Chapter 13

Coherent Sheaves and Vector Bundles

In this chapter, k is an arbitrary field.

a. Coherent sheaves

13.1. Let V be a k-ringed space. Suppose that, for each open subset U of V, we have an $\mathcal{O}_V(U)$-module $\mathcal{M}(U)$ and, for each pair of open subsets $U' \subset U$, a "restriction" map $\mathrm{res}_{U',U}\colon \mathcal{M}(U) \to \mathcal{M}(U')$ compatible with the module structures. The system is a ***sheaf of*** $\mathcal{O}_V$***-modules*** if (a) $U \rightsquigarrow \mathcal{M}(U)$ is a functor from the category of open subsets of V and (b) $\mathcal{M}$ satisfies the sheaf condition. The first condition means that

$$\begin{cases} \mathrm{res}_{U,U} \text{ is the identity map} & \text{for } U \subset U, \\ \mathrm{res}_{U'',U'} \circ \mathrm{res}_{U' \circ U} = \mathrm{res}_{U'',U} & \text{for } U'' \subset U' \subset U. \end{cases}$$

The second condition means that, for any open covering $U = \bigcup U_i$ of an open subset U,

$$\mathcal{M}(U) \simeq \{(m_i) \in \prod \mathcal{M}(U_i) \mid \mathrm{res}_{U_i \cap U_j, U_i}(m_i) = \mathrm{res}_{U_i \cap U_j, U_j}(m_j) \text{ for all } i, j\}.$$

With the obvious notion of morphism, the sheaves of $\mathcal{O}_V$-modules become a category.

13.2. Now let $V = \operatorname{Spm} A$ be an affine algebraic scheme over k, and let M be a finitely generated A-module. There is a unique sheaf of $\mathcal{O}_V$-modules $\mathcal{M}$ on V such that, for all $f \in A$,

$$\Gamma(D(f), \mathcal{M}) = M_f \quad (= A_f \otimes_A M)$$

(apply 10.5). Such an $\mathcal{O}_V$-module $\mathcal{M}$ is said to be ***coherent***. A homomorphism $M \to N$ of A-modules defines a homomorphism $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}_V$-modules, and $M \rightsquigarrow \mathcal{M}$ is a fully faithful functor from the category of finitely generated A-modules to the category of coherent $\mathcal{O}_V$-modules, with quasi-inverse $\mathcal{M} \rightsquigarrow \Gamma(V, \mathcal{M})$. We sometimes write $\tilde{M}$ for the coherent $\mathcal{O}_V$-module defined by M.

Now consider an algebraic scheme V over k. An $\mathcal{O}_V$-module $\mathcal{M}$ is said to be ***coherent*** if, for every open affine subset U of V, $\mathcal{M}|U$ is coherent. It suffices to check this condition for the sets in some open affine covering of V.

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For example, $\mathcal{O}_V^n$ is a coherent $\mathcal{O}_V$-module. An $\mathcal{O}_V$-module $\mathcal{M}$ is said to be ***locally free of rank*** n if it is locally isomorphic to $\mathcal{O}_V^n$, i.e., if every point $P \in V$ has an open neighbourhood such that $\mathcal{M}|U \approx \mathcal{O}_V^n$. A locally free $\mathcal{O}_V$-module of rank n is coherent.

13.3. For two coherent $\mathcal{O}_V$-modules $\mathcal{M}$ and $\mathcal{N}$, there is a unique coherent $\mathcal{O}_V$-module $\mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}$ such that

$$\Gamma(U, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}) = \Gamma(U, \mathcal{M}) \otimes_{\Gamma(U,\mathcal{O}_V)} \Gamma(U, \mathcal{N}) \qquad (*)$$

for all open affines U in V. Indeed, let $U = \bigcup U_i$ with the U_i open affines, and define $\Gamma(U, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N})$ to be the kernel of

$$\prod_i \Gamma(U_i, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}) \rightrightarrows \prod_{i,j} \Gamma(U_{ij}, \mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}), \quad U_{ij} \stackrel{\text{def}}{=} U_i \cap U_j.$$

If the restrictions of $\mathcal{M}$ and $\mathcal{N}$ to some open affine $U = \operatorname{Spm} A$ correspond to A-modules M and N, then $\mathcal{M} \otimes_{\mathcal{O}_V} \mathcal{N}|U$ corresponds to $M \otimes_A N$. The reader should be careful not to assume that (*) holds for nonaffine open subsets U (see example 13.10 below).

13.4. For coherent $\mathcal{O}_V$-modules $\mathcal{M}$ and $\mathcal{N}$, define $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{M}, \mathcal{N})$ to be the presheaf on V such that

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_V}(\mathcal{M}, \mathcal{N})) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{M}|U, \mathcal{N}|U)$$

for all open U in V. It is easy to see that this is a sheaf. If the restrictions of $\mathcal{M}$ and $\mathcal{N}$ to some open affine $U = \operatorname{Spm} A$ correspond to A-modules M and N, then $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{M}, \mathcal{N})|U$ is the sheaf of $\mathcal{O}_U$-modules defined by the A-module $\operatorname{Hom}_A(M, N)$. Hence, $\mathcal{H}om(\mathcal{M}, \mathcal{N})$ is again a coherent $\mathcal{O}_V$-module.

13.5. Let $v \in V$, and let $\mathcal{M}$ be a coherent $\mathcal{O}_V$-module. We define a $\kappa(v)$-module $\mathcal{M}(v)$ as follows: after replacing V with an open neighbourhood of v, we can assume that it is affine; hence we may suppose that $V = \operatorname{Spm}(A)$, that v corresponds to a maximal ideal $\mathfrak{m}$ in A (so that $\kappa(v) = A/\mathfrak{m}$), and that $\mathcal{M}$ corresponds to the A-module M; we then define

$$\mathcal{M}(v) = M \otimes_A \kappa(v) = M/\mathfrak{m}M.$$

It is a finitely generated vector space over $\kappa(v)$. Do not confuse $\mathcal{M}(v)$ with the stalk $\mathcal{M}_v$ of $\mathcal{M}$ which, with the above notation, is $M_\mathfrak{m} = M \otimes_A A_\mathfrak{m}$. Thus

$$\mathcal{M}(v) = \mathcal{M}_v/\mathfrak{m}\mathcal{M}_v = \kappa(v) \otimes_{A_\mathfrak{m}} \mathcal{M}_\mathfrak{m}.$$

Nakayama's lemma (1.3) shows that

$$\mathcal{M}(v) = 0 \Rightarrow \mathcal{M}_v = 0.$$

The ***support*** of a coherent $\mathcal{O}_V$-module $\mathcal{M}$ is

$$\operatorname{Supp}(\mathcal{M}) = \{v \in V \mid \mathcal{M}(v) \neq 0\} = \{v \in V \mid \mathcal{M}_v \neq 0\}.$$

Suppose that V is affine, and that $\mathcal{M}$ corresponds to the A-module M. Let $\mathfrak{a}$ be the annihilator of M:

$$\mathfrak{a} \stackrel{\text{def}}{=} \{f \in A \mid fM = 0\}.$$

Then $M/\mathfrak{m}M \neq 0 \iff \mathfrak{m} \supset \mathfrak{a}$ (for otherwise $A/\mathfrak{m}A$ contains a nonzero element annihilating $M/\mathfrak{m}M$), and so

$$\operatorname{Supp}(\mathcal{M}) = V(\mathfrak{a}).$$

Thus the support of a coherent module is a closed subset of V.

Note that if $\mathcal{M}$ is locally free of rank n, then $\mathcal{M}(v)$ is a vector space of dimension n for all v. There is a converse of this.

PROPOSITION 13.6. *Assume that V is reduced. If $\mathcal{M}$ is a coherent $\mathcal{O}_V$-module such that $\mathcal{M}(v)$ has constant dimension n for all $v \in V$, then $\mathcal{M}$ is a locally free of rank n.*

PROOF. We may assume that V is affine, say, $V = \mathrm{Spm}(A)$, and that $\mathcal{M}$ corresponds to the finitely generated A-module M. Fix a maximal ideal $\mathfrak{m}$ of A, and let $x_1, \dots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a basis for it over $\kappa(v)$. Consider the map

$$\gamma\colon A^n \to M, \quad (a_1, \dots, a_n) \mapsto \sum a_i x_i.$$

Its cokernel is a finitely generated A-module whose support does not contain v. Therefore there is an element $f \in A$, $f \notin \mathfrak{m}$, such that γ defines a surjection $A_f^n \to M_f$. After replacing A with A_f we may assume that γ itself is surjective. For every maximal ideal $\mathfrak{n}$ of A, the map $(A/\mathfrak{n})^n \to M/\mathfrak{n}M$ is surjective, and hence (because of the condition on the dimension of $\mathcal{M}(v)$) bijective. Therefore, the kernel of γ is contained in $\mathfrak{n}^n$ (meaning $\mathfrak{n} \times \cdots \times \mathfrak{n}$) for all maximal ideals $\mathfrak{n}$ in A, and Corollary 10.21[1] shows that this implies that the kernel is zero. □

13.7. In the above proof, we showed the following (without assume V to be reduced): let $v \in |V|$; if U is an open neighbourhood of v and $x_1, \dots, x_n \in \Gamma(U, \mathcal{M})$ are such that their images in $\mathcal{M}(v)$ generate it, then there is an open neighbourhood $U' \subset U$ of v such that $x_1|U', \dots, x_n|U'$ generate $\mathcal{M}|U'$.

13.8. With a little more effort, it is posssible to prove the following more precise result. Let $\mathcal{M}$ be a coherent $\mathcal{O}_V$-module on an algebraic scheme V over k. The function

$$v \mapsto \dim_{\kappa(v)} \mathcal{M}(v)\colon |V| \to \mathbb{Z}$$

is upper semicontinuous, i.e., the sets

$$U_r \stackrel{\text{def}}{=} \{v \mid \dim_{\kappa(v)} \mathcal{M}(v) \le r\}$$

are open for all $r \in \mathbb{N}$ (so the dimension is constant on an open subset, and jumps on closed subsets). Let r_0 be the smallest value such that U_{r_0} is nonempty. If V is reduced, then $\mathcal{M}|U_{r_0}$ is locally constant of rank r_0 (by 13.5).

b. Invertible sheaves.

An ***invertible sheaf*** on V is a locally free $\mathcal{O}_V$-module $\mathcal{L}$ of rank 1. The tensor product of two invertible sheaves is again an invertible sheaf. In this way, we get a product structure on the set of isomorphism classes of invertible sheaves:

$$[\mathcal{L}] \cdot [\mathcal{L}'] \stackrel{\text{def}}{=} [\mathcal{L} \otimes \mathcal{L}'].$$

The product structure is associative and commutative (because tensor products are associative and commutative, up to isomorphism), and $[\mathcal{O}_V]$ is an identity element. Define

$$\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_V).$$

[1]For a reduced k-algebra, the intersection of the maximal ideals is zero.

Clearly, $\mathcal{L}^\vee$ is free of rank 1 over any open set where $\mathcal{L}$ is free of rank 1, and so $\mathcal{L}^\vee$ is again an invertible sheaf. Moreover, the canonical map

$$\mathcal{L}^\vee \otimes \mathcal{L} \to \mathcal{O}_V, \quad (f,x) \mapsto f(x)$$

is an isomorphism (because it is an isomorphism over any open subset where $\mathcal{L}$ is free). Thus

$$[\mathcal{L}^\vee][\mathcal{L}] = [\mathcal{O}_V].$$

For this reason, we often write $\mathcal{L}^{-1}$ for $\mathcal{L}^\vee$.

From these remarks, we see that the set of isomorphism classes of invertible sheaves on V is a group — it is called the ***Picard group***, $\mathrm{Pic}(V)$, of V.

We say that an invertible sheaf $\mathcal{L}$ is ***trivial*** if it is isomorphic to $\mathcal{O}_V$ — then $\mathcal{L}$ represents the zero element in $\mathrm{Pic}(V)$.

PROPOSITION 13.9. *An invertible sheaf $\mathcal{L}$ on a complete variety V is trivial if and only if both it and its dual have nonzero global sections, i.e.,*

$$\Gamma(V,\mathcal{L}) \neq 0 \neq \Gamma(V,\mathcal{L}^\vee).$$

PROOF. We may assume that V is irreducible. Note first that, for any $\mathcal{O}_V$-module $\mathcal{M}$ on any variety V, the map

$$\mathrm{Hom}(\mathcal{O}_V,\mathcal{M}) \to \Gamma(V,\mathcal{M}), \quad \alpha \mapsto \alpha(1)$$

is an isomorphism.

Next recall that the only regular functions on a complete variety are the constant functions (10.142), i.e., $\Gamma(V,\mathcal{O}_V) = k'$ where k' is the algebraic closure of k in $k(V)$. Hence $\mathcal{H}om(\mathcal{O}_V,\mathcal{O}_V) = k'$, and so a homomorphism $\mathcal{O}_V \to \mathcal{O}_V$ is either 0 or an isomorphism.

We now prove the proposition. The sections define nonzero homomorphisms

$$s_1\colon \mathcal{O}_V \to \mathcal{L}, \quad s_2\colon \mathcal{O}_V \to \mathcal{L}^\vee.$$

We can take the dual of the second homomorphism, and so obtain nonzero homomorphisms

$$\mathcal{O}_V \xrightarrow{s_1} \mathcal{L} \xrightarrow{s_2^\vee} \mathcal{O}_V.$$

The composite is nonzero, and hence an isomorphism, which shows that $s_2^\vee$ is surjective, and this implies that it is an isomorphism (for any ring A, a surjective homomorphism of A-modules $A \to A$ is bijective because 1 must map to a unit). □

c. Invertible sheaves and divisors.

Now assume that V is nonsingular and irreducible. For a divisor D on V, the vector space $L(D)$ is defined to be

$$L(D) = \{f \in k(V)^\times \mid \mathrm{div}(f) + D \geq 0\}.$$

We make this definition local: define $\mathcal{L}(D)$ to be the sheaf on V such that, for any open set U,

$$\Gamma(U,\mathcal{L}(D)) = \{f \in k(V)^\times \mid \mathrm{div}(f) + D \geq 0 \text{ on } U\} \cup \{0\}.$$

The condition "$\operatorname{div}(f)+D \geq 0$ on U" means that, if $D=\sum n_Z Z$, then $\operatorname{ord}_Z(f)+n_Z \geq 0$ for all Z with $Z \cap U \neq \emptyset$. Thus, $\Gamma(U,\mathcal{L}(D))$ is a $\Gamma(U,\mathcal{O}_V)$-module, and if $U \subset U'$, then $\Gamma(U',\mathcal{L}(D)) \subset \Gamma(U,\mathcal{L}(D))$. We define the restriction map to be this inclusion. In this way, $\mathcal{L}(D)$ becomes a sheaf of $\mathcal{O}_V$-modules.

Suppose D is principal on an open subset U, say $D|U=\operatorname{div}(g)$, $g \in k(V)^\times$. Then

$$\Gamma(U,\mathcal{L}(D))=\{f \in k(V)^\times \mid \operatorname{div}(fg) \geq 0 \text{ on } U\} \cup \{0\}.$$

Therefore,

$$\Gamma(U,\mathcal{L}(D)) \to \Gamma(U,\mathcal{O}_V), \quad f \mapsto fg,$$

is an isomorphism. These isomorphisms clearly commute with the restriction maps for $U' \subset U$, and so we obtain an isomorphism $\mathcal{L}(D)|U \to \mathcal{O}_U$. Since every D is locally principal, this shows that $\mathcal{L}(D)$ is locally isomorphic to $\mathcal{O}_V$, i.e., that it is an invertible sheaf. If D itself is principal, then $\mathcal{L}(D)$ is trivial.

Next we note that the canonical map

$$\mathcal{L}(D) \otimes \mathcal{L}(D') \to \mathcal{L}(D+D'), \quad f \otimes g \mapsto fg$$

is an isomorphism on any open set where D and $D'$are principal, and hence it is an isomorphism globally. Therefore, we have a homomorphism

$$\operatorname{Div}(V) \to \operatorname{Pic}(V), \quad D \mapsto [\mathcal{L}(D)],$$

which is zero on the principal divisors.

EXAMPLE 13.10. Let V be an elliptic curve, and let P be the point at infinity. Let D be the divisor $D=P$. Then $\Gamma(V,\mathcal{L}(D))=k$, the ring of constant functions, but $\Gamma(V,\mathcal{L}(2D))$ contains a nonconstant function x. Therefore,

$$\Gamma(V,\mathcal{L}(2D)) \neq \Gamma(V,\mathcal{L}(D)) \otimes \Gamma(V,\mathcal{L}(D)),$$

— in other words, $\Gamma(V,\mathcal{L}(D) \otimes \mathcal{L}(D)) \neq \Gamma(V,\mathcal{L}(D)) \otimes \Gamma(V,\mathcal{L}(D))$.

PROPOSITION 13.11. *For an irreducible nonsingular variety, the map $D \mapsto [\mathcal{L}(D)]$ defines an isomorphism*

$$\operatorname{Div}(V)/\operatorname{PrinDiv}(V) \to \operatorname{Pic}(V).$$

PROOF. (Injectivity). If s is an isomorphism $\mathcal{O}_V \to \mathcal{L}(D)$, then $g=s(1)$ is an element of $k(V)^\times$ such that

(a) $\operatorname{div}(g)+D \geq 0$ (on the whole of V);

(b) if $\operatorname{div}(f)+D \geq 0$ on U, that is, if $f \in \Gamma(U,\mathcal{L}(D))$, then $f=h(g|U)$ for some $h \in \Gamma(U,\mathcal{O}_V)$.

Statement (a) says that $D \geq \operatorname{div}(-g)$ (on the whole of V). Suppose U is such that $D|U$ admits a local equation $f=0$. When we apply (b) to $-f$, then we see that $\operatorname{div}(-f) \leq \operatorname{div}(g)$ on U, so that $D|U+\operatorname{div}(g) \geq 0$. Since the U's cover V, together with (a) this implies that $D=\operatorname{div}(-g)$.

(Surjectivity). Define

$$\Gamma(U,\mathcal{K})=\begin{cases} k(V)^\times \text{ if } U \text{ is open and nonempty} \\ 0 \text{ if } U \text{ is empty.}\end{cases}$$

Because V is irreducible, $\mathcal{K}$ becomes a sheaf with the obvious restriction maps. On any open subset U where $\mathcal{L}|U \approx \mathcal{O}_U$, we have $\mathcal{L}|U \otimes \mathcal{K} \approx \mathcal{K}$. Since these open sets form a covering of V, V is irreducible, and the restriction maps are all the identity map, this implies that $\mathcal{L} \otimes \mathcal{K} \approx \mathcal{K}$ on the whole of V. Choose such an isomorphism, and identify $\mathcal{L}$ with a subsheaf of $\mathcal{K}$. On any U where $\mathcal{L} \approx \mathcal{O}_U$, $\mathcal{L}|U = g\mathcal{O}_U$ as a subsheaf of $\mathcal{K}$, where g is the image of $1 \in \Gamma(U, \mathcal{O}_V)$. Define D to be the divisor such that, on a U, g^{-1} is a local equation for D. □

EXAMPLE 13.12. Suppose V is affine, say $V = \operatorname{Spm} A$. We know that coherent $\mathcal{O}_V$-modules correspond to finitely generated A-modules, but what do the locally free sheaves of rank n correspond to? They correspond to finitely generated *projective* A-modules (CA, 12.6). The invertible sheaves correspond to finitely generated projective A-modules of rank 1. Suppose for example that V is a curve, so that A is a Dedekind domain. This gives a new interpretation of the ideal class group: it is the group of isomorphism classes of finitely generated projective A-modules of rank one (i.e., such that $M \otimes_A K$ is a vector space of dimension one).

This can be proved directly. First show that every (fractional) ideal is a projective A-module — it is obviously finitely generated of rank one; then show that two ideals are isomorphic as A-modules if and only if they differ by a principal divisor; finally, show that every finitely generated projective A-module of rank 1 is isomorphic to a fractional ideal (by assumption $M \otimes_A K \approx K$; when we choose an identification $M \otimes_A K = K$, then $M \subset M \otimes_A K$ becomes identified with a fractional ideal). [Exercise: Prove the statements in this last paragraph.]

REMARK 13.13. Quite a lot is known about $\operatorname{Pic}(V)$, the group of divisors modulo linear equivalence, or of invertible sheaves up to isomorphism. For example, for any complete nonsingular variety V, there is an abelian variety P canonically attached to V, called the ***Picard variety*** of V, and an exact sequence

$$0 \to P(k) \to \operatorname{Pic}(V) \to \operatorname{NS}(V) \to 0$$

where $\operatorname{NS}(V)$ is a finitely generated group called the Néron-Severi group.

Much less is known about algebraic cycles of codimension > 1, and about locally free sheaves of rank > 1 (and the two do not correspond exactly, although the Chern classes of locally free sheaves are algebraic cycles).

d. Direct images and inverse images of coherent sheaves.

Consider a homomorphism $A \to B$ of rings. From an A-module M, we get a B-module $B \otimes_A M$, which is finitely generated if M is finitely generated. Conversely, a B-module M can also be considered an A-module, but it usually won't be finitely generated (unless B is finitely generated as an A-module). Both these operations extend to maps of varieties.

Consider a regular map $\alpha \colon W \to V$, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_V$-modules. There is a unique coherent sheaf of $\mathcal{O}_W$-modules $\alpha^*\mathcal{F}$ with the following property: for any open affine subsets U' and U of W and V respectively such that $\alpha(U') \subset U$, $\alpha^*\mathcal{F}|U'$ is the sheaf corresponding to the $\Gamma(U', \mathcal{O}_W)$-module $\Gamma(U', \mathcal{O}_W) \otimes_{\Gamma(U, \mathcal{O}_V)} \Gamma(U, \mathcal{F})$.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_V$-modules. For any open subset U of V, we define $\Gamma(U, \alpha_*\mathcal{F}) = \Gamma(\alpha^{-1}U, \mathcal{F})$, regarded as a $\Gamma(U, \mathcal{O}_V)$-module via the map $\Gamma(U, \mathcal{O}_V) \to \Gamma(\alpha^{-1}U, \mathcal{O}_W)$. Then $U \mapsto \Gamma(U, \alpha_*\mathcal{F})$ is a sheaf of $\mathcal{O}_V$-modules. In general, $\alpha_*\mathcal{F}$ will not be coherent, even when $\mathcal{F}$ is.

LEMMA 13.14. (a) *For any regular maps $U \xrightarrow{\alpha} V \xrightarrow{\beta} W$ and coherent $\mathcal{O}_W$-module $\mathcal{F}$ on W, there is a canonical isomorphism*

$$(\beta\alpha)^*\mathcal{F} \xrightarrow{\approx} \alpha^*(\beta^*\mathcal{F}).$$

(b) *For any regular map $\alpha\colon V \to W$, α^* maps locally free sheaves of rank n to locally free sheaves of rank n (hence also invertible sheaves to invertible sheaves). It preserves tensor products, and, for an invertible sheaf $\mathcal{L}$, $\alpha^*(\mathcal{L}^{-1}) \simeq (\alpha^*\mathcal{L})^{-1}$.*

PROOF. (a) This follows from the fact that, given homomorphisms of rings $A \to B \to T$, $T \otimes_B (B \otimes_A M) = T \otimes_A M$.

(b) This again follows from well-known facts about tensor products of rings. □

e. Vector bundles

Let V be an algebraic variety over k. The ***trivial vector bundle of rank*** n over V is the variety $V \times \mathbb{A}^n$ equipped with the projection map $V \times \mathbb{A}^n \to V$. A pair (E, π) comprising an algebraic variety E and a (projection) morphism $\pi\colon E \to V$ is a ***vector bundle of rank*** n over V if it is locally isomorphic to the trivial vector bundle. More precisely, we require that, for some open covering $V = \bigcup U_i$, there exist isomorphisms $\varphi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{A}^n$ such that

(a) for each i, the diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{A}^n \arrow[dl, "p"] \\ & U_i & \end{tikzcd}$$

commutes;

(b) for each i, j, the isomorphism

$$\varphi_j \circ \varphi_i^{-1}\colon\ (U_i \cap U_j) \times \mathbb{A}^n \to (U_i \cap U_j) \times \mathbb{A}^n$$

is linear on the fibres, i.e., the map $(u, (a_1, \ldots, a_n)) \mapsto (u, (b_1, \ldots, b_n))$ is k-linear in the second variable.

A ***morphism of vector bundles*** $(E, p) \to (E', p')$ is a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & V & \end{tikzcd}$$

such that φ is linear on the fibres. A vector bundle of rank 1 is called a ***line bundle***.

Let V be an algebraic variety over k. A ***vector sheaf*** on V is a locally free sheaf $\mathcal{V}$ of $\mathcal{O}_V$-modules of finite rank. In order for a vector sheaf $\mathcal{W}$ to be a vector subsheaf of a vector sheaf $\mathcal{V}$, we require that the maps $\mathcal{W}_s \to \mathcal{V}_s$ be injective. The vector sheaves of rank n on V are exactly the sheaves of sections of vector bundles of rank n, and the vector subsheaves are exactly the sheaves of sections of the vector subbundles.

Examples

13.15 (THE TAUTOLOGICAL LINE BUNDLE). Let V be a closed subvariety of $\mathbb{P}^n$. A point P in V, when regarded as a point in $\mathbb{P}^n$, is a line l_P in $\mathbb{A}^{n+1}$. For the tautological line bundle, the fibre over P is the line l_P. As a set,

$$B \overset{\text{def}}{=} \{(a,P) \in \mathbb{A}^{n+1} \times V \mid a \in l_P\}.$$

A global section of B is a regular map $P \mapsto (P, s(P))$ with $s(P) \in \mathbb{A}^{n+1}$. As there are no nonconstant maps $V \to \mathbb{A}^{n+1}$ (10.142), such a section must be zero.

To be continued.