Chapter 14

Differentials (Outline)

In this subsection, we sketch the theory of Kähler differentials.[1] We allow k to be an arbitrary field.

Let A be a k-algebra, and let M be an A-module. Recall (from §5) that a k-derivation is a k-linear map $D\colon A \to M$ satisfying Leibniz's rule:

$$D(fg) = f\circ Dg + g\circ Df, \quad \text{all } f, g \in A.$$

DEFINITION 14.1. A pair $(\Omega^1_{A/k}, d)$ comprising an A-module $\Omega^1_{A/k}$ and a k-derivation $d\colon A \to \Omega^1_{A/k}$ is called the ***module of differential one-forms*** for A over k if it has the following universal property: for any k-derivation $D\colon A \to M$, there is a unique A-linear map $\alpha\colon \Omega^1_{A/k} \to M$ such that $D = \alpha\circ d$,

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "D"'] & \Omega^1_{A/k} \arrow[d, dashed, "\alpha"] \\ & M. \end{tikzcd}$$

Thus

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_{A\text{-linear}}(\Omega^1_{A/k}, M).$$

It can be defined to be the free A-module with basis the symbols df, $f \in A$, modulo the relations

$$d(f+g) = df + dg, \quad d(fg) = f\cdot dg + g\cdot df, \quad dc = 0 \text{ if } c \in k.$$

EXAMPLE 14.2. Let $A = k[X_1, \ldots, X_n]$; then $\Omega^1_{A/k}$ is the free A-module with basis the symbols $dX_1, \ldots, dX_n$, and

$$df = \sum \frac{\partial f}{\partial X_i} dX_i.$$

This is Chapter 14 of Algebraic Geometry by J.S. Milne. Last revised November 4, 2024

[1] Introduced by Erich Kähler in the 1930s, and later adopted by algebraic geometers. They avoid the double dualizing of other approaches. Note that

$$\mathrm{Der}_k(A, A) = \mathrm{Hom}_{A\text{-linear}}(\Omega^1_{A/k}, A),$$

so, if $\Omega^1_{A/k}$ is finitely generated and projective, then it is canonically isomorphic to the A-linear dual of $\mathrm{Der}_k(A, A)$. See Chapter 11.

EXAMPLE 14.3. Let $A = k[X_1, ..., X_n]/\mathfrak{a}$; then $\Omega^1_{A/k}$ is the free A-module with basis the symbols $dX_1, ..., dX_n$ modulo the relations:

$$df = 0 \text{ for all } f \in \mathfrak{a}.$$

EXAMPLE 14.4. A homomorphism $A \to A'$ of k-algebras gives rise to an isomorphism

$$A' \otimes_A \Omega^1_{A/k} \to \Omega^1_{A'/k}.$$

In particular, for any multiplicative subset S of A, we have

$$S^{-1}\Omega^1_{A/k} \simeq S^{-1}A \otimes_A \Omega^1_{A/k} \simeq \Omega^1_{S^{-1}A/k}.$$

PROPOSITION 14.5. *Let V be a variety. For each $n \geq 0$, there is a unique sheaf of $\mathcal{O}_V$-modules $\Omega^n_{V/k}$ on V such that $\Omega^n_{V/k}(U) = \bigwedge^n \Omega^1_{A/k}$ whenever $U = \operatorname{Spm} A$ is an open affine of V.*

PROOF. Omitted. □

The sheaf $\Omega^n_{V/k}$ is called the ***sheaf of differential n-forms on** V*.

EXAMPLE 14.6. Let E be the affine curve

$$Y^2 = X^3 + aX + b,$$

and assume $X^3 + aX + b$ has no repeated roots (so that E is nonsingular). Write x and y for regular functions on E defined by X and Y. On the open set $D(y)$ where $y \neq 0$, let $\omega_1 = dx/y$, and on the open set $D(3x^2 + a)$, let $\omega_2 = 2dy/(3x^2 + a)$. Since $y^2 = x^3 + ax + b$,

$$2ydy = (3x^2 + a)dx.$$

and so ω_1 and ω_2 agree on $D(y) \cap D(3x^2 + a)$. Since $E = D(y) \cup D(3x^2 + a)$, we see that there is a differential ω on E whose restrictions to $D(y)$ and $D(3x^2 + a)$ are ω_1 and ω_2 respectively. It is an easy exercise in working with projective coordinates to show that ω extends to a differential one-form on the whole projective curve

$$Y^2Z = X^3 + aXZ^2 + bZ^3.$$

In fact, $\Omega^1_{C/k}(C)$ is a one-dimensional vector space over k, with ω as basis. Note that

$$\omega = dx/y = dx/(x^3 + ax + b)^{\frac{1}{2}},$$

which cannot be integrated in terms of elementary functions. Its integral is called an elliptic integral (integrals of this form arise when one tries to find the arc length of an ellipse). The study of elliptic integrals was one of the starting points for the study of algebraic curves.

In general, if C is a complete nonsingular absolutely irreducible curve of genus g, then $\Omega^1_{C/k}(C)$ is a vector space of dimension g over k.

PROPOSITION 14.7. *If V is nonsingular, then $\Omega^1_{V/k}$ is a locally free sheaf of rank* $\dim(V)$ *(that is, every point P of V has a neighbourhood U such that $\Omega^1_{V/k}|U \approx (\mathcal{O}_V|U)^{\dim(V)}$).*

PROOF. Omitted. □

Let C be a complete nonsingular absolutely irreducible curve, and let ω be a nonzero element of $\Omega^1_{k(C)/k}$. We define the divisor (ω) of ω as follows: let $P \in C$; if t is a uniformizing parameter at P, then dt is a basis for $\Omega^1_{k(C)/k}$ as a $k(C)$-vector space, and so we can write $\omega = f\,dt$, $f \in k(V)^{\times}$; define $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$, and $(\omega) = \sum \mathrm{ord}_P(\omega)P$. Because $k(C)$ has transcendence degree 1 over k, $\Omega^1_{k(C)/k}$ is a $k(C)$-vector space of dimension one, and so the divisor (ω) is independent of the choice of ω up to linear equivalence. By an abuse of language, one calls (ω) for any nonzero element of $\Omega^1_{k(C)/k}$ a *canonical class* K on C. For a divisor D on C, let $\ell(D) = \dim_k(L(D))$.

THEOREM 14.8 (RIEMANN-ROCH). *Let C be a complete nonsingular absolutely irreducible curve over k.*

(a) *The degree of a canonical divisor is* $2g - 2$.

(b) *For any divisor D on C,*

$$\ell(D) - \ell(K - D) = 1 + g - \deg(D).$$

More generally, if V is a smooth complete variety of dimension d, it is possible to associate with the sheaf of differential d-forms on V a canonical linear equivalence class of divisors K. This divisor class determines a rational map to projective space, called the *canonical map*.

References

Shafarevich, 1994, III.5.

Mumford 1999, III.4.