Chapter 15

Algebraic Varieties over the Complex Numbers

This is only a brief outline.

It is not hard to show that there is a unique way to endow all algebraic varieties over $\mathbb{C}$ with a topology such that:

(a) on $\mathbb{A}^n = \mathbb{C}^n$ it is just the usual complex topology;

(b) on closed subsets of $\mathbb{A}^n$ it is the induced topology;

(c) all morphisms of algebraic varieties are continuous;

(d) it is finer than the Zariski topology.

We call this new topology the *complex topology* on V. Note that (a), (b), and (c) determine the topology uniquely for affine algebraic varieties ((c) implies that an isomorphism of algebraic varieties will be a homeomorphism for the complex topology), and (d) then determines it for all varieties.

Of course, the complex topology is *much* finer than the Zariski topology — this can be seen even on $\mathbb{A}^1$. In view of this, the next two propositions are a little surprising.

PROPOSITION 15.1. *If a nonsingular variety is connected for the Zariski topology, then it is connected for the complex topology.*

Consider, for example, $\mathbb{A}^1$. Then, certainly, it is connected for both the Zariski topology (that for which the nonempty open subsets are those that omit only finitely many points) and the complex topology (that for which V is homeomorphic to $\mathbb{R}^2$). When we remove a circle from V, it becomes disconnected for the complex topology, but remains connected for the Zariski topology. This does not contradict the proposition, because $\mathbb{A}^1_{\mathbb{C}}$ with a circle removed is not an algebraic variety.

Let V be a connected nonsingular (hence irreducible) curve. We prove that it is connected for the complex topology. Removing or adding a finite number of points to V will not change whether it is connected for the complex topology, and so we can assume that V is projective. Suppose V is the disjoint union of two nonempty open (hence closed) sets V_1 and V_2. According to the Riemann-Roch theorem (14.8), there exists a nonconstant rational function f on V having poles only in V_1. Therefore, its restriction to

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V_2 is holomorphic. Because V_2 is compact, f is constant on each connected component of V_2 (Cartan 1963, VI.4.5), say, $f(z) = a$ on some infinite connected component. Then $f(z) - a$ has infinitely many zeros, which contradicts the fact that it is a rational function.

The general case can be proved by induction on the dimension (Shafarevich 1994, VII.2).

PROPOSITION 15.2. *Let V be an algebraic variety over $\mathbb{C}$, and let C be a constructible subset of V (in the Zariski topology); then the closure of C in the Zariski topology equals its closure in the complex topology.*

PROOF. Mumford 1966, I, 10, Corollary 1, p. 60. □

For example, if U is an open dense subset of a closed subset Z of V (for the Zariski topology), then U is also dense in Z for the complex topology.

The next result helps explain why completeness is the analogue of compactness for topological spaces.

PROPOSITION 15.3. *Let V be an algebraic variety over $\mathbb{C}$; then V is complete (as an algebraic variety) if and only if it is compact for the complex topology.*

PROOF. Mumford 1966, I, 10, Theorem 2, p. 60. □

In general, there are many more holomorphic (complex analytic) functions than there are polynomial functions on a variety over $\mathbb{C}$. For example, by using the exponential function it is possible to construct many holomorphic functions on $\mathbb{C}$ that are not polynomials in z, but all these functions have nasty singularities at the point at infinity on the Riemann sphere. In fact, the only meromorphic functions on the Riemann sphere are the rational functions. This generalizes.

THEOREM 15.4. *Let V be a complete nonsingular variety over $\mathbb{C}$. Then V is, in a natural way, a complex manifold, and the field of meromorphic functions on V (as a complex manifold) is equal to the field of rational functions on V.*

PROOF. Shafarevich 1994, VIII, 3.1, Theorem 1. □

This provides one way of constructing compact complex manifolds that are not algebraic varieties: find such a manifold M of dimension n such that the transcendence degree of the field of meromorphic functions on M is $< n$. For a torus $\mathbb{C}^g/\Lambda$ of dimension $g > 1$, this is typically the case. However, when the transcendence degree of the field of meromorphic functions is equal to the dimension of manifold, then M can be given the structure, not necessarily of an algebraic variety, but of something more general, namely, that of an ***algebraic space in the sense of Artin***. Roughly speaking, an algebraic space is an object that is locally an affine algebraic variety, where locally means for the étale "topology" rather than the Zariski topology.[1]

One way to show that a complex manifold is algebraic is to embed it into projective space.

THEOREM 15.5. *Every closed analytic submanifold of $\mathbb{P}^n$ is algebraic.*

[1]Artin, Michael. Algebraic spaces. Whittemore Lectures given at Yale University, 1969. Yale Mathematical Monographs, 3. Yale University Press, New Haven, Conn.-London, 1971. vii+39 pp.

Knutson, Donald. Algebraic spaces. Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971. vi+261 pp.

PROOF. See Shafarevich 1994, VIII, 3.1, in the nonsingular case. □

COROLLARY 15.6. *Every holomorphic map from one nonsingular projective algebraic variety to a second nonsingular projective algebraic variety is algebraic (i.e., a morphism of algebraic varieties).*

PROOF. Let $\varphi : V \to W$ be the map. Then the graph Γ_φ of φ is a closed analytic subset of $V \times W$, and hence is algebraic according to the theorem. The projection map $\Gamma_\varphi \to V$ is an isomorphism of algebraic varieties by 8.60, which means that its inverse $V \to \Gamma_\varphi$ is algebraic. As φ is the composite of the isomorphism $V \to \Gamma_\varphi$ with the projection $\Gamma_\varphi \to W$, and both are algebraic, φ itself is algebraic. □

Since, in general, it is hopeless to write down a set of equations for a variety (it is a fairly hopeless task even for an abelian variety of dimension 3), the most powerful way we have for constructing varieties is to construct first a complex manifold and then prove that it has a natural structure as a algebraic variety. Sometimes one can then show that it has a canonical model over some number field, and then it is possible to reduce the equations defining it modulo a prime of the number field, and obtain a variety in characteristic p.

For example, it is known that $\mathbb{C}^g/\Lambda$ (Λ a lattice in $\mathbb{C}^g$) has the structure of an algebraic variety if and only if there is a skew-symmetric form ψ on $\mathbb{C}^g$ having certain simple properties relative to Λ. The variety is then an abelian variety, and all abelian varieties over $\mathbb{C}$ are of this form.

Bibliography

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