Appendix D

Derived Categories

Let $F\colon \mathsf{A} \to \mathsf{B}$ be a left exact functor of abelian categories, and assume that A has enough injectives. For an object A of A, we choose an injective resolution $A \to I^{\cdot}$ of A and define $(R^r F)(A) = H^r(FI^{\cdot})$. A different choice $A \to J^{\cdot}$ gives a different object $(R^r F)(A)' = H^r(FJ^{\cdot})$, but there exists a map $\alpha\colon I^{\cdot} \to J^{\cdot}$ of resolutions, and the corresponding map $(R^r F)(A) \to (R^r F)(A)'$ is an isomorphism which is independent of the choice of α. This means that the object $(R^r F)(A)$ of B is well-defined up to a unique isomorphism.

In passing from the complex $FI^{\cdot}$ to the family of objects $H^r(FI^{\cdot})$, we have lost information. However, $FI^{\cdot}$ is not well-defined as an object of the category $C(\mathsf{B})$ of complexes in B: different choices of $I^{\cdot}$ may give different (nonisomorphic) complexes $FI^{\cdot}$. The derived category $D(\mathsf{B})$ of B is defined to have the same objects as $C(\mathsf{B})$ but it has a new notion of morphism for which $FI^{\cdot}$ is well-defined up to a unique isomorphism.

The homotopy category

Let A be an abelian category. The category $C(\mathsf{A})$ of complexes in A is an abelian category, and we sometimes identify A with the full subcategory of $C(\mathsf{A})$ consisting of the complexes concentrated in degree 0. For a complex X, define $X[r]$ to be the complex with $(X[r])^n = X^{n+r}$ and $d^n_{X[r]} = (-1)^r d^n_X$. The functor

$$X \rightsquigarrow X[1]\colon C(\mathsf{A}) \to C(\mathsf{A})$$

is called the ***translation*** (or ***shift***) ***functor***, and is denoted by T.

Consider two maps $f, g\colon X \to Y$ of complexes. A ***homotopy*** from f to g is a family of maps $s^n\colon X^n \to Y^{n-1}$ such that

$$f^n - g^n = s^{n+1} \circ d^n_X + d^{n-1}_Y \circ s^n.$$

We say that f and g are ***homotopic*** if there exists a homotopy from one to the other, i.e., if there exists a diagram

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$$\begin{tikzcd}
X^{n-1} \arrow[r, "d_X^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[dl, "s^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] & X^{n+1} \arrow[dl, "s^{n+1}"'] \\
Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1}
\end{tikzcd}$$

in which the vertical maps from X^n to Y^n differ by the sum of the two maps constructed using the diagonal arrows.

z1 EXAMPLE D.1 Let A and B be objects of A, and let $B \to I^{\cdot}$ be an injective resolution of B. A map of complexes $A \to I^{\cdot}[r]$ is simply a map $f\colon A \to I^r$ such that $d_I^r \circ f = 0$. Moreover, ϕ is homotopic to zero if and only if there exists a map $s\colon B \to I^{r-1}$ such that $d_I^{r-1} \circ s = f$. Thus, the group of homotopy classes of maps $B \to I^{\cdot}[r]$ is precisely $H^r(\mathrm{Hom}(B, I^{\cdot})) \stackrel{\text{def}}{=} \mathrm{Ext}^r(B, C)$.

A map of complexes homotopic to zero remains homotopic to zero when composed with another map. Therefore, we can define a new category $K(\mathsf{A})$ as follows

$$\begin{cases} \mathrm{ob}(K(\mathsf{A})) & = & \mathrm{ob}(C(\mathsf{A})) \\ \mathrm{Hom}_{K(\mathsf{A})}(X,Y) & = & \mathrm{Hom}_{C(\mathsf{A})}(X,Y)/\{f \mid f \text{ homotopic to zero}\}. \end{cases}$$

In other words, the objects of $K(\mathsf{A})$ are the complexes in A and the morphisms are the homotopy classes of maps of complexes.

For each r, the functor $X^{\cdot} \rightsquigarrow H^r(X^{\cdot})$ factors through $K(\mathsf{A})$.

Triangulated categories

The category $K(\mathsf{A})$ is additive but not abelian (in contrast to $C(\mathsf{A})$. In particular, there is no notion of an exact sequence in $K(\mathsf{A})$. Of course, one can say that a sequence of objects of $K(\mathsf{A})$ is exact if it is exact as a sequence in $C(\mathsf{A})$, but this notion is not intrinsic to the category $K(\mathsf{A})$ — it depends on how we have described it — and the sequences arising in this way are not well-behaved. So what extra structure does $K(\mathsf{A})$ have? It has a distinguished class of triangles.

For the moment, let K be an additive category with an automorphism $T\colon \mathsf{K} \to \mathsf{K}$ (the translation functor), and write $X[r]$ for $T^r X$. A ***triangle*** in K is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]. \qquad (1)$$ ez6

Sometimes, this triangle is written

$$\begin{tikzcd}
 & X \arrow[dl] & \\
Y \arrow[rr] & & Z, \arrow[ul, "+"']
\end{tikzcd}$$

which explains the name. A ***morphism of triangles*** is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, "w"] & X[1] \arrow[d, "u[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd} \tag{2}$$ ez7

in K.

Now return to $C(\mathsf{A})$. The ***cone*** of a map $f\colon X \to Y$ of complexes is the complex

$$C(f) = X[1] \oplus Y, \quad d_{C(f)} = \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix}$$

i.e.,

$$\cdots \longrightarrow X^n \oplus Y^{n-1} \xrightarrow{d^{n-1}_{C(f)}} X^{n+1} \oplus Y^n \longrightarrow \cdots$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} -d^n_X x \\ f^n x + d^{n-1}_Y y \end{pmatrix}$$

There are projection maps $y \mapsto (0, y)\colon Y \to C(f)$ and $(x, y) \mapsto -x\colon C(f) \to X[1]$, and hence a triangle

$$X \to Y \to C(f) \to X[1]. \tag{3}$$ ez3

A triangle in $K(\mathsf{A})$ isomorphic to one of this form is said to be ***distinguished***.

The collection of distinguished triangles in $K(\mathsf{A})$ satisfies the following conditions.[1]

TR1 (Existence axiom). For every object X in A, the triangle $X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$ is distinguished; for every morphism $X \to Y$ in $K(\mathsf{A})$, there exists a distinguished triangle $X \to Y \to Z \to X[1]$; every triangle isomorphic to a distinguished triangle is distinguished.

TR2 (Rotation axiom). A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished.

TR3 (Morphisms axiom). Every diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, dashed] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1], \end{tikzcd}$$

in which the rows are distinguished triangles and the square at left commutes can be completed to a morphism of triangles.

TR4 (octahedral axiom). For every pair of composable morphisms f_1, f_2 and every triple of distinguished triangles

$$X \xrightarrow{f_1} Y \xrightarrow{g_1} A \xrightarrow{h_1} X[1]$$

$$Y \xrightarrow{f_2} Z \xrightarrow{g_2} B \xrightarrow{h_2} Y[1]$$

$$X \xrightarrow{f_2 \circ f_1} Z \xrightarrow{g'} C \xrightarrow{h'} X[1]$$

[1] We follow Verdier's original numbering. Not all authors are so scrupulous.

based on f_1, f_2, $f_2 \circ f_1$, there exists a octahedron[2] based on f_1, f_2 and containing the distinguished triangles as faces:

$$\begin{tikzcd}
X \arrow[rr, "f_1"] \arrow[rrrr, bend left, "f_2\circ f_1"] & & Y \arrow[rr, "f_2"] \arrow[dl, "g_1"] & & Z \arrow[dl, "g_2"] \arrow[ddll, bend left, "g'"] \\
& A \arrow[ul, "h_1", "+"'] \arrow[dr, dashed, "f"] & & B \arrow[ll, dashed, "h", "+"'] \arrow[ul, "h_2", "+"'] & \\
& & C \arrow[uull, bend left, "h'", "+"'] \arrow[ur, dashed, "g"'] & &
\end{tikzcd} \tag{4}$$

ez4

Each triangle either commutes or is distinguished. In other words, there exists a distinguished triangle

$$A \xrightarrow{f} C \xrightarrow{g} B \xrightarrow{h} A[1]$$

such that

$$h_1 = h' \circ f, \quad g_2 = g \circ g', \quad h = g_1 \circ h_2$$
$$h_2 \circ g = f_1 \circ h', \quad f \circ g_1 = g' \circ f_2.$$

An additive category together with a translation functor (automorphism $T\colon X \rightsquigarrow X[1]$) and a collection of (distinguished) triangles satisfying TR1–TR4 is called a ***triangulated category.*** Let $F\colon \mathsf{K} \to \mathsf{K}'$ be a functor of triangulated categories together with an isomorphism $F \circ T \to T' \circ F$. Then F is said to be ***exact*** (or ***triangulated***) if it maps distinguished triangles to distinguished triangles.

Let K be a triangulated category. A ***cohomological functor*** on K is an additive functor $H\colon \mathsf{K} \to \mathsf{A}$ to an abelian category such that, for every distinguished triangle $X \to Y \to Z \to X[1]$ in K, the sequence

$$H(X) \to H(Y) \to H(Z)$$

is exact. Then TR2 implies that $H(Y) \to H(Z) \to H(X[1])$ is exact, etc., and so there is a long exact sequence

$$\cdots \to H^r(X) \to H^r(Y) \to H^r(Z) \to H^{r+1}(X) \to \cdots$$

in which we have written $H^r(X)$ for $H(X[r])$.

For example, if $\mathsf{K} = K(\mathsf{A})$, then $X \rightsquigarrow H^0(X)$ is exact, and $H^r(X)$ has its usual meaning. In particular, from a morphism $f\colon X \to Y$, we obtain a long exact sequence

$$\cdots \to H^r(X) \xrightarrow{H^r(f)} H^r(Y) \to H^r(C(f)) \to H^{r+1}(X) \to \cdots.$$

If f is injective, then $H^r(C(f)) \simeq H^r(Z)$ where Z is the cokernel of f (cf. Remark D.11 below). If f is surjective, then $H^r(C(f)) \simeq H^{r+1}(Z)$ where Z is the kernel of f.

[2] I have flattened it. The reader should imagine it folded so that the outside triangle forms the back face.

Let K be a triangulated category. For every object A in K, $\mathrm{Hom}(A,-)$ is a cohomological functor, and so, from each distinguished triangle (1), we get a long exact sequence

$$\cdots \to \mathrm{Hom}(A,X[r]) \to \mathrm{Hom}(A,Y[r]) \to \mathrm{Hom}(A,Z[r]) \to \mathrm{Hom}(A,X[r+1]) \to \cdots$$

Using this and the usual five-lemma, it is possible to deduce the triangulated five-lemma: let (u,v,w) be a morphism of distinguished triangles (see (2)); if u and v are isomorphisms, then so also is w.

NOTES

z9 D.2 In a triangulated category, the distinguished triangles play the role of short exact sequences. We can think of the first three triangles in TR4 as giving isomorphisms $Y/X \simeq A$, $Z/Y \simeq B$, $Z/X \simeq C$; then the distinguished triangle (f,g,h) gives us an isomorphism $B \simeq C/A$, i.e.,

$$Z/Y \simeq (Z/X)/(Y/X).$$

Thus TR4 plays the role of one of the Noether isomorphism theorems.

z9a D.3 The cone of a map $f\colon X \to Y$ of complexes is often written $Y \oplus X[1]$, in which case the differential becomes

$$d_{C(f)} = \begin{pmatrix} d_Y & f[1] \\ 0 & d_{X[1]} \end{pmatrix}.$$

z9b D.4 Let $Y \to C$ be an injective morphism of complexes such that, for each n, the sequence

$$0 \to Y^n \to C^n \to C^n/Y^n \to 0$$

is split exact. Choose a splitting,

$$C^n = Y^n \oplus C^n/Y^n$$

for each n, and let $X = (C/Y)[-1]$. With respect to the decomposition

$$C = Y \oplus X[1], \qquad (5)$$ ez9

the differential on C becomes

$$d_C = \begin{pmatrix} d_Y & f[1] \\ 0 & d_{X[1]} \end{pmatrix}$$

with f a morphism of complexes $X \to Y$, and (5) identifies Y with the cone of f.

z10 D.5 A weak variant of TR4 states that, for every pair of composable morphisms f_1, f_2, there exists an octahedron (4) based on f_1, f_2. It is known that, in the presence of TR1,2,3, the weak form of TR4 implies the usual form. On the other hand, it is known that, in the presence of TR1,2, the usual form of TR4 implies TR3. (See the MR review 1867203 by Balmer.)

Differential graded categories

yThe passage from $C(\mathsf{A})$ to $K(\mathsf{A})$ described above can be abstracted as follows. A ***graded category*** is an additive category C equipped with gradations

$$\mathrm{Hom}(A,B) = \bigoplus\nolimits_n \mathrm{Hom}^n(A,B)$$

on the Hom groups that are compatible with composition of morphisms; in particular, $\mathrm{id}_A \in \mathrm{Hom}^0(A,A)$. Such a category is a ***differential graded (DG) category*** if, in addition, it is equipped with differentials $d\colon \mathrm{Hom}^n(A,B) \to \mathrm{Hom}^{n+1}(A,B)$ of degree one such that $d \circ d = 0$ and

$$d(g \circ f) = dg \circ f + (-1)^{\deg g}(g \circ df)$$

whenever g is homogeneous and $g \circ f$ is defined. Thus, for every pair (A,B) of objects in a DG-category, we get a complex

$$\cdots \to \mathrm{Hom}^n(A,B) \xrightarrow{d} \mathrm{Hom}^{n+1}(A,B) \to \cdots.$$

A ***DG morphism*** $A \to B$ is an element of $Z^0(\mathrm{Hom}^{\cdot}(A,B))$. For example, the category $C^{\cdot}(\mathsf{A})$ with the same objects as $C(\mathsf{A})$ but with the morphisms

$$\mathrm{Hom}_{C^{\cdot}(\mathsf{A})}(A,B) = \bigoplus\nolimits_n \mathrm{Hom}_{C(\mathsf{A})}(A,B[n])$$

is a graded category; with the differential

$$\begin{aligned} d\colon \mathrm{Hom}^n(A,B) &\to \mathrm{Hom}^{n+1}(A,B) \\ d(f) &= d_B \circ f + (-1)^n f \circ d_A, \end{aligned}$$

it becomes DG category. Note that, in this case, $Z^0(\mathrm{Hom}^{\cdot}(A,B))$ consists of the usual (degree 0) homomorphisms of complexes $A \to B$ and $H^0(\mathrm{Hom}^{\cdot}(A,B))$ consists of the homotopy classes of homomorphisms of complexes $A \to B$.

A ***strongly pre-triangulated category*** is a DG category C such that:

◇ for each object A of C and $m \in \mathbb{Z}$, there exists an object, denoted $A[m]$, representing the functor $C \rightsquigarrow \mathrm{Hom}^m(C,A)$), so

$$\mathrm{Hom}(C,A[m]) = \mathrm{Hom}^m(C,A), \text{ all } C \in \mathrm{ob}(\mathsf{C});$$

◇ for each morphism $f\colon A \to B$ in C with $d \circ f = 0$, there exists an object, denoted $\mathrm{Cone}(f)$, representing the functor sending each $C \in \mathrm{ob}(\mathsf{C})$ to the cone on

$$\mathrm{Hom}(C,A) \xrightarrow{f\circ -} \mathrm{Hom}(C,B).$$

Let C be a DG category. The associated ***homotopy category*** $\mathrm{Ho}(\mathsf{C})$ is the graded additive category with the same objects as C but with $\mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(A,B) = H^0(\mathrm{Hom}^{\cdot}_{\mathsf{C}}(A,B))$. If C is strongly pre-triangulated, then $\mathrm{Ho}(\mathsf{C})$ has a translation functor, namely, $A \rightsquigarrow A[1]$, and a class of distinguished triangles, namely, those isomorphic to one of the form

$$A \xrightarrow{f} B \to \mathrm{Cone}(f) \to A[1].$$

With this structure, $\mathrm{Ho}(\mathsf{C})$ becomes a triangulated category.

Categories of fractions

Return to the situation at the start of this appendix. If $A \to I^{\cdot}$ and $A \to J^{\cdot}$ are two injective resolutions of A, then there exists a map $I^{\cdot} \to J^{\cdot}$ inducing the identity map on A, and any two such maps are homotopic. Therefore they give a well-defined map $FI^{\cdot} \to FJ^{\cdot}$ in $K(\mathsf{A})$, but this map is not an isomorphism, only a quasi-isomorphism. To complete the program and obtain the derived category $D(\mathsf{A})$, we have to invert the quasi-isomorphisms. We first consider this abstractly.

Let K be an additive category. A class S of morphisms in K is a ***multiplicative system*** if it satisfies the following three conditions.

FR1 The identity map of every object of K belongs to S, and if f and g are composable morphisms in S, then their composite $g \circ f$ belongs to S.

FR2 Every diagram with solid arrows and s in S can be completed to a commutative square with t in S:

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed, "t"] & \bullet \arrow[d, "s"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d, "s"] & \bullet \arrow[d, dashed, "t"] \\ \bullet \arrow[r, dashed] & \bullet \end{tikzcd}$$

FR3 Let f, g be morphisms with the same source and target. If there exists a morphism t such that $t \circ f = t \circ g$, then there exists a morphism s in S with $f \circ s = g \circ s$. Similarly, with the arrows reversed.

A multiplicative system S is said to be ***saturated*** if a morphism f lies in S whenever there exist morphisms h and k such that $f \circ h$ and $k \circ f$ lie in S.

We now let S be a multiplicative system for K, and we define the category $\mathsf{K}S^{-1}$. The objects of $\mathsf{K}S^{-1}$ are the same as the objects of K.

Let X and Y be objects of $K(\mathsf{A})$. Each diagram

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \tag{6}$$

ez1

with s in S defines a morphism $X \to Y$ in $D(\mathsf{A})$, which we denote "$f \circ s^{-1}$". Two diagrams (X', s, f) and (X'', t, g) define the same morphism if and only if there exists a commutative diagram

$$\begin{tikzcd} & & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & X'' \arrow[dlll, "t" near end] \arrow[dr, "g"] & \\ X & & & & Y \end{tikzcd} \tag{7}$$

ez2

with r in S. In other words, the morphisms from X to Y in $D(\mathsf{A})$ are certain equivalence classes of "roofs" (6).

Consider morphisms $X \to Y$ and $Y \to Z$ in $D(\mathsf{A})$:

$$\begin{tikzcd}[column sep=small] & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

This diagram can be completed to a diagram

$$\begin{tikzcd}[column sep=small] & & X'' \arrow[dl, "t'"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

in which t' is in S (by FR2). The composite of the the two morphisms is defined by the diagram (X'', st', gh). In other words,

$$\text{“}g \circ t^{-1}\text{”} \circ \text{“}f \circ s^{-1}\text{”} = \text{“}(gh) \circ \left(st'\right)^{-1}\text{”}.$$

In this way, $\mathsf{K}S^{-1}$ becomes an additive category. The natural functor $Q\colon \mathsf{K} \to \mathsf{K}S^{-1}$ sends elements of S is to isomorphisms, and is universal among additive functors with this property. If S is saturated, then the only morphisms in K mapped to isomorphisms are those in S.

Now let K be a triangulated category with translation functor T, and let S be a multiplicative system of morphisms in K satisfying the following conditions:

FR4 if $s \in S$, then so also does $T(s)$;

FR5 if in TR3, the morphisms u, v are in S, then the diagram can be completed by a morphism in S.

Then $S^{-1}\mathsf{K}$ has a unique structure of triangulated category for which the functor $\mathsf{K} \to S^{-1}\mathsf{K}$ is exact.

NOTES

D.6 Let K be a triangulated category, and let H be a cohomological functor on K. Then the morphisms f whose cone $C(f)$ is such that $H^i(C(f)) = 0$ form a multiplicative system in K (cf. Verdier 1977, 4-1.

D.7 Let K be an additive category. A class of morphisms S in K is said to ***admit a calculus of right fractions*** if it satisfies FR1, FR2 with only the right hand diagram, and the first statement of FR3. Then $\mathsf{K}S^{-1}$ can be constructed as above. Similarly, a class S is said to ***admit a calculus of left fractions*** if it satisfies FR1, FR2 with only the left hand diagram, and the second statement of FR3. For such a class, we can construct a category $S^{-1}\mathsf{K}$ whose morphisms are equivalence classes of diagrams

$$\begin{tikzcd}[column sep=small] X \arrow[dr, "f"] & & Y \arrow[dl, "s"'] \\ & Y' & \end{tikzcd}$$

with s in S. Now let S be an arbitrary class of morphisms in K. It is not difficult to show that there always exists a functor from K to an additive category $\mathsf{K}[S^{-1}]$ sending elements of S to isomorphisms, and universal with this property. However, there is not a pleasant description of the morphisms in $\mathsf{K}[S^{-1}]$ unless S admits a calculus of right (resp. left) fractions, in which case $\mathsf{K}[S^{-1}]$ equals $\mathsf{K}S^{-1}$ (resp. $S^{-1}\mathsf{K}$).[3]

D.8 Let K be a category with a single object X. The conditions FR1,2,3 on a set S of morphisms K correspond to the Ore conditions on S as a subset of the ring $\mathrm{End}(X)$.

Derived categories

Let A be an abelian category. The class of quasi-isomorphisms in $K(\mathsf{A})$ is a multiplicative system[4], and the corresponding category of fractions is called the ***derived category*** $D(\mathsf{A})$ of A. Thus, the objects of $D(\mathsf{A})$ are just the complexes in A.

The functor $T\colon X \rightsquigarrow X[1]$ defines an automorphism T of $D(\mathsf{A})$, and $D(\mathsf{A})$ becomes a triangulated category when we take the distinguished triangles to be those isomorphic (in $D(\mathsf{A})$) to one of the form (3). Now $Q\colon K(\mathsf{A}) \to D(\mathsf{A})$ is an exact functor of triangulated categories.

A sequence of morphisms in $K(\mathsf{A})$,

$$X_1 \xleftarrow{s_1} Y_1 \xrightarrow{f_1} X_2 \xleftarrow{s_2} \cdots \xrightarrow{f_n} X_n,$$

in which the s_i's are quasi-isomorphisms, defines a morphism

$$\text{“}f_n \circ s_n^{-1}\text{”} \circ \cdots \circ \text{“}f_1 \circ s_1^{-1}\text{”}\colon X_1 \to X_n$$

in $D(\mathsf{A})$. It lifts the family of morphisms in A,

$$H^r(f_n) \circ \cdots \circ H^r(s_1)^{-1}\colon H^r(X_1) \to H^r(X_n), \quad r \in \mathbb{Z},$$

to the derived category.

Write Q for the functor $K(\mathsf{A}) \to D(\mathsf{A})$ (or $C(\mathsf{A}) \to D(\mathsf{A})$). Then “$f \circ s^{-1}$”$= Q(f) \circ Q(s)^{-1}$.

The subcategories $C^+(\mathsf{A})$, $C^-(\mathsf{A})$, $C^b(\mathsf{A})$ of $C(\mathsf{A})$ define categories $K^+(\mathsf{A})$, $K^-(\mathsf{A})$, $K^b(\mathsf{A})$ and $D^+(\mathsf{A})$, $D^-(\mathsf{A})$, $D^b(\mathsf{A})$. The inclusion functor

$$D^b(\mathsf{A}) \to D(\mathsf{A})$$

is an equivalence from $D^b(\mathsf{A})$ onto the full subcategory of $D(\mathsf{A})$ whose objects X are such that $H^r(X) = 0$ except possibly for finitely many values of r.

NOTES

z3 D.9 Let X be a complex in A. Then $Q(X) = 0$ in $D(\mathsf{A})$ if and only if $H^r(X) = 0$ for all r. Let $f\colon X \to Y$ be a morphism in $C(\mathsf{A})$. Then $Q(f) = 0$ if and only if there exists a quasi-isomorphism $s\colon X' \to X$ such that $f \circ s = 0$ in $K(\mathsf{A})$ (equivalently, there exists a quasi-isomorphism $t\colon Y \to Y'$ such that $t \circ f = 0$ in $K(\mathsf{A})$). There need not exist an s such that $f \circ s = 0$ in $C(\mathsf{A})$.

[3] For more on such things, see Krause, Henning. Localization theory for triangulated categories. In Triangulated categories, 161–235, London Math. Soc. Lecture Note Ser., 375, Cambridge Univ. Press, Cambridge, 2010.

[4] This is not true in $C(\mathsf{A})$, which why we first pass to the homotopy category.

z4 D.10 Let X be a complex in A. The endomorphisms of X homotopic to zero form an ideal in $\mathrm{End}_{C(\mathsf{A})}(X)$, and the quotient of $\mathrm{End}_{C(\mathsf{A})}(X)$ by this ideal is $\mathrm{End}_{K(\mathsf{A})}(X)$. The quasi-isomorphisms in $\mathrm{End}_{K(\mathsf{A})}(X)$ form multiplicative subset S satisfying the Ore condition (q.v. Wikipedia). Therefore, we can form the quotient ring $S^{-1}\,\mathrm{End}_{K(\mathsf{A})}(X)$. The natural map $\mathrm{End}_{K(\mathsf{A})}(X) \to \mathrm{End}_{D(\mathsf{A})}(X)$ factors through $S^{-1}\,\mathrm{End}_{K(\mathsf{A})}(X)$.

z5 D.11 Let

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

be an exact sequence of complexes in A. The mapping cone $C(p)$ of p is $Y[1] \oplus Z$. There are obvious maps

$$Z \xrightarrow{j} C[p] \xleftarrow{s} X[1],$$

and s is a quasi-isomorphism, and so we get a triangle

$$X \xrightarrow{Qi} Y \xrightarrow{Qp} Z \xrightarrow{(Qs)^{-1}\circ Q(j)} X[1]$$

in $D(\mathsf{A})$. Such a triangle is said to be ***standard.*** The long exact cohomology sequence of this triangle is the cohomology sequence of the original exact sequence. In other words, the map "$s^{-1} \circ j$" lifts the family of connecting morphisms $H^r(Z) \to H^{r+1}(X)$, $r \in \mathbb{Z}$, to the derived category. The distinguished triangles in $D(\mathsf{A})$ are exactly the triangles isomorphic to a standard triangle (Keller 1998, 3.1).[5]

z6 D.12 The Yoneda Ext group $\mathrm{Ext}^r_{\mathsf{A}}(A,B)$ of two objects A and B of an abelian category A is defined to be the group of equivalence classes of exact sequences

$$0 \to B \to X_{r-1} \to \cdots \to X_0 \to A \to 0 \qquad (8) \quad \text{ez5}$$

(see Mitchell 1965, VII). When A has enough projectives (resp. injectives) this agrees with the usual definition defined in terms of projective (resp. injective) resolutions (ibid.). From (8), we get a maps

$$\begin{tikzcd} & (B \arrow[r] \arrow[dl, "f"'] & X_{r-1} \arrow[r] & \cdots \arrow[r] & X_0) \arrow[dr, "s"] & \\ B[r] & & & & & A \end{tikzcd}$$

of complexes. As s is a quasi-isomorphism, this gives a morphism "$f \circ s^{-1}$": $A \to B[r]$ in $D(\mathsf{A})$. In this way, we get a map[6]

$$\mathrm{Ext}^r_{\mathsf{A}}(A,B) \to \mathrm{Hom}_{D(\mathsf{A})}(A, B[r]),$$

which is known to be an isomorphism (Verdier 1996, 3.2.12). [7] In particular,

$$\mathrm{Hom}_{\mathsf{A}}(A,B) \simeq \mathrm{Hom}_{D(\mathsf{A})}(A,B),$$

and so the natural functor $\mathsf{A} \to D(\mathsf{A})$ is fully faithful.

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[5] Keller, Bernhard. Introduction to abelian and derived categories. Representations of reductive groups, 41–61, Publ. Newton Inst., Cambridge Univ. Press, Cambridge, 1998

[6] In fact, in order to get the boundary maps to agree, it is necessary to multiply this map by $(-1)^{r(r+1)/2}$.

[7] Verdier, Jean-Louis. Des catégories dérivées des catégories abéliennes. Edited by Georges Maltsiniotis. Astérisque No. 239 (1996)

Derived subcategories defined by thick subcategories

A full subcategory C of an abelian category A is said to be ***thick*** if it is abelian, the inclusion functor is exact, and if the class of its objects is closed under extension.

Let C be a thick subcategory of A. The category $D_{\mathsf{C}}(\mathsf{A})$ is defined to be the full subcategory of $D(\mathsf{A})$ consisting of the objects X such that $H^r(X) \in \mathsf{C}$ for all r. It is again a triangulated category.

The canonical functor $D^+(\mathsf{C}) \to D^+_{\mathsf{C}}(\mathsf{A})$ is an equivalence if, for every monomorphism $C \to A$ with C in C, there exists a morphism $A \to C'$ with C' in C such that the composite $C \to C'$ is a monomorphism. A similar statement holds with b for $+$.

NOTES

z2 D.13 The construction of the derived category $D(\mathsf{A})$ can be abstracted as follows. Let K be a triangulated category. A ***null system*** in K is a class of objects $\mathcal{N}$ such that (a) $0 \in \mathcal{N}$; (b) $X \in \mathcal{N}$ if and only if $X[1] \in \mathcal{N}$; and (c) if $X, Y \in \mathcal{N}$ and $X \to Y \to Z \to X[1]$ is distinguished, then $Z \in \mathcal{N}$. Let $S(\mathcal{N})$ be the set of morphisms $s\colon X \to Y$ such that there exists a distinguished triangle

$$X \xrightarrow{s} Y \to Z \to X[1]$$

with $Z \in \mathcal{N}$. Then $S(\mathcal{N})$ is a multiplicative system, and we define $\mathsf{K}/\mathcal{N}$ to be the corresponding category of fractions. Then $\mathsf{K}/\mathcal{N}$ is a triangulated category. The natural functor $Q\colon \mathsf{K} \to \mathsf{K}/\mathcal{N}$ has the property that $Q(X) \approx 0$ for all $X \in \mathcal{N}$, and every functor of triangulated categories with this property factors uniquely through Q.

In the construction of the derived category, $\mathcal{N}$ consists of the complexes X such that $H^r(X) = 0$ for all r, and $S(\mathcal{N})$ consists of the quasi-isomorphisms.

z7 D.14 Let B be a full additive subcategory of A such that every object A of A is a subobject of an object of B, i.e., there exists an exact sequence $0 \to A \to B$ with B in B. Then, for every complex X in $C^+(\mathsf{A})$, there exists quasi-isomorphism $X \to X'$ with X' in $C^+(\mathsf{B})$. The class

$$\mathcal{N}' = \mathcal{N} \cap \mathrm{ob}(K^+(\mathsf{B})) = \{X \in \mathrm{ob}(K^+(\mathsf{B}) \mid H^r(X) = 0 \text{ for all } r\}$$

is a null system in $K^+(\mathsf{B})$, and the natural functor

$$K^+(\mathsf{B})/\mathcal{N}' \to D^+(\mathsf{A})$$

is an equivalence of categories.

z8 D.15 Let A be an abelian category, and let I be the full additive subcategory of A whose objects are injective. If I is a complex in $C^+(\mathsf{I})$ such that $H^r(I) = 0$ for all r, then id_I is homotopic to the zero map, and so $I \approx 0$ in $K^+(\mathsf{A})$. The natural functor

$$K^+(\mathsf{I}) \to D^+(\mathsf{A})$$

is always fully faithful, and it is an equivalence of categories if A has enough injectives.

Let A and B be objects of A, and let $s\colon B \to I^{\cdot}$ be an injective resolution of B. Then

$$f \mapsto \text{“}s[r]^{-1} \circ f\text{”}\colon \mathrm{Hom}_{K^+(\mathsf{A})}(A, I^{\cdot}[r]) \to \mathrm{Hom}_{D^+(\mathsf{A})}(A, B[r])$$

is an isomorphism. Therefore (see Example 1),

$$\mathrm{Hom}_{D^+(\mathsf{A})}(A, B[r]) \simeq \mathrm{Ext}^r_{\mathsf{A}}(A, B),$$

in agreement with (D.12).

z11 D.16 Let K be a triangulated category. A ***triangulated subcategory*** K' of K is a subcategory that admits a triangulated structure for which the inclusion is exact. A full subcategory K' of K is triangulated if and only if it is invariant under translation and every distinguished triangle with two objects in K' is isomorphic to a triangle with all of its objects in K'. A full triangulated subcategory M of K is ***thick***[8] if every direct summand of an object of M is isomorphic to an object of M.

Let M be a thick subcategory of K, and let S be the class of morphisms f such that the cone on f lies in M (equivalently, the third vertex of every distinguished triangle based on f lies in M). Then S is a saturated multiplicative subset of K, and every saturated multiplicative subset of K arises in this way. The category $S^{-1}\mathsf{K}$ has a natural structure of triangulated category for which $\mathsf{K} \to S^{-1}\mathsf{K}$ is exact.

Derived functors

Let $F\colon \mathsf{A} \to \mathsf{B}$ be a functor of abelian categories. Then F defines a functor $C^+(\mathsf{A}) \to C^+(\mathsf{B})$ that sends homotopic maps to homotopic maps, and so it defines a functor $K^+(F)\colon K^+(\mathsf{A}) \to K^+(\mathsf{B})$. However, F does not preserve quasi-isomorphisms unless it is exact, and so, in general, there is **not** a functor $RF\colon D^+(\mathsf{A}) \to D^+(\mathsf{B})$ making the following diagram commute

$$\begin{array}{ccc} K^+(\mathsf{A}) & \xrightarrow{K^+(F)} & K^+(\mathsf{B}) \\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle Q} \\ D^+(\mathsf{A}) & \xrightarrow{RF} & K^+(\mathsf{B}). \end{array}$$

Instead, one defines the ***right derived functor*** of F to be a functor $RF\colon D^+(\mathsf{A}) \to D^+(\mathsf{B})$ together a natural transformation $s\colon Q \circ K^+(F) \to RF \circ Q$ such that the pair (RF, s) is universal. When it exists, the pair (RF, s) is unique, up to a unique isomorphism, and $X \rightsquigarrow H^r(RF(X))$ is called the ***rth right derived functor*** of F.

Let $F\colon \mathsf{A} \to \mathsf{B}$ be a left exact functor of abelian categories. A full additive subcategory I of A is said to be ***F-injective*** if

(a) every object of A is a subobject of an object of I;

(b) if $0 \to A' \to A \to A'' \to 0$ is exact in A, and A' and A are in I, then A'' is in I;

(c) if $0 \to A' \to A \to A'' \to 0$ is exact in A, and A' is in I, then

$$0 \to F(A') \to F(A) \to F(A'') \to 0$$

is exact in B.

[8] Verdier's original definition is that M is thick (épaisse) if it satisfies the following condition: let $X \to Y \to Z \to X[1]$ be a distinguished triangle in K; if Z is in M and the map $X \to Y$ factors through an object of M, then X, Y, and Z are all objects of M. It is not difficult to show that the two definitions are equivalent (Rickard, Jeremy. Derived categories and stable equivalence. J. Pure Appl. Algebra 61 (1989), no. 3, 303–317, Proposition 1.3).

The conditions (b) and (c) imply that F maps objects in $\mathcal{N}' \overset{\text{def}}{=} \mathcal{N} \cap K^+(\mathsf{I})$ to the zero object, and so the composite $K^+(\mathsf{I}) \xrightarrow{K^+(F)} K^+(\mathsf{A}) \xrightarrow{Q} D^+(\mathsf{A})$ factors through $K^+(\mathsf{I})/\mathcal{N}'$. On the other hand (see D.14), (a) implies that the natural functor $K^+(\mathsf{I})/\mathcal{N}' \to D^+(\mathsf{A})$ is an equivalence of categories, and so we get a diagram

$$\begin{tikzcd} & K^+(\mathsf{I})/\mathcal{N}' \arrow[dl, "\text{equivalence}"'] \arrow[dr] & \\ D^+(\mathsf{A}) \arrow[rr, dashed, "RF"] & & D^+(\mathsf{B}). \end{tikzcd}$$

Define RF by choosing a quasi-inverse to the equivalence; then RF is the right derived functor of F. Explicitly, let X be a complex in $C^+(\mathsf{A})$; according to (D.14), there exists a quasi-isomorphism $X \to I$ with I in $C^+(\mathsf{I})$; the complex FI, regarded as an object in $D^+(\mathsf{B})$, is independent of the choice of the quasi-isomorphism up to a unique isomorphism, and is defined to be $RF(X)$.

Note that the full subcategory I of A consisting of the injective objects satisfies the conditions (b) and (c) because every short exact sequence whose first object is injective splits. Therefore, if A has enough injectives, the subcategory I is F-injective, the right derived functor RF exists, and $RF(X) = F(I)$ for any quasi-isomorphism $X \to I$ with X in $C^+(\mathsf{A})$ and I in $C^+(\mathsf{I})$.

Let $F\colon \mathsf{A} \to \mathsf{B}$ be left exact, and assume that RF exists. An object X of A is said to be ***F-acyclic*** if $R^rF(X) = 0$ for all $r > 0$. The full subcategory of A whose objects are ***F-acyclic*** if F-injective (and contains every F-injective subcategory). Therefore, if RF exists, $RF(X)$ can always be computed by choosing a quasi-isomorphism $X \to I$ with I a bounded-below complex of F-acyclics, and setting $RF(X) = FI$.

Left derived functors are defined similarly. Let $F\colon \mathsf{A} \to \mathsf{B}$ be a left exact functor of abelian categories. If A has enough projectives, then the left derived functor LF of F exists; moreover, $LF(X) = F(P)$ for any quasi-isomorphism $P \to X$ with P a bounded-above complex of projectives.

Composites of derived functors

Let

$$\mathsf{A} \xrightarrow{F} \mathsf{B} \xrightarrow{G} \mathsf{C}$$

be left exact functors of abelian categories. If RF, RG, and $R(G \circ F)$ all exist, then it follows from the definition of "right derived functor" that there is a canonical morphism of functors

$$R(G \circ F) \to RG \circ RF.$$

This is an isomorphism, for example, if A has enough injectives and F maps injective objects in A to F-acyclic objects in B. Indeed, let X be a complex in $C^+(\mathsf{A})$, and choose a quasi-isomorphism $X \to I$ with I a bounded-below complex of injective objects; then $R(G \circ F)(X) = (G \circ F)(I)$ and $(RG \circ RF)(X) = RG(FI) = G(FI)$.

Derived Homs

Let A be an abelian category. For complexes X and Y in A with Y bounded below, define $\mathrm{Hom}^{\cdot}(X, Y)$ to be the complex of abelian groups with $\mathrm{Hom}^r(X, Y)$ equal to the group of homomorphisms $X \to Y$ of degree r and with the differential

$$\varphi \mapsto d_Y \circ \varphi - (-1)^r \varphi \circ d_X.$$

In this way, we get a bifunctor

$$\mathrm{Hom}^{\cdot}\colon K(\mathsf{A})^{\mathrm{opp}} \times K(\mathsf{A}) \to K(\mathsf{Ab}).$$

If A has enough injectives, this derives to a bifunctor

$$R\,\mathrm{Hom}\colon D(\mathsf{A})^{\mathrm{opp}} \times D^{+}(\mathsf{A}) \to D(\mathsf{Ab}).$$

Moreover,

$$\mathrm{Hom}_{D(\mathsf{A})}(X,Y) \simeq H^0(R\,\mathrm{Hom}(X,Y)).$$

This suggest defining

$$\begin{aligned} \mathrm{Ext}^r_{D(\mathsf{A})}(X,Y) &= H^r(R\,\mathrm{Hom}(X,Y)) \\ &\simeq H^0(R\,\mathrm{Hom}(X,Y[r])) \\ &\simeq \mathrm{Hom}_{D(\mathsf{A})}(X,Y[r]). \end{aligned}$$

When $X, Y \in \mathrm{ob}(\mathsf{A})$,

$$\mathrm{Ext}^r_{D(\mathsf{A})}(X,Y) \simeq \mathrm{Ext}^r_{\mathsf{A}}(X,Y).$$

Derived tensor products

Let A be an abelian category with a tensor product structure. For bounded-above complexes X and Y in A, define $X \otimes Y$ to be the complex with

$$(X \otimes Y)^r = \bigoplus_{p+q=r} X^p \otimes Y^q$$

and with the differential

$$d_X \otimes 1 + 1 \otimes d_Y\colon (X \otimes Y)^r \to (X \otimes Y)^{r+1}.$$

If A has enough flat objects, then the bifunctor

$$\otimes\colon K^{-}(\mathsf{A}) \times K^{-}(\mathsf{A}) \to K^{-}(\mathsf{A})$$

derives to a functor

$$\otimes^{L}\colon D^{-}(\mathsf{A}) \times D^{-}(\mathsf{A}) \to D^{-}(\mathsf{A}).$$

To compute it on a pair of complexes X, Y, choose a quasi-isomorphism $P \to X$ (or $P \to Y$) with P a bounded-above complex of flat objects, and then

$$X \otimes^{L} Y = P \otimes Y \text{ (or } X \otimes P).$$

NOTES There is a friendly introduction to the theory of derived categories in Iversen 1986[9]. The first chapter of Kashiwara and Schapira 1990[10] is an excellent survey of the homological algebra one needs for sheaf theory, and their book 2006[11] is a very complete account of the same topic. See also Holm and Jørgensen 2010[12].

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[9] Iversen, Birger. Cohomology of sheaves. Universitext. Springer-Verlag, Berlin, 1986.

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