group G^{ad} is a product, $G^{\mathrm{ad}} = \prod_{i=1}^{s} G_i$, of $\mathbf{Q}$-simple adjoint groups G_i. Each G_i can be written $G_i = \mathrm{Res}_{F_i/\mathbf{Q}} G^i$ where G^i is absolutely simple and the F_i are totally real (Deligne [3, 2.3.4]). For each embedding $v\colon F_i \hookrightarrow \mathbf{R}$ we obtain a group G^i_v over $\mathbf{R}$, and $G^i_v(\mathbf{R})$ is either compact or has exactly two connected components (Deligne [3, 1.2.8]). In the latter case we write $G^i_v(\mathbf{R})^+$ (or simply $+$) for the component containing 1 and $G^i_v(\mathbf{R})^-$ (or simply $-$) for the other component. Note that $G_i \otimes_{\mathbf{Q}} \mathbf{R} = \prod G^i_v$. Define:

$$G^{\mathrm{ad}}(\mathbf{R})^+ = \{g \in G^{\mathrm{ad}}(\mathbf{R}) \mid g \longmapsto + \text{ for all } i \text{ and } v \text{ with } G^i_v(\mathbf{R}) \text{ non-compact}\} ,$$
$$G^{\mathrm{ad}}(\mathbf{R})^- = \{g \in G^{\mathrm{ad}}(\mathbf{R}) \mid g \longmapsto - \text{ for all } i \text{ and } v \text{ with } G^i_v(\mathbf{R}) \text{ non-compact}\} ,$$
$$G^{\mathrm{ad}}(\mathbf{R})^\pm = G^{\mathrm{ad}}(\mathbf{R})^+ \cup G^{\mathrm{ad}}(\mathbf{R})^- .$$

Clearly, $G^{\mathrm{ad}}(\mathbf{R})^\pm$ is a normal subgroup of $G^{\mathrm{ad}}(\mathbf{R})$, and there is an exact sequence

$$1 \longrightarrow G^{\mathrm{ad}}(\mathbf{R})^+ \longrightarrow G^{\mathrm{ad}}(\mathbf{R})^+ \longrightarrow \{\pm\} \longrightarrow 1 .$$

For $* = +, -$ or $\pm$, we define

$$G^{\mathrm{ad}}(\mathbf{Q})^* = G^{\mathrm{ad}}(\mathbf{R})^* \cap G(\mathbf{Q}) ;$$
$$G(\mathbf{R})_* = \text{inverse image of } G^{\mathrm{ad}}(\mathbf{R})^* \text{ in } G(\mathbf{R}) ;$$
$$G(\mathbf{Q})_* = G(R)_* \cap G(\mathbf{Q}) .$$

The real approximation theorem shows that there is an exact sequence

$$1 \longrightarrow G^{\mathrm{ad}}(\mathbf{Q})^{+} \longrightarrow G^{\mathrm{ad}}(\mathbf{Q})^{\pm} \longrightarrow \{\pm\} \longrightarrow 1 .$$

Remark 3.1. Consider $\tilde{G} \xrightarrow{\rho} G \to G^{\mathrm{ad}}$. The group $\tilde{G}(\mathbf{R})$ is connected (Borel-Tits [1, § 4]) and so the image of $\tilde{G}(\mathbf{R})$ in $G^{\mathrm{ad}}(\mathbf{R})$ is $G^{\mathrm{ad}}(\mathbf{R})^{+}$. Thus an element g of $G(\mathbf{R})$ is in $G(\mathbf{R})_{+}$ if and only if $g = \rho(\tilde{g})c$ for some $\tilde{g} \in \tilde{G}(\mathbf{R})$ and $c \in Z(\mathbf{R})$, where $Z = Z(G)$. Define T by the exact sequence

$$1 \longrightarrow G^{\mathrm{der}} \longrightarrow G \xrightarrow{\nu} T \longrightarrow 1 .$$

If $G^{\mathrm{der}} = \tilde{G}$ then an element $g \in G(\mathbf{R})$ is in $G(\mathbf{R})_{+}$ if and only if $\nu(g) \in \nu(Z(\mathbf{R}))$.

Now let $h \in X$ be special and $\mu = \mu_h$. Choose a $\mathbf{Q}$-rational maximal torus T in G such that h factors through $T_{\mathbf{R}}$, and let N be the normalizer of T in G.

LEMMA 3.2. *There exist $n \in N(\mathbf{R})$ and $w \in \rho(\tilde{G}(\mathbf{C}))$ such that* $\mathbf{ad}(n) \circ \mu = \iota\mu$ *and* $\mu(-1) = wn$.

Proof. Choose a maximal set of strongly orthogonal noncompact roots $\{\gamma_1, \cdots, \gamma_r\}$ of $\mathfrak{g}_{\mathbf{C}}^{\mathrm{der}}$ with respect to $\mathfrak{t}_{\mathbf{C}}^{\mathrm{der}}$ in the sense of Harish-Chandra, and use it to define a homomorphism ϕ of SL_2 to $\tilde{G}$ over $\mathbf{R}$ as usual (Ash et al. [1, III.2]). We can choose the γ_i's in such a way that $\langle \gamma_i, \mu \rangle = 1$ for all $i = 1, \cdots, r$. Put $w = (\rho \circ \phi)\left(\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right)$. Then $w \in N(\mathbf{C}) \cap \rho(\tilde{G}(\mathbf{C}))$. Furthermore,

$$w^{-1} \cdot \iota(w) = (\rho \circ \phi)\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = (\textstyle\sum_{i=1}^{r} \gamma_i^{\vee})(-1) = (\mu - \iota\mu)(-1) ,$$

where $\gamma_i^{\vee}$ denotes the coroot of γ_i. Hence $n = w\mu(-1) \in N(\mathbf{R})$ and it has the as required properties.

Let $'h$ be the element of X corresponding to $\iota\mu$. If n is as in Lemma 3.2, then $\mathbf{ad}(n) \circ h = {}'h$. Since $'h$ and h^{-1} become equal when composed with $G_{\mathbf{R}} \to G_{\mathbf{R}}^{\mathrm{ad}}$, n belongs to $G(\mathbf{R})_{-}$ in view of Deligne [3, 1.2.7]. In particular, we see that $G(\mathbf{R})_{\pm} \to \{\pm\}$ is surjective, and the real approximation theorem shows that there is an exact sequence

$$1 \longrightarrow G(\mathbf{Q})_{+} \longrightarrow G(\mathbf{Q})_{\pm} \longrightarrow \{\pm\} \longrightarrow 1 .$$

Now suppose that $\mathrm{Sh}(G, X)$ has a weakly canonical model over a real field containing $E(G, X)$. Then ι defines an antiholomorphic involution of $\mathrm{Sh}(G, X)$. One of the conjectures of Langlands gives an explicit description of this involution.

To state this conjecture of Langlands, let $h \in X$ be as above and $n \in G(\mathbf{R})_{-}$ be as in Lemma 3.2. Let $K_{\infty} \subset G(\mathbf{R})$ be the isotropy subgroup of h. Since K_{∞} is the centralizer of $h(i)$, and of $'h(i)$, we see that n normalizes K_{∞}.

Therefore we can define an antiholomorphic automorphism η of X by $\eta(\mathbf{ad}\, g \circ h) = \mathbf{ad}(gn) \circ h$.

Conjecture B. (Langlands [1, p. 418], [2, p. 2.7, Conjecture B], [3, p. 234]). The involution of $\mathrm{Sh}(G, X)$ defined by ι is $[x, g] \mapsto [\eta(x), g]$.

Remark 3.3. (a) If $h' = \mathbf{ad}(g) \circ h$ with $g \in G(\mathbf{R})$ then

$$\mu_{h'} = \mathbf{ad}(g) \circ \mu_h \text{ and } \iota\mu_{h'} = \iota(\mathbf{ad}\, g \circ \mu_h) = \mathbf{ad}(g) \circ \iota\mu_h = \mathbf{ad}(gn) \circ \mu_h .$$

Thus $\eta(h') = {}^{\iota}h'$. In particular the validity of the conjecture is independent of the choice of the special point h.

(b) Since the two automorphisms of $\mathrm{Sh}(G, X)$, $[x, g] \mapsto \iota[x, g]$ and $[x, g] \mapsto [\eta(x), g]$, are continuous and commute with the Hecke operators they will be equal if they agree at one point (Deligne [1, 5.2]). Thus, to prove conjecture B, it suffices to show that $\iota[h, 1] = [\eta(h), 1]$ $(= [{}^{\iota}h, 1])$ for a single special h.

(c) In the case that the canonical model of $\mathrm{Sh}(G, X)$ is a moduli variety over $E(G, X)$, it is easy to verify conjecture B. Consider for example $\mathrm{Sh}(C\mathrm{Sp}(V), S^{\pm})$ where $C\mathrm{Sp}(V)$ is the group of symplectic similitudes corresponding to some non-degenerate skew-symmetric form ψ on V. We have already observed that there is a bijection $\mathrm{Sh}(C\mathrm{Sp}(V), S^{\pm}) \xrightarrow{\approx} \mathcal{A}(V, \psi)$ where $\mathcal{A}(V, \psi)$ consists of certain isomorphism classes of triples (A, t, k). In fact $\mathrm{Sh}(C\mathrm{Sp}(V), S^{\pm})$ is the solution of a moduli problem over $\mathbf{C}$. The moduli problem is defined over $\mathbf{Q}$ $(= E(G, X))$ and so $\mathrm{Sh}(C\mathrm{Sp}(V), S^{\pm})$ has a model M over $\mathbf{Q}$, which the main theorem of complex multiplication shows to be a canonical model. If we set $\tau[A, t, k] = [\tau A, \tau t, k\tau^{-1}]$ for $\tau \in \mathrm{Aut}(\mathbf{C})$ and $[A, t, k] \in \mathcal{A}(V, \psi)$, then this action agrees with the action of τ on $\mathrm{Sh}(C\mathrm{Sp}(V), S^{\pm})$ defined by the identification $\mathrm{Sh}(C\mathrm{Sp}(V), S^{\pm}) = M_{\mathbf{C}}$.

Let $h \in X$. Then $[h, 1] \in \mathrm{Sh}(C\mathrm{Sp}(V), S^{\pm})$ corresponds to $[A, t, k]$ where A is the abelian variety defined by the $\mathbf{Q}$-rational Hodge structure (V, h), $t = \psi$, and k is $V^f(A) = V(\mathbf{A}^f) \xrightarrow{1} V(\mathbf{A}^f)$. Since $\iota\colon (\iota A)(\mathbf{C}) \to A(\mathbf{C})$ is a homeomorphism, it defines an isomorphism $f\colon H_1(\iota A, \mathbf{Q}) \xrightarrow{\approx} H_1(A, \mathbf{Q}) = V$. The canonical isomorphism $H^1_{dR}(A) \otimes_{\mathbf{C},\iota} \mathbf{C} \xrightarrow{\approx} H^1_{dR}(\iota A)$ of de Rham groups preserves the Hodge filtrations, from which it follows easily that ${}^{\iota}h = f \circ h' \circ f^{-1}$ where h' defines the Hodge structure on $H_1(\iota A, \mathbf{Q})$. Since t corresponds to ιt under f, and the map $f \otimes 1\colon V^f(\iota A) \to V(\mathbf{A}^f)$ is ι^{-1}, we see that $[A, \iota t, k\iota^{-1}]$ corresponds to $[{}^{\iota}h, 1] \in \mathrm{Sh}(G, X)$. Thus $\iota[h, 1] = [\iota h, 1]$ which, according to the above remark, proves conjecture B.

A similar argument proves the conjecture in the case that there is an embedding $(G, X) \hookrightarrow (C\mathrm{Sp}(V), S^{\pm})$ for then also the canonical model is a moduli variety over $E(G, X)$. It is however clear from (b) above and Deligne [1, 1.15] that if conjecture B is true for $\mathrm{Sh}(G', X')$ and (G, X) embeds in (G', X'), then conjecture B is true for $\mathrm{Sh}(G, X)$.

Recall (Deligne [3, 2.1.14]) that the action of $G(\mathbf{A}^f)$ on $\mathrm{Sh}(G, X)$ (on the right) induces an action of $G(\mathbf{A}^f)$ on $\pi_0(\mathrm{Sh}(G, X))$ under which $\pi_0(\mathrm{Sh}(G, X))$ becomes a principal homogeneous space for $\bar{\pi}_0\pi(G) = G(\mathbf{A}^f)/G(\mathbf{Q})^{\wedge}_{+}$. The image of $G(\mathbf{Q})_-$ in $G(\mathbf{A}^f)/G(\mathbf{Q})^{\wedge}_{+} \subset \mathrm{Aut}(\pi_0(\mathrm{Sh}(G, X)))$ is therefore an element of order 2. On the other hand, if $\mathrm{Sh}(G, X)$ has a weakly canonical model over a real field then ι acts on $\mathrm{Sh}(G, X)$ and hence on $\pi_0(\mathrm{Sh}(G, X))$.

PROPOSITION 3.4. *Assume that* $\mathrm{Sh}(G, X)$ *has a weakly canonical model over a real field E containing $E(G, X)$. Then for any $\alpha \in G(\mathbf{Q})_-$, the image of α in $G(\mathbf{A}^f)$ acts on* $\pi_0(\mathrm{Sh}(G, X))$ *as ι.*

Proof. According to Deligne [3, 2.6.3], ι acts on $\pi_0(\mathrm{Sh}(G, X))$ as $(\pi_0 N_{E/\mathbf{Q}} q_M)(\bar{e})$, where $\bar{e} \in \pi_0\pi(\mathbf{G}_{mE}) = \pi_0(\mathbf{A}_E^\times/E^\times)$ maps to $\iota \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$, M denotes the $G(\mathbf{C})$-conjugacy class of maps $\mu\colon \mathbf{G}_m \to G_{\mathbf{C}}$ corresponding to X, $q_M\colon \pi(\mathbf{G}_{mE}) \to \pi(G_E)$ and $N_{E/\mathbf{Q}}\colon \pi(G_E) \to \pi(G)$ are the maps defined in Deligne [3, 2.4], and $\pi_0 N_{E/\mathbf{Q}} q_M$ is the composite

$$\pi_0\pi(\mathbf{G}_{mE}) \xrightarrow{\pi_0(N_{E/\mathbf{Q}}\circ q_M)} \pi_0\pi(G) \longrightarrow \bar{\pi}_0\pi(G) = \pi_0\pi(G)/\pi_0(G(\mathbf{R})_+) .$$

The problem is to elucidate these maps.

Assume first that $G^{\mathrm{der}} = \tilde{G}$. By definition M is defined over E. For any $k \supset E$, there is a map

$$\tilde{q}_M\colon \mathbf{G}_m(k) \longrightarrow (G/\tilde{G})(k)$$

with the following property (see Deligne [3, 2.4]): let $\mu \in M$ be defined over $k' \supset k$, and denote the composite

$$\mathbf{G}_m(k') \xrightarrow{\mu} G(k') \longrightarrow (G/\tilde{G})(k')$$

by $\tilde{q}_\mu$; then $\tilde{q}_M = \tilde{q}_\mu$ over k'. Consider

$$\begin{array}{ccccccc} & & k^\times & & & & \\ & & \downarrow & \searrow^{\tilde{q}_M} & & & \\ 1 \longrightarrow & & G(k)/\tilde{G}(k) & \longrightarrow & (G/\tilde{G})(k) & \xrightarrow{\partial} & H^1(k, \tilde{G}) . \end{array}$$

We see that the restriction of $\tilde{q}_M$ to the kernel $(k^\times)_0$ of $\partial \circ \tilde{q}_M$ factors through $G(k)/\tilde{G}(k)$. Since $(E_v^\times)_0 = E_v^\times$ for v a finite prime of E and $(E_v^\times)_0 = (\mathbf{R}^\times)^+$ for v a real prime, on forming a restricted product we obtain a map $q_M\colon (\mathbf{A}_E)^+ \to G(\mathbf{A}_E)/\tilde{G}(\mathbf{A}_E)$. On passing to a quotient we obtain the map q_M from $\mathbf{A}_E^\times/E^\times = (\mathbf{A}_E^\times)^+/(E^\times)^+$ to $G(\mathbf{A}_E)/\tilde{G}(\mathbf{A}_E)G(E) = \pi(G_E)$.

Let $e = (1, \cdots, 1; 1, \cdots, 1, -1) \in \mathbf{A}_E^\times$, where the final place corresponds to the real prime v_0 of E defined by the given embedding $E \hookrightarrow \mathbf{R}$. Then e represents $\bar{e} \in \pi_0(\mathbf{A}_E^\times/E^\times)$. We compute $q_M([e]) \in \pi(G_E)$, where $[e]$ is the image of e in $\mathbf{A}_E^\times/E^\times$.

For $v \neq v_0$, $e_v = 1$, and $\tilde{q}_M(e_v)$ is represented by $1 \in G(E_v)$. For $v = v_0$, let $h \in X$ be special, let $\mu = \mu_h$ and choose n and w as in Lemma 3.2. Then $\tilde{q}_M = \tilde{q}$ over C. Since $\mu(e_v) = \mu(-1) = wn$, $\mu(e_v)$ and n have the same image in $(G/\tilde{G})(\mathbf{R})$; thus $q_M(e_v)$ is defined and can be represented by $n \in G(E_v)$. We conclude that $(1, \cdots, 1; 1, \cdots, 1, n) \in G(\mathbf{A}_E)$ represents $q_M([e])$. It follows that $(N_{E/\mathbf{Q}}q_M)([e])$ is represented by $\xi = (1, \cdots, 1; n) \in G(\mathbf{A})$, and $(\pi_0 N_{E/\mathbf{Q}} q_M)(\bar{e})$ is represented by the image $\bar{\xi}$ of ξ in $\bar{\pi}_0\pi(G) = \pi_0\pi(G)/\pi_0(G(\mathbf{R})_+)$:

$$\iota \text{ acts on } \pi_0(\mathrm{Sh}(G, X)) \text{ as } \bar{\xi} .$$

Now for $\alpha \in G(\mathbf{Q})_-$, let $\alpha_0 = (\alpha, \cdots, \alpha; 1) \in G(\mathbf{A})$. Then $\alpha_0\xi^{-1} \in G(\mathbf{Q})G(\mathbf{R})_+$, and so the image $\bar{\alpha}_0$ of α_0 in $\bar{\pi}_0\pi(G)$ is $\bar{\xi}$. Therefore $(\pi_0 N_{E/\mathbf{Q}} q_M)(\bar{e})$ is also represented by $\bar{\alpha}_0$. To complete the proof of this case, we observe that, when $\bar{\pi}_0\pi(G)$ is identified with $G(\mathbf{A}^f)/G(\mathbf{Q})^\wedge_+$, $\bar{\alpha}_0$ is the image of $\alpha \in G(\mathbf{Q}) \subset G(\mathbf{A}^f)$.

For the general case, one can repeat the argument with the group $(G/\tilde{G})(k)$ replaced by $\mathbf{H}^1(\tilde{G} \to G)$ (see Deligne [3, 2.4]).

4. Definition of $\varepsilon\colon G(\mathbf{Q})^{\pm\wedge}(\mathrm{rel}\, G') \to \mathfrak{E}_E(G, G', X^+)$

Let (G, G', X^+) define a connected Shimura variety, as in Section 1. Recall that $E(G, X^+)$ is defined to be $E(G, X)$, where X is the $G(\mathbf{R})$-conjugacy class of maps $\mathbf{S} \to G_{\mathbf{R}}$ containing X^+. Assume that $E(G, X^+)$ is real. Then the discussion in Section 3 applies to $G = G^{\mathrm{ad}}$ and we have groups $G(\mathbf{R})^+$, $G(\mathbf{R})^-$, $\cdots$, $G(\mathbf{Q})^\pm$, and exact sequences

$$1 \longrightarrow G(\mathbf{R})^+ \longrightarrow G(\mathbf{R})^\pm \longrightarrow \{\pm\} \longrightarrow 1 ,$$
$$1 \longrightarrow G(\mathbf{Q})^+ \longrightarrow G(\mathbf{Q})^\pm \longrightarrow \{\pm\} \longrightarrow 1 .$$

Recall (Deligne [3, 2.5.7]) that for any $E \subset \overline{\mathbf{Q}}$ that is finite over $E(G, X)$, there is a canonical extension

$$1 \longrightarrow G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') \longrightarrow \mathfrak{E}_E(G, G', X^+) \longrightarrow \mathrm{Gal}(\overline{\mathbf{Q}}/E) \longrightarrow 1 .$$

In the following we assume $E \subset \mathbf{R}$.

PROPOSITION 4.1. *With the above assumptions and notations, there exists a canonical embedding*

$$\varepsilon\colon G(\mathbf{Q})^{\pm\wedge}(\mathrm{rel}\, G') \longrightarrow \mathfrak{E}_E(G, G', X^+) ,$$

rendering

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\operatorname{rel} G') & \xrightarrow{\varepsilon} & \mathfrak{E}_E(G, G', X^+) & \xrightarrow{\pi} & \operatorname{Gal}(\bar{\mathbf{Q}}/E) & \longrightarrow & 1 \\ & & \big\downarrow & \nearrow_{\varepsilon} & & & & & \\ & & G(\mathbf{Q})^{\pm\wedge}(\operatorname{rel} G') & & & & & & \end{array}$$

commutative and such that $(\pi\varepsilon)^{-1}(\iota) = G(\mathbf{Q})^{-\wedge}(\operatorname{rel} G')$.

Proof. We first review Deligne's construction [3, 2.5] of the canonical extension. Choose a pair (G_1, X_1) satisfying (1.1) and such that $(G_1^{\mathrm{ad}}, G_1^{\mathrm{der}}, X_1^+) = (G, G', X^+)$ for some $X_1^+ \subset X_1$; it is possible to do this in such a way that $E(G_1, X_1) = E(G, X^+)$; see Deligne [3, 2.5.5]. The canonical extension is defined by the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\operatorname{rel} G') & \longrightarrow & \mathfrak{E}_E(G, G', X^+) & \xrightarrow{\pi} & \operatorname{Gal}(\bar{\mathbf{Q}}/E) & \longrightarrow & 1 \\ & & \| & & \big\downarrow f & & \big\downarrow r_{G_1, X_1} & & \\ 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\operatorname{rel} G') & \longrightarrow & \dfrac{G_1(\mathbf{A}^f)}{Z(\mathbf{Q})^\wedge} *_{G_1(\mathbf{Q})_+/Z(\mathbf{Q})} G(\mathbf{Q})^+ & \longrightarrow & \bar{\pi}_0\pi(G_1) & \longrightarrow & 1 \end{array} \tag{4.2}$$

in which r_{G_1, X_1} is the reciprocity law and Z is the center of G_1. The calculation made in the proof of (3.4) shows that

$$\begin{aligned} \pi^{-1}(\{1, \iota\}) \underset{\approx}{\xrightarrow{f}} & \frac{G_1(\mathbf{Q})^\wedge_\pm}{Z(\mathbf{Q})^\wedge} *_{G_1(\mathbf{Q})_+/Z(\mathbf{Q})} G(\mathbf{Q})^+ \\ \approx & \frac{G_1(\mathbf{Q})^\wedge_\pm}{Z(\mathbf{Q})^\wedge} *_{G_1(\mathbf{Q})_\pm/Z(\mathbf{Q})} G(\mathbf{Q})^\pm , \end{aligned}$$

which can be identified with $G(\mathbf{Q})^{\pm\wedge}(\operatorname{rel} G')$; Deligne [3, 2.1.15.1]. We define ε to be the inverse isomorphism.

To see that ε is independent of the choice of (G_1, X_1), take another (G_2, X_2) with the same properties as (G_1, X_1). Let G_3 be the identity component of the fiber product $G_1 \times_G G_2$, and $X_3 = X_1 \times_X X_2$. Then (G_3, X_3) also has the same properties as (G_1, X_1). We see easily that, via the projections $G_3 \to G_1$ and $G_3 \to G_2$, (G_3, X_3), (G_1, X_1) and (G_2, X_2) all define the same ε.

Remark 4.3. Let the notations be as in the above proof. For simplicity, put $\mathcal{G} = (G_1(\mathbf{A}^f)/Z(\mathbf{Q})^\wedge) *_{G_1(\mathbf{Q})_+/Z(\mathbf{Q})} G(\mathbf{Q})^+$. Note that in the identification

$$G(\mathbf{Q})^{\pm\wedge}(\operatorname{rel} G') = \frac{G_1(\mathbf{Q})^\wedge_\pm}{Z(\mathbf{Q})^\wedge} *_{G_1(\mathbf{Q})/Z(\mathbf{Q})_\pm} G(\mathbf{Q})^\pm ,$$

$\alpha \in G(\mathbf{Q})^\pm$ is identified with $1 * \alpha$. Therefore, if $\alpha \in G(\mathbf{Q})^-$ lifts to $\alpha_1 \in G_1(\mathbf{Q})_-$, then $\varepsilon(\alpha)$ is the element of $\mathfrak{E}_E(G, G', X^+)$ such that

$$f(\varepsilon(\alpha)) = \alpha_1 * 1 \in \mathcal{G} \quad \text{and} \quad \pi(\varepsilon(\alpha)) = \iota \in \operatorname{Gal}(\bar{\mathbf{Q}}/E) .$$

In general, let γ_1 be an element of $G_1(\mathbf{Q})^-$, and let γ be its image in $G(\mathbf{Q})_-$. Then for any $\alpha \in G(\mathbf{Q})^-$, $\varepsilon(\alpha)$ is the element of $\mathfrak{E}_E(G, G', X^+)$ such that

$$f(\varepsilon(\alpha)) = \gamma_1 * \gamma^{-1}\alpha \in \mathcal{G} \quad \text{and} \quad \pi(\varepsilon(\alpha)) = \iota \in \mathrm{Gal}(\overline{\mathbf{Q}}/E) .$$

Assume (G, G', X) is of primitive abelian type. Then the pair (G_2, X_2) constructed in the proof of Deligne [3, 2.3.10] satisfies the conditions $G_2^{\mathrm{ad}} = G$, $G_2^{\mathrm{der}} = G'$, $(G_2, X_2) \to (G, X)$ and $E(G_2, X_2) = E(G, X)$, and so can be chosen as (G_1, X_1) in the proof of Proposition 4.1. However, this is not the most convenient one to use. We shall use a group G_3 that is larger than G_2. Let the notations be as in (1.3) and (1.4). Recall that V is a vector space over F and $G_2 \subset \mathrm{GL}(V)$. We take G_3 to be the $\mathbf{Q}$-algebraic group generated by G_2 and $F^\times$. Then (G_3, X_2) can be used instead of (G_2, X_2) as our (G_1, X_1). The extra properties (G_3, X_2) enjoys, which are established in the proof of Deligne [3, 2.3.10], are summarized in the following proposition, in which (G_3, X_2) is denoted by (G_2, X_2).

PROPOSITION 4.4. *Let the notations and assumptions be as in* (1.4). *Then there exists a diagram*

$$(G_2, X_2) \longrightarrow (G, X) \longleftarrow (G_1, X_1) \hookrightarrow (C\mathrm{Sp}(V), S^{\pm})$$

such that $G_1^{\mathrm{ad}} = G_2^{\mathrm{ad}} = G$, $G_1^{\mathrm{der}} = G_2^{\mathrm{der}} = G'$, $E(G_1, X_1) = E(G, X)E(F^\times, h_\Sigma)$, $E(G_2, X_2) = E(G, X)$, $G_1 \subset G_2$, $Z(G_2) \supset F^\times$ *and* $X_1 = \{h_2 h_\Sigma \mid h_2 \in X_2\}$.

5. Statement of conjecture B^0; equivalence with conjecture B

Let (G, G', X^+) define a connected Shimura variety as in Section 1. Recall (Deligne [3, 2.7.10]) that a weakly canonical model for $\mathrm{Sh}^0(G, G', X^+)$ over $E \supset E(G, X^+)$ is a scheme $\mathrm{Sh}^0(G, G', X^+)_{\overline{\mathbf{Q}}}$ over $\overline{\mathbf{Q}}$ together with a left action of $\mathcal{E}_E(G, G', X^+)$ satisfying certain properties.

Let X be the conjugacy class of maps $\mathbf{S} \to G_{\mathbf{R}}$ containing X^+. Assume that $E(G, X^+) = E(G, X)$ is totally real. Fix a special $h_0 \in X$, and let $n \in N(\mathbf{R})$ and $\eta\colon X \to X$ be as in Section 3. Since $n \in G(\mathbf{R})^-$, as was remarked in Section 3, we see that $\mathbf{ad}\,\alpha \circ \eta(x) \in X^+$ for all $\alpha \in G(\mathbf{Q})^-$ and $x \in X^+$.

Conjecture B^0. Assume that $\mathrm{Sh}^0(G, G', X^+)$ has a weakly canonical model over a field $E \subset \mathbf{R}$; then for all $\alpha \in G(\mathbf{Q})^-$, the element $\varepsilon(\alpha) \in \mathcal{E}_E(G, G', X^+)$ acts on $\mathrm{Sh}^0(G, G', X^+) = \varprojlim \Gamma\backslash X^+$ as follows: $[x] \mapsto [\mathbf{ad}\,\alpha \circ \eta(x)]$ for all $x \in X^+$.

Remark 5.1. Suppose α_1 and α_2 are both in $G(\mathbf{Q})^-$. Then $\alpha^+ = \alpha_1\alpha_2^{-1} \in G(\mathbf{Q})^+$. Hence $\varepsilon(\alpha_1) = \varepsilon(\alpha^+)\varepsilon(\alpha_2)$ and

$$[\mathbf{ad}\,\alpha_1 \circ \eta(x)] = [\mathbf{ad}\,\alpha^+ \circ \mathbf{ad}\,\alpha_2 \circ \eta(x)] = \varepsilon(\alpha^+)[\mathbf{ad}\,\alpha_2 \circ \eta(x)] .$$

Thus, conjecture B^0 holds for all $\alpha \in G(\mathbf{Q})^-$ if and only if it does for one α.

PROPOSITION 5.2. *Let* (G, X) *satisfy* (1.1), *and assume* $\mathrm{Sh}(G, X)$ *has a weakly canonical model over some field* $E \subset \mathbf{R}$. *Then conjecture* B *holds for*

$\mathrm{Sh}(G, X)$ *if and only if conjecture* B^0 *holds for* $\mathrm{Sh}^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$.

Proof. The proof is straightforward, but it is convenient first to review the various group actions on $\mathrm{Sh}(G, X)$ and $\mathrm{Sh}^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$.

The group $G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}})$ acts canonically on $\mathrm{Sh}^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ on the left. When $\mathrm{Sh}^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ is identified with the connected component $\mathrm{Sh}^0(G, X)$ of $\mathrm{Sh}(G, X)$ containing the image of $X^+ \times 1$, then the action of $\gamma \in G^{\mathrm{ad}}(\mathbf{Q})^+$ is the restriction of

$$[x, g] \longmapsto \gamma[x, g] = [\gamma(x), \mathbf{ad}(\gamma)(g)],\ x \in X,\ g \in G(\mathbf{A}^f)\ .$$

By transport of structure, there is also a right action of $G^{\mathrm{ad}}(\mathbf{Q})^+$ on $\mathrm{Sh}(G, X)$:

$$[x, g]\gamma = \gamma^{-1}[x, g],\ \gamma \in G^{\mathrm{ad}}(\mathbf{Q})^+,\ x \in X,\ g \in G(\mathbf{A}^f)\ .$$

The group $G(\mathbf{A}^f)$ acts on $\mathrm{Sh}(G, X)$ on the right, via the Hecke operators. If $\gamma \in G^{\mathrm{ad}}(\mathbf{Q})^+$ is the image of $\delta \in G(\mathbf{Q})_+$, then the actions of γ and δ (considered as an element of $G(\mathbf{A}^f)$) agree. Thus there is a right action of

$$\mathcal{G} = \frac{G(\mathbf{A}^f)}{Z(\mathbf{Q})^\wedge} *_{G(\mathbf{Q})_+/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})^+$$

on $\mathrm{Sh}(G, X)$: $[x, g](g' * \gamma) = [\gamma^{-1}(x), \mathbf{ad}\, \gamma^{-1}(gg')]$.

When $\mathcal{G}$ is made to act on $\pi_0(\mathrm{Sh}(G, X))$, the stabilizer of the image of $\mathrm{Sh}^0(G, X)$ is $G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}})$, and $\pi_0(\mathrm{Sh}(G, X))$ becomes a principal homogeneous space for the abelian quotient $\bar{\pi}_0\pi(G) = G(\mathbf{A}^f)/G(\mathbf{Q})_+^\wedge$ of $\mathcal{G}$. These facts are summarized by an exact sequence:

$$1 \longrightarrow G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}}) \longrightarrow \mathcal{G} \longrightarrow \bar{\pi}_0\pi(G) \longrightarrow 1\ .$$

Now assume $\mathrm{Sh}(G, X)$ has a weakly canonical model over a finite extension E of $E(G, X)$. Then $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ acts on $\pi_0(\mathrm{Sh}(G, X))$ on the left. Since $\pi_0(\mathrm{Sh}(G, X))$ is a principal homogeneous space for $\bar{\pi}_0\pi(G)$, the action of $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ is described by a homomorphism $r\colon \mathrm{Gal}(\bar{\mathbf{Q}}/E) \to \bar{\pi}_0\pi(G)$ such that $\sigma \cdot x = x \cdot r(\sigma)$. The map r has an explicit description (Deligne [3, 2.6]), and there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}}) & \longrightarrow & \mathcal{E}_E(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+) & \xrightarrow{\pi} & \mathrm{Gal}(\bar{\mathbf{Q}}/E) & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r} & & \\
1 & \longrightarrow & G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}}) & \longrightarrow & \mathcal{G} & \longrightarrow & \bar{\pi}_0\pi(G) & \longrightarrow & 1\ .
\end{array}$$

Convert the right action of $\mathcal{G}$ on $\mathrm{Sh}(G, X)$ to a left action, and consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G^{\mathrm{der}}) & \longrightarrow & \mathcal{E}_E(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+) & \xrightarrow{\pi} & \mathrm{Gal}(\bar{\mathbf{Q}}/E) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle f \times \pi} & & \| & & \\
1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G} \times \mathrm{Gal}(\bar{\mathbf{Q}}/E) & \longrightarrow & \mathrm{Gal}(\bar{\mathbf{Q}}/E) & \longrightarrow & 1\ .
\end{array}$$

The action of $\mathcal{E}_E = \mathcal{E}_E(G^{ad}, G^{der}, X^+)$ on $Sh^0(G, X)$ arising, via this diagram, from the left actions of $\mathcal{G}$ and $Gal(\bar{Q}/E)$ on $Sh(G, X)$, corresponds to the given action of $\mathcal{E}_E$ on the weakly canonical model of $Sh^0(G^{ad}, G^{der}, X^+)$ over E.

Now we prove the proposition. Recall that E is assumed to be real. Fix an element $\alpha \in G^{ad}(\mathbf{Q})^-$ which lifts to an element α_1 of $G(\mathbf{Q})_-$. Then $\pi(\varepsilon(\alpha)) = \iota \in Gal(\bar{\mathbf{Q}}/E)$ and $f(\varepsilon(\alpha)) = \alpha_1 * 1 \in \mathcal{G}$, see (4.3). Hence

$$\varepsilon(\alpha)[h, 1] = (\iota[h, 1])\mathcal{T}(\alpha_1)^{-1} \qquad \text{for} \quad h \in X^+ .$$

Since conjecture B holds if and only if

$$\iota[h, 1] = [\eta(h), 1] = [\mathbf{ad}\, \alpha_1 \circ \eta(h), \alpha_1] = [\mathbf{ad}\, \alpha \circ \eta(h), 1]\mathcal{T}(\alpha_1) ,$$

this shows

$$\begin{aligned} \text{conjecture B holds} &\iff \varepsilon(\alpha)[h, 1] = [\mathbf{ad}\, \alpha \circ \eta(h), 1] \\ &\iff \text{conjecture } B^0 \text{ holds for } \alpha. \end{aligned}$$

This completes the proof, in view of (5.1).

6. A relation between conjectures B and CM

We consider the situation of (1.4). Thus (G, G', X) defines a connected Shimura variety of primitive abelian type. Write $G = \mathrm{Res}_{F_0/\mathbf{Q}} G^s$ with F_0 totally real and G^s absolutely simple, and let I_c and I_{nc} be as in (1.3). Denote by F_0' the totally real number field corresponding to the subgroup of $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ that stabilizes I_c. We have $F_0' \subset E(G, X)$.

Let $h \in X$ be special, and let $T \subset G$ be a $\mathbf{Q}$-rational torus such that h factors through $T_{\mathbf{R}}$. Let F be a quadratic totally imaginary extension of F_0 and let Σ be some family $(\sigma')_{\sigma \in I_c}$ of embeddings $\sigma' \colon F \to \mathbf{C}$ such that $\sigma' | F_0 = \sigma$. Denote by h_Σ the Hodge structure on F defined by Σ; see (1.3). We shall assume that (T, h) and (F, Σ) are such that there exists an automorphism τ of $\mathbf{C}$ with

$$\tau = \begin{cases} \mathrm{id} & \text{on } E(F^\times, h_\Sigma) , \\ \iota & \text{on } E(T, h) . \end{cases}$$

This is the case, for example, if $E(F^\times, h_\Sigma)$ and $E(T, h)$ are linearly disjoint over F_0'. Using Deligne [1, 6.5], we know that for a given (T, h) there is always an (F, Σ) such that this holds. On the other hand, we can also start with an (F, Σ) and choose a (T, h) such that $E(T, h)$ is linearly disjoint from $E(F^\times, h_\Sigma)E(G, X)$ over $E(G, X)$; see Deligne [1, 5.1]. Then (T, h) and (F, Σ) satisfy our assumption if $E(G, X)$ is totally real.

Remark 6.1. If (T, h) and (F, Σ) satisfy the assumption, then so do (T', h') and (F, Σ), where $T' = \mathbf{ad}\,\gamma(T)$ and $h' = \mathbf{ad}\,\gamma \circ h$ with $\gamma \in G(\mathbf{Q})$. This follows from the fact that $E(T', h') = E(T, h)$.

We assume now that $E(G, X)$ is totally real. Let (T, h), (F, Σ) and τ be as above. Consider the diagram

$$(G, X) \longleftarrow (G_1, X_1) \longrightarrow (C\mathrm{Sp}(V), S^{\pm})$$

constructed in Deligne [3, 2.3.10]. The information we need concerning this diagram is collected in Proposition 4.4. Lift (T, h) to $(T_1, h_1) \subset (G_1, X_1)$ as in (1.5); then $E(T_1, h_1) = E(T, h)E(F^{\times}, h_{\Sigma})$. To simplify the notations, we put $E = E(G, X)$, $E_1 = E(G_1, X_1)$, $E(h) = E(T, h)$, $E_1(h) = E(T_1, h_1)$ and $F' = E(F^{\times}, h_{\Sigma})$. Thus we have $E_1 = EF'$ and $E_1(h) = E(h)F'$. Fix a component X^+ of X. We identify $\mathrm{Sh}^0(G, G', X^+)$ (resp. $\mathrm{Sh}(G_1, X_1)$) with its canonical model over E (resp. E_1).

Note that τ fixes E_1, because it fixes both F' and E, E being a totally real subfield of $E(h)$. Thus we are in the situation of (2.8). Let μ_1 be the cocharacter of T_1 associated to h_1, and define $v \in G_1(\bar{\mathbf{Q}})$ and $\beta_1(\tau, \mu_1) \in G_1(\mathbf{A}^f)$ as in (2.9).

PROPOSITION 6.2. *Conjecture* B^0 *holds for* (G, G', X^+) *if and only if* $\tau[h_1, 1] = [\mathbf{ad}\, v \circ {}^{\tau}h_1, \beta_1(\tau, \mu_1)^{-1}]$.

First we show that Proposition 6.2 is a consequence of the following assertion.

PROPOSITION 6.3. *Let the notations and assumptions be as above. Then* $\varepsilon(\alpha)[h] = [\mathbf{ad}\, \alpha \circ \eta(h)]$ *for all* $\alpha \in G(\mathbf{Q})^-$ *(and for the given* h*) if and only if* $\tau[h_1, 1] = [\mathbf{ad}\, v \circ {}^{\tau}h_1, \beta_1(\tau, \mu_1)^{-1}]$.

In fact, note that the $G(\mathbf{Q})^+$-orbit of $[h]$ is dense in $\mathrm{Sh}^0(G, G', X^+)$. Therefore conjecture B^0 holds for $\mathrm{Sh}^0(G, G', X^+)$ if and only if $\varepsilon(\alpha)[h'] = [\mathbf{ad}\, \alpha \circ \eta(h')]$ for all $\alpha \in G(\mathbf{Q})^-$ and all $[h']$ in the $G(\mathbf{Q})^+$-orbit of $[h]$. Let $\gamma \in G(\mathbf{Q})^+$, and consider $T' = \mathbf{ad}\, \gamma(T)$ and $h' = \mathbf{ad}\, \gamma \circ h$. By Remark 6.1, Proposition 6.3 also applies to (T', h'). Since (T, h) lifts to $(T_1, h_1) \subset (G_1, X_1)$, (T', h') lifts to (T_1', h_1'), where $T_1' = \mathbf{ad}\, \gamma(T_1)$ and $h_1' = \mathbf{ad}\, \gamma \circ h_1$. Moreover, ${}^{\tau}h_1' = \mathbf{ad}\, \gamma \circ {}^{\tau}h_1$, and we can take $\mathbf{ad}\, \gamma(\beta_1(\tau, \mu_1))$ as $\beta_1(\tau, \mu_1')$, where μ_1' is the cocharacter of T_1' associated to h_1', and take $\mathbf{ad}\, \gamma(v)$ as the v for (T_1', h_1'). Therefore, by Proposition 6.3, $\varepsilon(\alpha)[h'] = [\mathbf{ad}\, \alpha \circ \eta(h')]$ for all $\alpha \in G(\mathbf{Q})^-$ if and only if

(6.4) $$\tau[h_1', 1] = [\mathbf{ad}\, \gamma(v) \circ {}^{\tau}h_1', \beta_1(\tau, \mu_1')^{-1}]\,.$$

But we have

$$\begin{aligned} \tau[h_1', 1] &= \tau[\mathbf{ad}\, \gamma \circ h_1, 1] = \tau([h_1, 1](\gamma * 1)) \\ &= (\tau[h_1, 1])(\gamma * 1)\,, \end{aligned}$$

and

$$\begin{aligned}[\mathbf{ad}\,\gamma(v)\circ{}^{\tau}h_1',\ \beta_1(\tau,\mu_1')^{-1}] &= [\mathbf{ad}\,\gamma(v)\circ\mathbf{ad}\,\gamma\circ{}^{\tau}h_1,\ \mathbf{ad}\,\gamma(\beta_1(\gamma,\mu_1)^{-1})]\\ &= [\mathbf{ad}\,\gamma\circ\mathbf{ad}\,v\circ{}^{\tau}h_1,\ \mathbf{ad}\,\gamma(\beta_1(\tau,\mu_1)^{-1})]\\ &= [\mathbf{ad}\,v\circ{}^{\tau}h_1,\ \beta_1(\tau,\mu_1)^{-1}](\gamma * 1)\ .\end{aligned}$$

In other words, (6.4) holds for all h_1' in the $G(\mathbf{Q})^+$-orbit of h_1 if and only if it holds for h_1. Putting these observations together, we obtain Proposition 6.2.

It remains to prove Proposition 6.3. Let $(G_2, X_2)\to(G, X)$ be as in Proposition 4.4; thus $E(G_2, X_2)=E$, $G_2\supset G_1$, $Z(G_2)\supset F^\times$ and $X_2=\{x_1h_\Sigma^{-1}\mid x_1\in X_1\}$. Lift (T, h) to $(T_2, h_2)\subset(G_2, X_2)$. Then $T_2\supset T_1F^\times$. Furthermore, using Lemma 1.2, one shows that $E(T_2, h_2) = E(T, h) = E(h)$ and $h_2 = h_1h_\Sigma^{-1}$. Therefore h_2 factors through $T_2^* \overset{\text{df}}{=} T_1F^\times$.

Let $\bar r = \bar\beta(\iota\tau, \mu_2)^{-1}$, where μ_2 is the cocharacter of T_2^* corresponding to h_2, and $\bar s = \bar\beta(\tau, \mu_\Sigma)^{-1}$, where μ_Σ is the cocharacter of $F^\times$ corresponding to h_Σ. As $\iota\tau$ fixes $E(h)$, $\bar r = r_{E(h)}(T_2^*, h_2)(\tau)$, and as τ fixes F', $\bar s = r_{F'}(F^\times, h_\Sigma)(\tau)$; see (2.5). Moreover, as $\mu_1=\mu_2\mu_\Sigma$, the computation of (2.7) shows $\bar\beta(\tau,\mu_1)^{-1}=\bar r\bar s$.

Let L be a Galois extension of $\mathbf{Q}$ that splits T_1, $F^\times$ and T_2^*. Since w_{h_1} is defined over $\mathbf{Q}$, we can define $\beta(\tau,\mu_1)\in T_1(\mathbf{A}_L^f)$, and choose v and $\beta_1(\tau,\mu_1)$ so that $\beta_1(\tau,\mu_1) = \beta(\tau,\mu_1)v^{-1}$; see (2.9). We have $\bar r\in T_2^*(\mathbf{A}^f)/T_2^*(\mathbf{Q})^\wedge$ and $\bar s\in F^\times(\mathbf{A}^f)/F^\times(\mathbf{Q})^\wedge$; let $r\in T_2^*(\mathbf{A}^f)$ and $s\in F^\times(\mathbf{A}^f)$ be their respective representatives. Since $\bar\beta(\tau,\mu_1)^{-1}=\bar r\bar s$, we can choose r, s in such a way that $rs = z\beta(\tau,\mu_1)^{-1}$ with $z\in T_2^*(L)$. Note that $zv^{-1}\in G_2(\mathbf{Q})$.

(a) Let $\mathcal{G}_2 = \big(G_2(\mathbf{A}^f)/Z_2(\mathbf{Q})^\wedge\big) *_{G_2(\mathbf{Q})_+/Z_2(\mathbf{Q})} G(\mathbf{Q})^+$, where $Z_2 = Z(G_2)$, and consider the following diagram (Deligne [3, 2.5.3, 2.5.8, 2.5.10]):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\,G') & \longrightarrow & \mathcal{G}_2 & \xrightarrow{\pi_2} & \bar\pi_0\pi(G_2) & \longrightarrow & 1\\ & & \| & & \uparrow{\scriptstyle f_2} & & \uparrow{\scriptstyle r_{G_2,X_2}} & & \\ 1 & \longrightarrow & G(\mathbf{Q})^{+}(\mathrm{rel}\,G') & \longrightarrow & \mathcal{E}_E(G, G', X^+) & \longrightarrow & \mathrm{Gal}(\bar{\mathbf{Q}}/E) & \longrightarrow & 1\\ & & \cup & & \cup & & \cup & & \\ 1 & \longrightarrow & T(\mathbf{Q}) & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathrm{Gal}(\bar{\mathbf{Q}}/E(h)) & \longrightarrow & 1\ . \end{array}$$

Since $r*1\in\mathcal{G}_2$ and $\iota\tau\in\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ map to the same element in $\bar\pi_0\pi(G_2)$, they are both the image of an element $\lambda=\lambda(h)\in\mathcal{E}_E(G, G', X^+)$. As $\iota\tau$ lies in $\mathrm{Gal}(\bar{\mathbf{Q}}/E(h))$ and $r*1$ lies in

$$\frac{T_2(\mathbf{A}^f)}{Z_2'(\mathbf{Q})^\wedge} *_{T_2(\mathbf{Q})/Z_2'(\mathbf{Q})} T(\mathbf{Q})\ ,$$

where $Z_2' = Z_2\cap T_2$, the element $\lambda(h)$ lies in $\mathcal{E}'$. Therefore $\lambda(h)$ fixes the point $[h] \in \mathrm{Sh}^0(G, G', X^+)$.

(b) Now consider $\varepsilon(\alpha) \in \mathcal{E}_E(G, G', X^+)$. As remarked in 4.3, we can use the diagram in (a) to define the map $\varepsilon: G(\mathbf{Q})^{\pm\wedge}(\mathrm{rel}\, G') \to \mathcal{E}_E$. Fix an element γ_1 of $G_1(\mathbf{Q})^-$ and let γ be its image in $G(\mathbf{Q})_-$. Since $G_1(\mathbf{Q}) \subset G_2(\mathbf{Q})$, the image of $\varepsilon(\alpha)$ in $\mathcal{G}_2$ is $\gamma_1 * \gamma^{-1}\alpha$; see 4.3. Therefore $\varepsilon(\alpha)\lambda(h) \in \mathcal{E}_E$ maps to $(\gamma_1 * \gamma^{-1}\alpha)(r * 1)$ in $\mathcal{G}_2$ and to $\iota(\iota\tau) = \tau$ in $\mathrm{Gal}(\overline{\mathbf{Q}}/E)$.

Consider the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') & \longrightarrow & \mathcal{E}_E & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{Q}}/E) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') & \longrightarrow & \mathcal{E}_{E_1} & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{Q}}/E_1) & \longrightarrow & 1 \end{array}$$

where $\mathcal{E}_{E_1} = \mathcal{E}_{E_1}(G, G', X^+)$. Since τ lies in $\mathrm{Gal}(\overline{\mathbf{Q}}/E_1)$, $\varepsilon(\alpha)\lambda(h) \in \mathcal{E}_E$ arises from an element $\varepsilon_1(\alpha, h) \in \mathcal{E}_{E_1}$. We have $\varepsilon_1(\alpha, h)[h] = \varepsilon(\alpha)\lambda(h)[h] = \varepsilon(\alpha)[h]$.

(c) Observe that (G_2, X_1) defines a Shimura variety, $(G_2, X_1) \to (G, X)$, and $E(G_2, X_1) = E(G_1, X_1) = E_1$. Thus we have an exact commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') & \longrightarrow & \mathcal{G}_2 & \xrightarrow{\pi_2} & \bar{\pi}_0\pi(G_2) & \longrightarrow & 1 \\ & & \| & & \uparrow{\scriptstyle \tilde{f}_2} & & \uparrow{\scriptstyle r_{G_2, X_1}} & & \\ 1 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') & \longrightarrow & \mathcal{E}_{E_1} & \longrightarrow & \mathrm{Gal}(\overline{\mathbf{Q}}/E_1) & \longrightarrow & 1\,. \end{array}$$

We show that $\tilde{f}_2(\varepsilon_1(\alpha, h)) = (\gamma_1 * \gamma^{-1}\alpha)(z\beta(\tau, \mu_1)^{-1} * 1)$.

We have a map

$$r_{F'}(F^\times, h_\Sigma): \mathrm{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow \bar{\pi}_0\pi(F^\times) = F^\times(\mathbf{A}^f)/F^\times(\mathbf{Q})^\wedge\,;$$

composing this map with $\bar{\pi}_0\pi(F^\times) \to \bar{\pi}_0\pi(G_2)$ (resp. $F^\times(\mathbf{A}^f)/F^\times(\mathbf{Q})^\wedge \to \mathcal{G}_2$), we obtain a map

$$r_{F'}: \mathrm{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow \bar{\pi}_0\pi(G_2) \left(\text{resp. } \tilde{r}_{F'}: \mathrm{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow \mathcal{G}_2\right).$$

Denote the product map of

$$\mathcal{E}_{E_1} \hookrightarrow \mathcal{E}_E \quad \text{and} \quad \mathcal{E}_{E_1} \xrightarrow{\pi_1} \mathrm{Gal}(\overline{\mathbf{Q}}/E_1) \hookrightarrow \mathrm{Gal}(\overline{\mathbf{Q}}/F')$$

by i, and the natural injection of $\mathrm{Gal}(\overline{\mathbf{Q}}/E_1)$ into $\mathrm{Gal}(\overline{\mathbf{Q}}/E) \times \mathrm{Gal}(\overline{\mathbf{Q}}/F')$ by j. Then the diagram

$$\begin{array}{ccccc} \mathcal{E}_{E_1} & \xrightarrow{i} & \mathcal{E}_E \times \mathrm{Gal}(\overline{\mathbf{Q}}/F') & \xrightarrow{f_2 \times \tilde{r}_{F'}} & \mathcal{G}_2 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathrm{Gal}(\overline{\mathbf{Q}}/E_1) & \xrightarrow{j} & \mathrm{Gal}(\overline{\mathbf{Q}}/E) \times \mathrm{Gal}(\overline{\mathbf{Q}}/F') & \xrightarrow{r_{G_2, X_2} \times r_{F'}} & \bar{\pi}_0\pi(G_2) \end{array}$$

is commutative. Since $X_1 = \{x_2 h_\Sigma \mid x_2 \in X_2\}$, we have $(r_{G_2, X_2} \times r_{F'}) \circ j = r_{G_2, X_1}$ and $(f_2 \times \tilde{r}_{F'}) \circ i = \tilde{f}_2$. Thus

$$\begin{aligned} \tilde{f}_2(\varepsilon_1(\alpha, h)) &= f_2(\varepsilon(\alpha)\lambda(h)) \cdot r_{F'}(\tau) = (\gamma_1 * \gamma^{-1}\alpha)(r * 1)(s * 1) \\ &= (\gamma_1 * \gamma^{-1}\alpha)(z\beta(\tau, \mu_1)^{-1} * 1)\,. \end{aligned}$$

(d) Next we show that on the canonical model of the Shimura variety $\mathrm{Sh}(G_2, X_1)$,

$$[\mathbf{ad}\,\alpha\circ{}^{\tau}h_1,\ 1]\tilde{f}_2(\varepsilon_1(\alpha, h)) = [\mathbf{ad}\,v\circ{}^{\tau}h_1,\ \beta_1^{-1}]\ ,$$

where $\beta_1 = \beta_1(\tau, \mu_1) = \beta(\tau, \mu_1)v^{-1}$. In fact, for any $\delta_1\in(G_2(\mathbf{A}^f)/Z_2(\mathbf{Q})^{\wedge})$,

$$(\gamma_1*\gamma^{-1}\alpha)(\delta_1*1) = (\gamma_1\cdot(\mathbf{ad}\,\gamma^{-1}\alpha)(\delta_1)*1)(1*\gamma^{-1}\alpha)\ .$$

Therefore, for $x_1\in X_1$,

$$\begin{aligned}[\mathbf{ad}\,\alpha\circ x_1,\ 1]&(\gamma_1*\gamma^{-1}\alpha)(\delta_1*1)\\ &= [\mathbf{ad}\,\alpha\circ x_1,\ \gamma_1\cdot(\mathbf{ad}\,\gamma^{-1}\alpha)(\delta_1)](1*\gamma^{-1}\alpha)\\ &= [\mathbf{ad}\,(\alpha^{-1}\gamma)\circ\mathbf{ad}\,\alpha\circ x_1,\ (\mathbf{ad}\,\alpha^{-1}\gamma)(\gamma_1)\cdot\delta_1]\\ &= [\mathbf{ad}\,((\mathbf{ad}\,\alpha^{-1})(\gamma_1))\circ x_1,\ (\mathbf{ad}\,\alpha^{-1})(\gamma_1)\cdot\delta_1]\\ &= [x_1,\ \delta_1]\end{aligned}$$

because $(\mathbf{ad}\,\alpha^{-1})(\gamma_1)\in G_2(\mathbf{Q})$. Especially, in view of (c),

$$\begin{aligned}[\mathbf{ad}\,\alpha\circ{}^{\tau}h_1,\ 1]&\tilde{f}_2(\varepsilon_1(\alpha, h))\\ &= [\mathbf{ad}\,\alpha\circ{}^{\tau}h_1,\ 1](\gamma_1*\gamma^{-1}\alpha)(z\beta(\tau, \mu_1)^{-1}*1)\\ &= [\mathbf{ad}\,\alpha\circ{}^{\tau}h_1,\ 1](\gamma_1*\gamma^{-1}\alpha)(zv^{-1}\beta_1^{-1}*1)\\ &= [{}^{\tau}h_1,\ zv^{-1}\beta_1^{-1}]\\ &= [\mathbf{ad}(vz^{-1})\circ{}^{\tau}h_1,\ \beta_1^{-1}] && (\text{as } zv^{-1}\in G_2(\mathbf{Q}))\\ &= [\mathbf{ad}\,v\circ{}^{\tau}h_1,\ \beta_1^{-1}] && (\text{as } z\in T_2(\mathbf{C}))\ .\end{aligned}$$

(e) The inclusion $(G_1, X_1)\hookrightarrow(G_2, X_1)$ induces maps $\mathrm{Sh}(G_1, X_1)\hookrightarrow\mathrm{Sh}(G_2, X_1)$, $\mathcal{G}_1\hookrightarrow\mathcal{G}_2$ and $\bar{\pi}_0\pi(G_1)\hookrightarrow\bar{\pi}_0\pi(G_2)$ (Deligne [1, 1.15.3]). Note that the composite $\mathcal{E}_{E_1}\xrightarrow{f_1}\mathcal{G}_1\to\mathcal{G}_2$ coincides with $\tilde{f}_2$. Since both $[\mathbf{ad}\,\alpha\circ{}^{\tau}h_1,\ 1]$ and $[\mathbf{ad}\,v\circ{}^{\tau}h_1,\ \beta_1^{-1}]$ are on the canonical model of the Shimura variety $\mathrm{Sh}(G_1, X_1)$, the result (d) shows

$$[\mathbf{ad}\,\alpha\circ{}^{\tau}h_1,\ 1]f_1(\varepsilon_1(\alpha, h)) = [\mathbf{ad}\,v\circ{}^{\tau}h_1,\ \beta_1^{-1}]\ .$$

(f) Finally we observe that $\tau\mu_1 = \tau\mu_2\cdot\tau\mu_\Sigma = \iota\mu_2\cdot\mu_\Sigma$ projects to $\iota\mu$ in $X_*(T)$. Thus $(T_1, {}^{\tau}h_1)$ is the lift of $(T, \eta(h))$ to (G_1, X_1). We also recall that $\pi_1(\varepsilon_1(\alpha, h)) = \tau$.

Therefore, for $\alpha\in G(\mathbf{Q})^-$,

$$\begin{aligned}\varepsilon(\alpha)[h] &= [\mathbf{ad}\,\alpha\circ\eta(h)]\\ &\Longleftrightarrow \varepsilon_1(\alpha, h)[h] = [\mathbf{ad}\,\alpha\circ\eta(h)] && (\text{by (b)})\\ &\Longleftrightarrow \tau[h_1,\ 1] = [\mathbf{ad}\,\alpha\circ{}^{\tau}h_1,\ 1]f_1(\varepsilon_1(\alpha, h)) && (\text{by (f)})\\ &\Longleftrightarrow \tau[h_1,\ 1] = [\mathbf{ad}\,v\circ{}^{\tau}h_1,\ \beta_1^{-1}] && (\text{by (e)})\end{aligned}$$

This completes the proof of Proposition 4.3.

7. Proof of conjecture B^0

In this section we prove conjecture B^0 for (G, G', X^+) of primitive abelian type. For (G, G', X^+) of type C, this is done in Shih [1]. We shall use this result to prove conjecture B^0 for all other cases. For completeness' sake, we start with a sketch of the proof for the type C case.

Every (G, G', X^+) of type C is obtained in the following fashion. Let F_0 be a totally real number field and B a quaternion algebra over F_0. We use σ to denote the main involution of B. Denote by I the set of embeddings of F_0 into $\mathbf{R}$, by I_{nc} the set of $\tau \in I$ at which B splits, and by I_c the complement of I_{nc}. Let Φ be a non-degenerate F_0-bilinear symmetric form on a free left B-module Λ of rank n such that

$$\Phi(bx, y) = \Phi(x, b^\sigma y) \quad \text{for} \quad x, y \in \Lambda \text{ and } b \in B .$$

Let G_* be the similitude group of Φ, considered as an algebraic group over F_0, and let $G_0 = \mathrm{Res}_{F_0/\mathbf{Q}} G_*$. There is a natural way of defining a $G_{0\mathbf{R}}$-conjugacy class X_0 of homomorphisms of S into $G_{0\mathbf{R}}$ such that (G_0, X_0) defines a Shimura variety; see Deligne [1, 6.3]. The reflex field $E(G_0, X_0)$ is totally real. Let $G = G_0^{\mathrm{ad}}$ and $G' = G_0^{\mathrm{der}}$. Let X_0^+ be a component of X_0. We can identify X_0^+ with a $G(\mathbf{R})^+$-conjugacy class X^+ of homomorphisms of S into $G_{\mathbf{R}}$. The triple (G, G', X^+) is of type C. The center Z_0 of G_0 is $\mathrm{Res}_{F_0/\mathbf{Q}}\mathbf{G}_m$. Thus

$$1 \longrightarrow F_0^\times \longrightarrow G_0(\mathbf{Q}) \longrightarrow G(\mathbf{Q}) \longrightarrow 1$$

is exact. In particular, $G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G') = G_0(\mathbf{Q})_+^\wedge / Z_0(\mathbf{Q})^\wedge$.

The first step towards proving conjecture B^0 is to show that there is $t \in Z_0(\mathbf{A}^f) \cap G'(\mathbf{A}^f)$ such that

$$[\mathbf{ad}\,\alpha \circ \eta(h)] = \varepsilon(\alpha\lambda)[h] \quad \text{for all} \quad \alpha \in G(\mathbf{Q})^- \text{ and } h \in X^+ , \tag{7.1}$$

where λ denotes the image of t in $G_0(\mathbf{Q})_+^\wedge / Z_0(\mathbf{Q})^\wedge = G(\mathbf{Q})^{+\wedge}(\mathrm{rel}\, G')$. (Note that an element t of $Z_0(\mathbf{A}^f)$ is in $G'(\mathbf{A}^f)$ if and only if $t^2 = 1$.) Two essential ingredients we need in proving the above claim are (i) uniqueness of canonical models and (ii) a concrete description of the automorphism group of $\mathrm{Sh}^0(G, G', X^+)_{\mathbf{C}}$. For the former, we refer to Deligne [3, 2.7.19], and the latter, to T. Miyake [1], or to Milne-Shih [3]. The element t is unique modulo ± 1.

Let F be a quadratic totally imaginary extension of F_0, and consider the diagram

$$(G, X) \longleftarrow (G_1, X_1) \hookrightarrow (C\,\mathrm{Sp}(V), S_\pm^\pm)$$

as in (1.4). Let $h \in X^+$ be special, and let $T \subset G$ be a $\mathbf{Q}$-rational torus such that h factors through $T_{\mathbf{R}}$. Lift (T, h) to $(T_1, h_1) \subset (G_1, X_1)$. Consider $(F^\times, h_\Sigma)$

and an automorphism τ of C as in Section 6. Since $(T_1, h_1) \hookrightarrow (C\mathrm{Sp}(V), S^{\pm})$, we have a diagram

$$\begin{array}{ccccc} [h_1, g_1] & \mathrm{Sh}(T_1, h_1) & \hookrightarrow & \mathcal{A}(V, \psi) \\ \downarrow & \downarrow \approx & & \approx \downarrow \chi_\tau \\ [{}^\tau h_1, g_1] & \mathrm{Sh}(T_1, {}_\tau h_1) & \hookrightarrow & \mathcal{A}({}^\tau V, {}^\tau\psi) \end{array}$$

as in conjecture CM (see § 2). Using (7.1) and the argument of Section 6, we can show that the diagram is commutative if the left vertical map is replaced by $[h_1, g_1] \mapsto [{}^\tau h_1, g_1]\lambda$. The image of $\mathrm{Sh}(T_1, h_1)$ in $\mathcal{A}(V, \psi)$ is a family of abelian varieties, which we denote by $\mathcal{A}(T_1, \{h_1\}, V)$.

Note that $\lambda = 1$ (i.e., $t = \pm 1$) if I_c is empty, because in this case $E(F^\times, h_\Sigma) = \mathbf{Q}$, so τ fixes the reflex field of (T_1, h_1) and conjecture CM holds.

To get a more precise statement, we assume that (G_1, X_1) is constructed using Shimura's original method [1] (see also Deligne [1, § 6]). Thus $V = \Lambda \otimes_{F_0} F$ and we have an exact sequence

$$1 \longrightarrow Z_0 \longrightarrow G_0 \times \mathrm{Res}_{F/\mathbf{Q}} G_m \longrightarrow G_1 \longrightarrow 1$$
$$a \longmapsto (a, a^{-1}) \ .$$

Note that t, when considered as an element of $G_1(\mathbf{A}^f)$, is in the center of $G_1(\mathbf{A}^f)$. We shall write $t = t(B, n)$ to emphasize its dependence on B and n. We choose (T, h) in the following way: Let P be a quadratic totally imaginary extension of F_0 that splits B. Then $T_0 = (\mathrm{Res}_{P/\mathbf{Q}} \mathbf{G}_m)^n$ can be embedded in G_0 and there is an $h_0 \in X_0^+$ that factors through $T_{0\mathbf{R}}$. We let (T, h) be the projection of (T_0, h_0) to (G, X^+).

With this choice of (G_1, X_1) and (T, h), T_1 is simply n copies of $\mathrm{Res}_{FP/\mathbf{Q}} G_m$, and the abelian varieties (up to isogeny) that appear in the family $\mathcal{A}(T_1, \{h_1\}, V)$ are n-fold products of an abelian variety with FP as its field of complex multiplication. The conjugate of the family under τ is described by the map

$$[h_1, g_1] \longmapsto [h_1, g_1]\lambda = [{}^\tau h_1, t(B, n) g_1] \ .$$

From this we conclude that $t(B, n) \in (F_0 \otimes \mathbf{A}^f)^\times$ modulo ± 1 is independent of n. Actually it only depends on $\Sigma = I_{nc}$, the set of infinite places where B splits, and not on B; see Shih [1, Proposition 11].

Thus to a totally real number field k and a non-empty set Σ of embeddings of k into $\mathbf{R}$, we can associate a well-defined element $t(k, \Sigma)$ of $(k \otimes \mathbf{A}^f)^\times$ modulo ± 1. We remark that the above considerations show that the statement at the end of Example 2.7 is correct if $\beta(\tau, \mu)$ is replaced by $t(F_0, \Sigma_0)\beta(\tau, \mu)$. Our goal is to prove that $t(k, \Sigma) = \pm 1$ for all k and Σ. This would complete the proof of conjecture B^0 for (G, G', X^+) of type C, and also the proof of Example 2.7. We noted already that $t(k, \Sigma) = \pm 1$ if

$\Sigma = I$, the set of all embeddings of k into $\mathbf{R}$.

By considering various families of the form $\mathcal{A}(T_1, \{h_1\}, V)$ and their conjugates, we obtain the following relations between $t(k, \Sigma)$'s. For simplicity, we shall use $t \equiv t'$ to mean that t is congruent to t' modulo ± 1. The fields k and k_1 are totally real.

(i) If (k_1, Σ_1) is an extension of (k, Σ), then $t(k, \Sigma) \equiv t(k_1, \Sigma_1)$ in $(k_1 \otimes \mathbf{A}^f)^\times$.

(ii) If $\gamma: k \to k_1$ is an isomorphism, and Σ is the pull back of Σ_1 by γ, then $t(k_1, \Sigma_1) \equiv \gamma(t(k, \Sigma))$ in $(k_1 \otimes \mathbf{A}^f)^\times$.

(iii) Assume that k is normal over $\mathbf{Q}$, and Σ_1 and Σ_2 are two disjoint sets of embeddings of k into $\mathbf{R}$. Then $t(k, \Sigma_1)t(k, \Sigma_2) \equiv t(k, \Sigma_1 \cup \Sigma_2)$.

These functorial properties are all we need to conclude that $t(k, \Sigma) = \pm 1$ for any k and Σ. For details, see Shih [1, Theorem 16]. Thus we have shown that conjecture B^0 holds for groups of type C, as well as the statement in Example 2.7.

Now turn to the proof of conjecture B^0 in general. For each (G, G', X^+) of primitive abelian type, we shall take the corresponding (G_0, X_0) as given in the appendix. We have $E(G_0, X_0) = E(G, X^+)$. In view of Proposition 5.2, we can either prove conjecture B^0 for (G, G', X^+) or prove conjecture B for (G_0, X_0). Recall that only those (G_0, X_0) with $E(G_0, X_0)$ totally real are under consideration.

(A) This is a trivial case, because (G_0, X_0) is embeddable in some $(C\mathrm{Sp}(V), S^\pm)$; see K. Miyake [1] and Remark 3.3(c).

($B, D^{\mathbf{R}}$) According to Shih [3], in this case (G_0, X_0) can be embedded in some (G_1, X_1) such that $(G_1^{\mathrm{ad}}, G_1^{\mathrm{der}}, X_1^+)$ is of type C. Since conjecture B holds for (G_1, X_1), it also holds for (G_0, X_0).

($D^{\mathbf{H}}$) We use Proposition 6.2 here. Let the notations be as in case ($D^{\mathbf{H}}$) of the appendix. Let

$$q \sim \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} \qquad (\varepsilon_i \in B)$$

be a diagonalization of q. Then for each i, $P_i = F_0(\varepsilon_i)$ is a CM-field, and $T_0 = \prod_{i=1}^n \mathrm{Res}_{P_i/\mathbf{Q}} \mathbf{G}_m$ can be embedded in G_0. Denote $\mathrm{Res}_{P_i/\mathbf{Q}} \mathbf{G}_m$ simply by $T_0^{(i)}$ so $T_0 = T_0^{(1)} \times \cdots \times T_0^{(n)}$. Let X_0^+ be a connected component of X_0. We can embed T_0 in G_0 in such a way that some $h_0 \in X_0^+$ factors through $T_{0\mathbf{R}}$. Let $h_0^{(i)}: \mathbf{S} \to T_{0\mathbf{R}}^{(i)}$ be the i^{th} factor of h_0, and let $E^{(i)} = E(T_0^{(i)}, h_0^{(i)})$. Then $E(T_0, h_0)$ is the composite of $E^{(1)}, \cdots, E^{(n)}$.

Put $G = G_0^{\mathrm{ad}}$, $G' = G_0^{\mathrm{der}}$ and let X^+ be the $G(\mathbf{R})^+$-conjugacy class of homomorphisms of $\mathbf{S}$ into $G_{\mathbf{R}}$ induced by X_0^+. Let (T, h) be the image of

(T_0, h_0) in (G, X^+). We have $E(T, h) = E(T_0, h_0)$, which is the composite of $E^{(1)}, \cdots, E^{(n)}$.

Let F be a quadratic totally imaginary extension of F_0, and let Σ and h_Σ be as in (1.4). Consider the usual diagram

$$(G, X) \longleftarrow (G_1, X_1) \hookrightarrow (C\mathrm{Sp}(V), S^\pm) .$$

As in the type C case, we can choose G_1 so that there is an exact sequence

$$\begin{array}{c} 1 \longrightarrow \mathrm{Res}_{F_0/\mathbf{Q}}\mathbf{G}_m \longrightarrow G_0 \times \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m \longrightarrow G_1 \longrightarrow 1 \\ a \longmapsto (a, a^{-1}) \end{array}$$

and take $\Lambda \otimes_{F_0} F$ as V. Let (T_1, h_1) be the lift of (T, h) to (G_1, X_1). Then $T_1 = \prod_{i=1}^n T_1^{(i)}$, where $T_1^{(i)} = \mathrm{Res}_{FP_i/\mathbf{Q}}\mathbf{G}_m$. We choose (T_0, h_0) and $(F^\times, h_\Sigma)$ in such a way that there exists an automorphism τ of $\mathbf{C}$ which induces the identity map on $E(F^\times, h_\Sigma)$, and the complex conjugation on $E(T, h)$; see Section 6.

The inclusion $(T_1, h_1) \hookrightarrow (C\mathrm{Sp}(V), S^\pm)$ identifies the Shimura variety $\mathrm{Sh}(T_1, h_1)$ with a family $\mathcal{A}(T_1, \{h_1\}, V)$ of abelian varieties. We show that conjecture CM holds for (T_1, h_1) and τ. In view of Propositions 2.10 and 6.2, this would prove that conjecture B^0 holds for (G, G', X^+).

Members of $\mathcal{A}(T_1, \{h_1\}, V)$ are (isogenous to) products $A_1 \times \cdots \times A_n$, where A_i is an abelian variety with complex multiplication by FP_i. Since $\mathrm{Sh}(T_1, h_1)$ is the product of $\mathrm{Sh}(T_1^{(i)}, h_1^{(i)})$, $i = 1, \cdots, n$, we only have to prove that conjecture CM holds for τ and each individual $(T_1^{(i)}, h_1^{(i)})$. As $E^{(i)} = E(T_0^{(i)}, h_0^{(i)})$ is a CM-subfield of $E(T, h)$, τ acts as ι on $E^{(i)}$. Therefore conjecture CM for $(T_1^{(i)}, h_1^{(i)})$ and τ is equivalent to the statement of Example 2.7. As we have established this statement while proving conjecture B^0 for groups of type C, the proof of conjecture B^0 for groups of type D^{H} is now completed.

Let (G, X) be of abelian type. By definition (see § 1), there exist $(G_i, G'_i, X_i^+)_i$ of primitive abelian type such that $G^{\mathrm{ad}} = \prod G_i$, G^{der} is a quotient of $\prod G'_i$, and $X^+ \approx \prod X_i^+$ for a suitable component X^+ of X. Assume $E(G, X)$ is totally real. Then $E(G^{\mathrm{ad}}, X^+)$ and all $E(G_i, X_i^+)$ are totally real. As conjecture B^0 holds for $\mathrm{Sh}^0(G_i, G'_i, X'_i)$ for each i, it holds for $\mathrm{Sh}^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$. Therefore conjecture B holds for $\mathrm{Sh}(G, X)$ in view of Proposition 5.2.

THEOREM 7.2. *Conjecture* B *holds for all* $\mathrm{Sh}(G, X)$ *of abelian type* (*such that* $E(G, X)$ *is totally real*).

Appendix

We give a list of classical reductive groups G_0 such that (G_0, X_0) defines a Shimura variety for a suitable X_0, and such that $(G_0^{\mathrm{ad}}, G_0^{\mathrm{der}})$ is of primitive abelian type. Every (G, G', X^+) of primitive abelian type is of the form $(G_0^{\mathrm{ad}}, G_0^{\mathrm{der}}, X_0^+)$ with some (G_0, X_0) from the following list. These (G_0, X_0) all have the property $E(G_0, X_0) = E(G_0^{\mathrm{ad}}, X_0^+)$.

In the following, F_0 is a totally real number field, and I is the set of all embeddings of F_0 into $\mathbf{R}$. We use $\bar{z}$ to denote the complex conjugate of $z \in \mathbf{C}$.

(A) Let K be a quadratic totally imaginary extension of F_0, and A a central simple algebra over K, together with an involution σ of the second kind. Then $\{x \in A^\times \mid xx^\sigma \in F_0^\times\}$ defines a reductive group G_* over F_0. We put $G_0 = \mathrm{Res}_{F_0/\mathbf{Q}} G_*$. The center of G_0 is $\mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_m$.

For non-negative integers r and s, we put

$$I_{r,s} = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix},$$

and

$$GU(r, s) = \{g \in \mathrm{GL}_{r+s}(\mathbf{C}) \mid g I_{r,s} {}^t\bar{g} = \nu(g) I_{r,s},\ \nu(g) \in \mathbf{R}^\times\}.$$

Then for each $\nu \in I$, there are non-negative integers r_ν and s_ν such that

$$G_0(\mathbf{R}) \cong \textstyle\prod_\nu GU(r_\nu, s_\nu).$$

Let $I_{nc} = \{\nu \in I \mid r_\nu \cdot s_\nu \neq 0\}$ and let I_c be the complement of I_{nc}. Define $h_\nu \colon \mathbf{S} \cong \mathbf{C}^\times \to GU(r_\nu, s_\nu)$ by

$$h_\nu(z) = \begin{cases} \begin{pmatrix} zI_{r_\nu} & 0 \\ 0 & \bar{z} I_{s_\nu} \end{pmatrix} & \text{if } \nu \in I_{nc} \\ 1 & \text{if } \nu \in I_c, \end{cases}$$

and define $h_0 \colon \mathbf{S} \to G_0(\mathbf{R})$ to be the product of h_ν's. Let X_0 be the $G_0(\mathbf{R})$-conjugacy class of h_0. Then (G_0, X_0) defines a Shimura variety. For any connected component X_0^+ of X_0, $(G_0^{\mathrm{ad}}, G_0^{\mathrm{der}}, X_0^+)$ is of type A.

The reflex field $E(G_0, X_0)$ is either $\mathbf{Q}$ or a CM-field. The former case happens if and only if $r_\nu = s_\nu$ for all $\nu \in I$. In this case the map η defined in Section 3 takes h_0 to $h_0' = \prod_\nu h_\nu'$, where

$$h_\nu'(z) = \begin{cases} \begin{pmatrix} \bar{z} I_r & 0 \\ 0 & zI_r \end{pmatrix}, & r = r_\nu = s_\nu, \quad \text{if } \nu \in I_{nc} \\ 1, & \text{if } \nu \in I_c. \end{cases}$$

(B) Let $n \geqq 3$ be an odd integer and q a quadratic form on an n-dimensional vector space over F_0 such that the signature of q at a $\nu \in I$ is $(n, 0)$, $(0, n)$, $(n-2, 2)$ or $(2, n-2)$. The special Clifford group of q defines a reductive group G_* over F_0. We put $G_0 = \mathrm{Res}_{F_0/\mathbf{Q}} G_*$. The center of G_0 is $\mathrm{Res}_{F_0/\mathbf{Q}} G_m$.

We refer to Shih [3] for the description of X_0 such that (G_0, X_0) defines a Shimura variety. The reflex field $E(G_0, X_0)$ is totally real. The derived group G_0^{der} is the spin group of q. $(G_0^{\mathrm{ad}}, G_0^{\mathrm{der}}, X_0^+)$ is of type B for any connected component X_0^+ of X_0.

(C) G_0 is the similitude group of a hermitian form over a quaternion algebra whose center is F_0; see Section 7.

(D^{R}) There are two cases:

(1) Same as type B, except $n \geqq 4$ is even.

(2) Let B be a totally indefinite quaternion algebra over F_0 and denote by σ the main involution of B. Let q be a σ-antihermitian form on a left free B-module of rank $n \geqq 2$. At each $\tau \in I$, q defines a quadratic form on a $2n$-dimensional real vector space. We assume that its signature is $(2n, 0)$, $(0, 2n)$, $(2n-2, 2)$ or $(2, 2n-2)$. Let G_* be the algebraic group over F_0 defined by the special Clifford group of q, and let $G_0 = \mathrm{Res}_{F_0/\mathbf{Q}} G_*$. We define X_0 as in Shih [3]. Then (G_0, X_0) defines a Shimura variety and $(G_0^{\mathrm{ad}}, G_0^{\mathrm{der}})$ is of type D^{R}.

In both cases $E(G_0, X_0)$ is totally real, and the center of G_0 is $\mathrm{Res}_{F_0/\mathbf{Q}} Z_*$, where Z_* is an extension of μ_2 by $\mathbf{G}_m$ over F_0.

(D^{H}) Let B be a quaternion algebra over F_0 with main involution σ. Let q be a σ-anti-hermitian form on a free left B-module Λ of rank $n \geqq 4$. Let I_{nc} be the set of $\tau \in I$ where B does not split, and let I_c be the complement of I_{nc}. As usual, we assume I_{nc} is non-empty; let r be its cardinality. We assume also that at every $\tau \in I_c$, the real quadratic form defined by q is definite. Let G_0 be the algebraic group over $\mathbf{Q}$ such that

$$G_0(\mathbf{Q}) = \{g \in \mathrm{GL}_B(\Lambda) \mid gq{}^tg^\sigma = \nu(g)q,\ \nu(g) \in F_0^\times \quad \text{and} \quad N(g) = \nu(g)^n\},$$

where N denotes the reduced norm from $\mathrm{End}_B(\Lambda)$ to F_0. Then $G_0(\mathbf{R})$ is isomorphic to the product of r copies of $GO^*(2n)$, where

$$GO^*(2n) = \left\{ g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbf{C}) \,\middle|\, g \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} {}^t\bar{g} = \nu(g) \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \right.$$
$$\left. \nu(g) \in \mathbf{R}^\times \quad \text{and} \quad \det(g) = \nu(g)^n \right\}.$$

Define $h_0\colon \mathbf{S} \cong \mathbf{C}^\times \to G_0(\mathbf{R}) \cong \big(GO^*(2n)\big)^r$ so that each component of h_0 is given by

$$z \longmapsto \begin{pmatrix} zI_n & 0 \\ 0 & \bar{z}I_n \end{pmatrix}$$

and define X_0 to be the $G_0(\mathbf{R})$-conjugacy class of h_0. Then (G_0, X_0) defines a Shimura variety. The center of G_0 is $\mathrm{Res}_{F_0/\mathbf{Q}} G_m$.

The reflex field $E(G_0, X_0)$ is either a CM-field or a totally real field, depending on whether n is odd or even. Let $h_0'\colon \mathbf{S} \cong \mathbf{C}^\times \to G_0(\mathbf{R}) \cong (GO^*(2n))^r$ be a homomorphism such that each component of h_0' is given by

$$z \longmapsto \begin{pmatrix} \bar{z}I_n & 0 \\ 0 & zI_n \end{pmatrix}.$$

Then h_0' belongs to X_0 if and only if n is even. In this case the map η defined in Section 3 takes h_0 to h_0'.

When $n = 4$, we also allow G_0 of the "mixed type". We let I_c be the set of $\tau \in I$ such that B splits at τ and the quadratic form over $\mathbf{R}$ determined by q at τ is definite. Denote the complement of I_c by I_{nc}. Let s (resp. r) be the number of $\tau \in I_{nc}$ at which B splits (resp. does not split). We assume $r > 0$. If B splits at a $\tau \in I_{nc}$, we assume that the signature of the real quadratic form determined by q at τ is (6, 2) or (2, 6). Then

$$G_0(\mathbf{R}) \cong (GO^*(8))^r \times (GO(6, 2)^+)^s\,,$$

where

$$GO(6, 2)^+ = \left\{ g \in \mathrm{GL}_8(\mathbf{R}) \mid g \begin{pmatrix} I_6 & 0 \\ 0 & -I_2 \end{pmatrix} {}^t g = \nu(g) \begin{pmatrix} I_6 & 0 \\ 0 & -I_2 \end{pmatrix}, \right.$$
$$\left. \nu(g) \in \mathbf{R}^\times \quad \text{and} \quad \det g > 0 \right\}.$$

We define $h_0\colon \mathbf{S} \to G_0(\mathbf{R})$: the homomorphism into the component $GO^*(8)$ is defined as before and the homomorphism into the component $GO(6, 2)^+$ is given by

$$z \longmapsto \begin{pmatrix} |z|^2 I_6 & & 0 \\ & \operatorname{Re} z^2 & \operatorname{Im} z^2 \\ 0 & -\operatorname{Im} z^2 & \operatorname{Re} z^2 \end{pmatrix}.$$

Let X_0 be the $G_0(\mathbf{R})$-conjugacy class of h_0. Then (G_0, X_0) defines a Shimura variety.

The reflex field $E(G_0, X_0)$ is totally real. Let h_0' be the image of h_0 under the map η of Section 3. Then the component of h_0' corresponding to the factor $GO^*(8)$ is

$$z \longmapsto \begin{pmatrix} \bar{z}I_4 & 0 \\ 0 & zI_4 \end{pmatrix},$$

and to the factor $GO(6, 2)^+$, it is

$$z \longmapsto \begin{pmatrix} |z|^2 I_6 & & 0 \\ & \operatorname{Re} z^2 & -\operatorname{Im} z^2 \\ 0 & \operatorname{Im} z^2 & \operatorname{Re} z^2 \end{pmatrix}.$$

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