PROOF (OF THEOREM 10.15) Lemma 10.16 shows that the map

$$\mathbb{Z}_\ell \otimes \operatorname{Hom}(A,B) \to \operatorname{Hom}(T_\ell A, T_\ell B) \tag{4}$$

has torsion-free cokernel.

We next show that it is injective in the case that A is simple and $B = A$. The elements of $\mathbb{Z}_\ell \otimes \operatorname{End}(A)$ are finite sums

$$\sum c_i \otimes a_i, \quad c_i \in \mathbb{Z}_\ell, \quad a_i \in \operatorname{End}(A),$$

and so it suffices to show that the map $\mathbb{Z}_\ell \otimes M \to \operatorname{End}(T_\ell A)$ is injective for any finitely generated submodule M of $\operatorname{End}(A)$. Let $e_1, \ldots, e_m$ be a basis for M; we have to show that $T_\ell(e_1), \ldots, T_\ell(e_m)$ are linearly independent over $\mathbb{Z}_\ell$ in $\operatorname{End}(T_\ell A)$. Let P be the polynomial function on $\operatorname{End}^0(A)$ such that $P(\alpha) = \deg(\alpha)$ for all $\alpha \in \operatorname{End}(A)$. Because A is simple, every nonzero endomorphism α of A is an isogeny, and so $P(\alpha)$ is an integer > 0. The map $P\colon \mathbb{Q}M \to \mathbb{Q}$ is continuous for the real topology because it is a polynomial function, and so $U = \{v \mid P(v) < 1\}$ is an open neighbourhood of 0. As

$$(\mathbb{Q}M \cap \operatorname{End}(A)) \cap U \subset \operatorname{End}(A) \cap U = 0,$$

we see that $\mathbb{Q}M \cap \operatorname{End}(A)$ is discrete in $\mathbb{Q}M$, and therefore is a finitely generated $\mathbb{Z}$-module (ANT 4.15). Hence there is a common denominator for the elements of $\mathbb{Q}M \cap \operatorname{End}(A)$:

(*) there exists an integer N such that $N(\mathbb{Q}M \cap \operatorname{End}(A)) \subset M$.

Suppose that $T_\ell(e_1), \ldots, T_\ell(e_m)$ are linearly dependent, so that there exist $a_i \in \mathbb{Z}_\ell$, not all zero, such that $\sum a_i T_\ell(e_i) = 0$. For any $n \in \mathbb{N}$, there exist $n_i \in \mathbb{Z}$ such that $\ell^n \mid (a_i - n_i)$ in $\mathbb{Z}_\ell$ for all i. Then $\sum n_i T_\ell(e_i)$ is divisible by ℓ^n in $\operatorname{End}(T_\ell A)$, and so $\sum n_i e_i$ is divisible by ℓ^n in $\operatorname{End}(A)$ (by 10.16). Hence $N\left(\sum n_i e_i/\ell^n\right) \in N(\mathbb{Q}M \cap \operatorname{End}(A))$.

When n is sufficiently large, $|n_i|_\ell = |a_i|_\ell$ and $|Na_i|_\ell > 1/\ell^n$ for some i with $a_i \neq 0$. Then $|Nn_i/\ell^n|_\ell = |Na_i|_\ell \cdot \ell^n > 1$, and so $Nn_i/\ell^n \notin \mathbb{Z}$. Therefore $N\left(\sum n_i e_i/\ell^n\right)$ does not lie in M, which contradicts (*). This completes the proof that (4) is injective when $A = B$ is simple.

For arbitrary A, B choose isogenies $\prod_i A_i \to A$ and $B \to \prod_j B_j$ with the A_i and B_j simple. Then

$$\operatorname{Hom}(A,B) \to \prod\nolimits_{i,j} \operatorname{Hom}(A_i, B_j)$$

is injective. As $\operatorname{Hom}(A_i, B_j) = 0$ if A_i and B_j are not isogenous, and $\operatorname{Hom}(A_i, B_j) \hookrightarrow \operatorname{End}(A_i)$ if there exists an isogeny $B_j \to A_i$, the natural map

$$\left(\prod\nolimits_{i,j} \operatorname{Hom}(A_i, B_j)\right) \otimes \mathbb{Z}_\ell \to \prod\nolimits_{i,j} \operatorname{Hom}(T_\ell A_i, T_\ell B_j)$$

is injective. It follows that (4) is injective for A and B. □